Combinatorial Physics: Schützenberger factorization and noncommutative differential equations

Gérard H. E. Duchamp

LIPN, ***Université de Paris XIII, France***

Collaborators

Pawel Blasiak, ***Instit. of Nucl. Phys., Krakow, Pologne***

Andrzej Horzela, ***Instit. of Nucl. Phys., Krakow, Pologne***

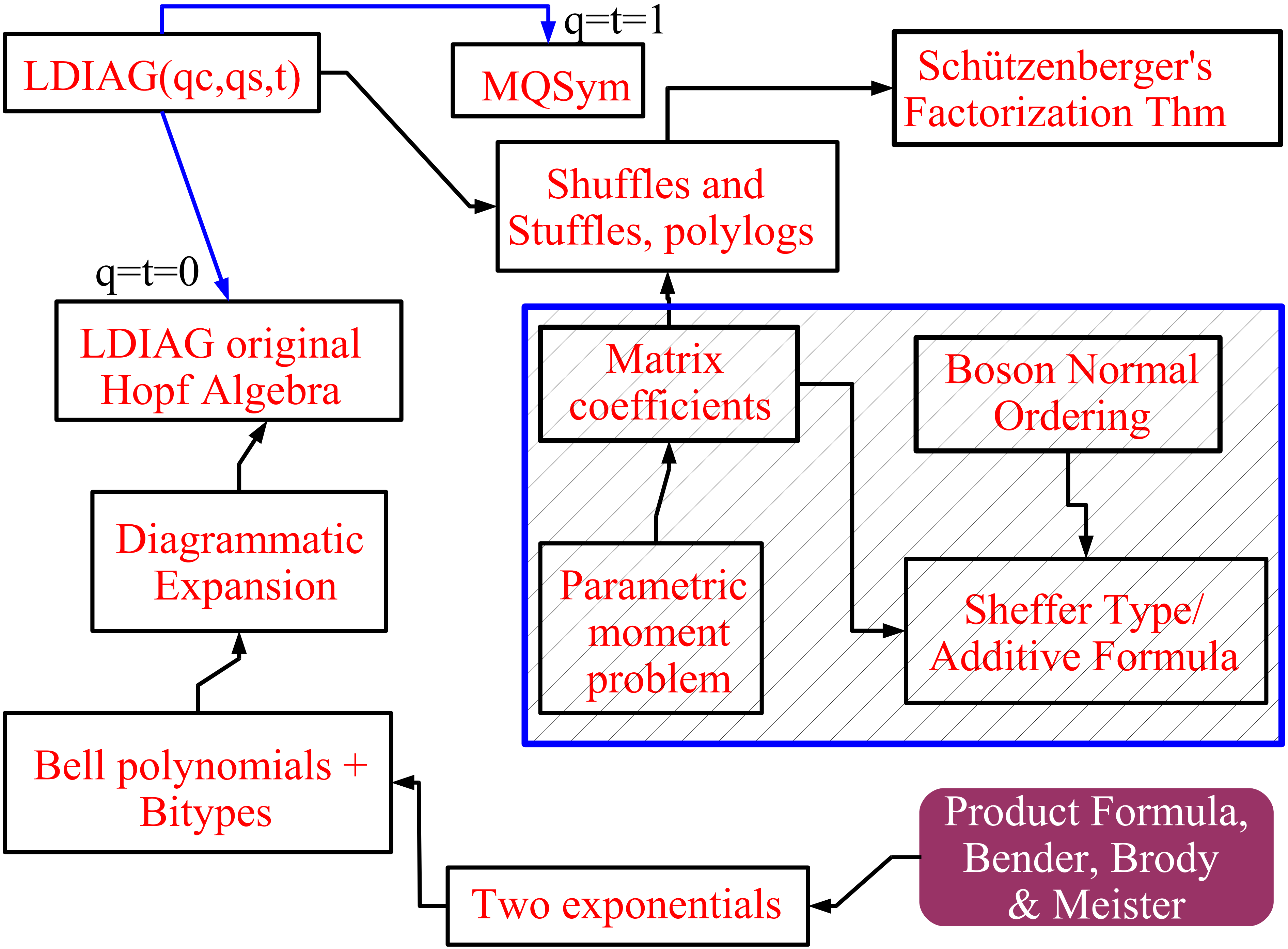
Karol A. Penson, **LPTMC**, ***Université de Paris VI, France***

Allan I. Solomon, ***The Open University, United Kingdom***

Hoang Ngoc Minh, **LIPN**, ***Université de Paris XIII, France***

Christophe Tollu, **LIPN**, ***Université de Paris XIII, France***

Moscou 01 Déc. 2010


LDIAG(qc,qs,t)
q=t=1
MQSym
Schützenberger's Factorization Thm
Shuffles and Stuffles, polylogs
q=t=0
LDIAG original Hopf Algebra
Matrix coefficients
Boson Normal Ordering
Diagrammatic Expansion
Parametric moment problem
Sheffer Type/ Additive Formula
Bell polynomials + Bitypes
Product Formula, Bender, Brody & Meister
Two exponentials

An interface between physics and number theory

Gérard H.E. Duchamp[1], **Hoang Ngoc Minh**[1,2], **Allan I. Solomon**[3,4], **Silvia Goodenough**[1]

[1] Université Paris 13 – LIPN - UMR 7030 CNRS, F-93430 Villetaneuse, France.
[2] Université Lille 2, 1 Place Déliot, 59024 Lille Cédex, France.
[3] Physics and Astronomy Department,The Open University, Milton Keynes MK7 6AA, UK
[4] Université Pierre et Marie Curie, LPTMC, UMR 7600 CNRS, 4 pl. Jussieu, 75252 Paris 5, France.

