Operational Methods in the Study of Sobolev-Jacobi Polynomials – additional materials

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MARIE CURIE ACTIONS

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Sobolev-Jacobi polynomials as umbral images of bi-variate Hermite polynomials [1]

Consider the polynomials $P_n(x,y)$ and $H_n(x,z)$ and their exponential generating functions:

$$P_n(x,y) := e^{-\frac{1}{2}\frac{1}{\hat{D}_x+\hat{D}_y-1}\partial_x^2}(xy)^n\,, \qquad \mathscr{G}(\lambda;x,y) := \sum_{n\geqslant 0}\frac{\lambda^n}{n!}P_n(x,y)$$

$$H_n(x,z) := e^{z\partial_x^2}x^n\,, \qquad \mathscr{H}(\lambda;x,z) := \sum_{n\geqslant 0}\frac{\lambda^n}{n!}H_n(x,z)\,.$$

Theorem 1 of [1]

The exponential generating function $\mathscr{G}(\lambda;x,y)$ is an **umbral image** of the generating function $\mathscr{H}(\lambda;x,z)$,

$$\mathscr{G}(\lambda;x,y) = \hat{\mathbb{I}}\left(\frac{1}{\sqrt{uv}}e^{(\lambda uvy)x+(\lambda uvy)^2\left(-\frac{1}{4u}\right)}\right) = \hat{\mathbb{I}}\left(\frac{1}{\sqrt{uv}}\mathscr{H}\left(\lambda uvy;x,-\frac{1}{4u}\right)\right)\,. \tag{1}$$

Thus the **monic SJ polynomials** $P_n(x) \equiv \tilde{P}_n^{(-1,-1)}(x) := P_n(x,1)$ may alternatively be expressed as

$$P_n(x) \equiv \tilde{P}_n^{(-1,-1)}(x) = \hat{\mathbb{I}}\left((uv)^{n-\frac{1}{2}}H_n(x,-\tfrac{1}{4u})\right) = \hat{\mathbb{I}}\left(\frac{1}{\sqrt{uv}}e^{-\frac{1}{4u}\partial_x^2}(xuv)^n\right)\,. \tag{2}$$

[1] Nicolas Behr, Giuseppe Dattoli, Gérard H.E. Duchamp, Silvia Licciardi, and Karol A. Penson. “Operational Methods in the Study of Sobolev-Jacobi Polynomials”. In: *arXiv:1810.09793* (today)

All-order lacunary exponential generating functions for the SJ polynomials

Consider the polynomials $P_n(x,y)$ and $H_n(x,z)$ and their **lacunary EGFs** ($K \geqslant 1, L \geqslant 0$):

$$\mathscr{G}_{K,L}(\lambda;x,y) := \sum_{n\geqslant 0} \frac{\lambda^n}{n!} P_{n\cdot K+L}(x,y)\,, \qquad \mathscr{H}_{K,L}(\lambda;x,z) := \sum_{n\geqslant 0} \frac{\lambda^n}{n!} H_{n\cdot K+L}(x,z)\,.$$

Theorem 2 of [1]

$$\mathscr{G}_{K=2T,0}(\lambda;x) = \sum_{\beta=0}^{T-1} \sum_{s\geqslant 0} \frac{\lambda^s}{s!} x^{K\cdot s-2\beta} \left(-\tfrac{1}{4}\right)^{\beta} \tilde{h}_{K\cdot s,\beta} \frac{\Gamma(K\cdot s-\beta-\frac{1}{2})}{\Gamma(K\cdot s-\frac{1}{2})} \cdot$$

$$\cdot\; {}_{(3T-1)}F_{(3T-1)}\left[\begin{matrix} \left(\left(s+\frac{j+1}{K}\right)_{0\leqslant j\leqslant K-2},\ \left(2s+\frac{2m-\beta-1}{K}\right)_{0\leqslant m\leqslant T-1}\right) \\ \left(\left(\frac{\beta+\ell+1}{T}\right)_{\substack{0\leqslant \ell\leqslant T-1,\\ \ell\neq T-1-\beta}},\ \left(s+\frac{2t-1}{2K}\right)_{0\leqslant t\leqslant K-1}\right)\end{matrix};\lambda\left(-\tfrac{1}{4}\right)^{T}\right]$$

Here, $\tilde{h}_{n,k}$ denote the so-called *matching coefficients* (of directed Hermite-configurations),

$$\tilde{h}_{n,k} := \begin{cases} \frac{n!}{(n-2k)!k!} & \text{if } 0\leqslant 2k\leqslant n \\ 0 & \text{otherwise.}\end{cases}$$

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$$\mathscr{G}_{K=2T+1,0}(\lambda;x) = \sum_{\beta=1}^{K-1} \sum_{s\geqslant 0} \frac{\lambda^s}{s!} x^{K\cdot s-2\beta} \left(-\tfrac{1}{4}\right)^{\beta} \tilde{h}_{K\cdot s,\beta} \frac{\Gamma(K\cdot s-\beta-\frac{1}{2})}{\Gamma(K\cdot s-\frac{1}{2})} \cdot$$

$$\cdot\ {}_{(3K-2)}F_{(3K-1)}\left[\begin{matrix}\left(\left(\frac{s}{2}+\frac{j+1}{2K}\right)_{\substack{0\leqslant j\leqslant 2K-2\\ j\neq K-1}},\ \left(s+\frac{2m-2\beta-1}{2K}\right)_{0\leqslant m\leqslant K-1}\right)\\ \left(\left(\frac{\beta+\ell+1}{K}\right)_{\substack{0\leqslant \ell\leqslant K-1\\ \ell\neq K-1-\beta}},\ \left(\frac{s}{2}+\frac{2t-1}{4K}\right)_{0\leqslant t\leqslant 2K-1}\right)\end{matrix};-\frac{\lambda^2}{4^{K+1}}\right]$$

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Theorem 2 of [1]

For integer parameters $K \in \mathbb{Z}_{\geqslant 1}$, denote by

$$\mathscr{G}_K(\mu;\lambda;x) := \sum_{L\geqslant 0} \frac{\mu^L}{L!}\, \mathscr{G}_{K,L}(\lambda;x) \tag{3}$$

the exponential generating function of lacunary shifts of the lacunary generating functions $\mathscr{G}_{K,0}(\lambda;x)$. Then $\mathscr{G}_K(\mu;\lambda;x)$ is given by

$$\mathscr{G}_K(\mu;\lambda;x) = \hat{\mathbb{I}}\left(\frac{1}{\sqrt{uv}}\, e^{\mu uvx - \frac{\mu^2 uv^2}{4}}\, \mathscr{H}_{K,0}\left(\lambda(uv)^K; x - \frac{\mu v}{2}, -\frac{1}{4u}\right)\right)\,. \tag{4}$$

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Examples for SJ-polynomial lacunary exponential generating functions

$$\mathscr{G}_{K,L}(\lambda;x) := \sum_{n\geqslant 0} \frac{\lambda^n}{n!} P_{n\cdot K+L}(x)$$

$$\mathscr{G}_{1,0}(\lambda;x) = \sum_{s\geqslant 0} \tfrac{\lambda^s}{s!} x^s \; {}_1F_2\left[\begin{matrix} s-\frac{1}{2} \\ \frac{s}{2}-\frac{1}{4},\; \frac{s}{2}+\frac{1}{4} \end{matrix}; -\frac{\lambda^2}{16}\right]$$

$$\mathscr{G}_{2,0}(\lambda;x) = \sum_{s\geqslant 0} \tfrac{\lambda^s}{s!}\, x^{2s} \; {}_2F_2\left[\begin{matrix} s+\frac{1}{2},\; 2s-\frac{1}{2} \\ s-\frac{1}{4},\; s+\frac{1}{4} \end{matrix}; -\frac{\lambda}{4}\right]$$

$$\mathscr{G}_{2,1}(\lambda;x) = \sum_{s\geqslant 0} \tfrac{\lambda^s}{s!}\, x^{2s+1} \; {}_2F_2\left[\begin{matrix} s+\frac{3}{2},\; 2s+\frac{1}{2} \\ s+\frac{1}{4},\; s+\frac{3}{4} \end{matrix}; -\frac{\lambda}{4}\right]$$

Examples for SJ-polynomial lacunary exponential generating functions

$$\mathscr{G}_{3,0}(\lambda;x)=\sum_{s\geqslant 0}\frac{\lambda^s}{s!}\,x^{3s}\,{}_7F_8\left[\begin{matrix}\frac{s}{2}+\frac{1}{6},\ \frac{s}{2}+\frac{1}{3},\ \frac{s}{2}+\frac{2}{3},\ \frac{s}{2}+\frac{5}{6},\ s-\frac{1}{6},\ s+\frac{1}{6},\ s+\frac{1}{2}\\ \frac{1}{3},\ \frac{2}{3},\ \frac{s}{2}-\frac{1}{12},\ \frac{s}{2}+\frac{1}{12},\ \frac{s}{2}+\frac{1}{4},\ \frac{s}{2}+\frac{5}{12},\ \frac{s}{2}+\frac{7}{12},\ \frac{s}{2}+\frac{3}{4}\end{matrix};-\frac{\lambda^2}{256}\right]$$

$$-\sum_{s\geqslant 0}\frac{\lambda^{s+1}}{(s+1)!}\,x^{3(s+1)-2}\left(\frac{\Gamma\left(3(s+1)-\frac{3}{2}\right)}{4\Gamma\left(3(s+1)-\frac{1}{2}\right)}\right)\left(\frac{(3(s+1))!}{(3(s+1)-2)!}\right)\cdot$$

$$\cdot\ {}_7F_8\left[\begin{matrix}\frac{s+1}{2}+\frac{1}{6},\ \frac{s+1}{2}+\frac{1}{3},\ \frac{s+1}{2}+\frac{2}{3},\ \frac{s+1}{2}+\frac{5}{6},\ (s+1)-\frac{1}{2},\ (s+1)-\frac{1}{6},\ (s+1)+\frac{1}{6}\\ \frac{2}{3},\ \frac{4}{3},\ \frac{s+1}{2}-\frac{1}{12},\ \frac{s+1}{2}+\frac{1}{12},\ \frac{s+1}{2}+\frac{1}{4},\ \frac{s+1}{2}+\frac{5}{12},\ \frac{s+1}{2}+\frac{7}{12},\ \frac{s+1}{2}+\frac{3}{4}\end{matrix};-\frac{\lambda^2}{256}\right]$$

$$+\sum_{s\geqslant 0}\frac{\lambda^{s+2}}{(s+2)!}\,x^{3(s+2)-4}\left(\frac{\Gamma\left(3(s+2)-\frac{5}{2}\right)}{16\Gamma\left(3(s+2)-\frac{1}{2}\right)}\right)\left(\frac{(3(s+2))!}{2!(3(s+2)-4)!}\right)\cdot$$

$$\cdot\ {}_7F_8\left[\begin{matrix}\frac{s+2}{2}+\frac{1}{6},\ \frac{s+2}{2}+\frac{1}{3},\ \frac{s+2}{2}+\frac{2}{3},\ \frac{s+2}{2}+\frac{5}{6},\ (s+2)-\frac{5}{6},\ (s+2)-\frac{1}{2},\ (s+2)-\frac{1}{6}\\ \frac{4}{3},\ \frac{5}{3},\ \frac{s+2}{2}-\frac{1}{12},\ \frac{s+2}{2}+\frac{1}{12},\ \frac{s+2}{2}+\frac{1}{4},\ \frac{s+2}{2}+\frac{5}{12},\ \frac{s+2}{2}+\frac{7}{12},\ \frac{s+2}{2}+\frac{3}{4}\end{matrix};-\frac{\lambda^2}{256}\right]$$

Examples for SJ-polynomial lacunary exponential generating functions

$$\mathscr{G}_{4,0}(\lambda;x) = \sum_{s\geqslant 0} \frac{\lambda^s}{s!}\, x^{4s}\; {}_5F_5\left[\begin{matrix} s+\frac{1}{4},\ s+\frac{1}{2},\ s+\frac{3}{4},\ 2s-\frac{1}{4},\ 2s+\frac{1}{4} \\ \frac{1}{2},\ s-\frac{1}{8},\ s+\frac{1}{8},\ s+\frac{3}{8},\ s+\frac{5}{8}\end{matrix};\frac{\lambda}{16}\right]$$

$$-\sum_{s\geqslant 0} \frac{\lambda^{s+1}}{(s+1)!}\, x^{4(s+1)-2} \left(\frac{\Gamma\big(4(s+1)-\frac{3}{2}\big)}{4\Gamma\big(4(s+1)-\frac{1}{2}\big)}\right)\left(\frac{(4(s+1))!}{(4(s+1)-2)!}\right)\,.$$

$$\cdot\ {}_5F_5\left[\begin{matrix} (s+1)+\frac{1}{4},\ (s+1)+\frac{1}{2},\ (s+1)+\frac{3}{4},\ 2(s+1)-\frac{3}{4},\ 2(s+1)-\frac{1}{4} \\ \frac{3}{2},\ (s+1)-\frac{1}{8},\ (s+1)+\frac{1}{8},\ (s+1)+\frac{3}{8},\ (s+1)+\frac{5}{8}\end{matrix};\frac{\lambda}{16}\right]$$

Thank you!

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