Abstract. We extend the Hopf algebra description of a simple quantum system given previously, to a more elaborate Hopf algebra, which is rich enough to encompass that related to a description of perturbative quantum field theory (pQFT). This provides a *mathematical* route from an algebraic description of non-relativistic, non-field theoretic quantum statistical mechanics to one of relativistic quantum field theory.

Such a description necessarily involves treating the algebra of polyzeta functions, extensions of the Riemann Zeta function, since these occur naturally in pQFT. This provides a link between physics, algebra and number theory. As a by-product of this approach, we are led to indicate *inter alia* a basis for concluding that the Euler gamma constant γ may be rational.

1. Introduction

In an introductory paper delivered at this Conference[1], *"From Quantum Mechanics to Quantum Field Theory: The Hopf route"*, Allan I. Solomon [1, 2] *et al.* start their exposition with the Bell numbers $B(n)$ which count the number of *set partitions* within a given set of n elements. It is shown there that these very elementary combinatorial ideas are in a sense generic within where G satisfies the following Fuchs differential equation with three regular singularities at $0, 1$ and ∞ :

$$dG(z) = [x_0\omega_0(z) + x_1\omega_1(z)]G(z), \tag{29}$$

with

$$\omega_0(z) := \frac{dz}{z} \quad \text{and} \quad \omega_1(z) := \frac{dz}{1-z}, \tag{30}$$

$$x_0 := \frac{t_{1,2}}{2\mathrm{i}\pi} \quad \text{and} \quad x_1 := -\frac{t_{2,3}}{2\mathrm{i}\pi}. \tag{31}$$

In the sequel, we set $X = \{x_0, x_1\}$; X^* denotes the set of words defined over X. The empty word is denoted by ε.

Proposition 2 ([11]) *If $G(z)$ and $H(z)$ are exponential solutions of (29) then there exists a Lie series $C \in \mathcal{L}ie_{\mathbb{C}}\langle\langle X\rangle\rangle$ such that $G(z) = H(z)\exp(C)$.*

$$dG(z) = [x_0\omega_0(z) + x_1\omega_1(z)]G(z), \tag{29}$$

with

$$\omega_0(z) := \frac{dz}{z} \quad \text{and} \quad \omega_1(z) := \frac{dz}{1-z}, \tag{30}$$

$$x_0 := \frac{t_{1,2}}{2\mathrm{i}\pi} \quad \text{and} \quad x_1 := -\frac{t_{2,3}}{2\mathrm{i}\pi}. \tag{31}$$

In the sequel, we set $X = \{x_0, x_1\}$; X^* denotes the set of words defined over X. The empty word is denoted by ε.

Proposition 2 ([11]) *If $G(z)$ and $H(z)$ are exponential solutions of (29) then there exists a Lie series $C \in \mathcal{L}ie_{\mathbb{C}}\langle\langle X \rangle\rangle$ such that $G(z) = H(z)\exp(C)$.*

4. Iterated integral and Chen generating series

The iterated integral associated with $w = x_{i_1} \cdots x_{i_k} \in X^*$, over ω_0 and ω_1 and along the path $z_0 \rightsquigarrow z$, is defined by the following multiple integral

$$\int_{z_0}^{z} \ldots \int_{z_0}^{z_{k-1}} \omega_{i_1}(t_1) \ldots \omega_{i_k}(t_k), \tag{32}$$

where $t_1 \cdots t_{r-1}$ is a subdivision of the path $z_0 \rightsquigarrow z$. In an abbreviated notation, we denote this integral by $\alpha_{z_0}^{z}(w)$ and $\alpha_{z_0}^{z}(\varepsilon) = 1$.

Example 1

$$\begin{aligned}
\alpha_0^z(x_0 x_1) &= \int_0^z \int_0^s \omega_0(s)\omega_1(t) \\
&= \int_0^z \int_0^s \frac{ds}{s}\frac{dt}{1-t} \\
&= \int_0^z \frac{ds}{s} \int_0^s dt \sum_{k \geq 0} t^k \\
&= \sum_{k \geq 1} \int_0^z ds \frac{s^{k-1}}{k} \\
&= \sum_{k \geq 1} \frac{z^k}{k^2}.
\end{aligned}$$

The last sum is nothing other than the Taylor expansion of the dilogarithm $\mathrm{Li}_2(z)$.

Series with variable coefficients.

Let V be a set (where will run the variable z), $\mathcal{H} \subset \mathbb{C}^V$, an algebra of fonctions and $\mathcal{M}$ a monoid (thought as a monooid of monomials). Every function $S : \mathcal{M} \to \mathcal{H}$ can be written

$$S := \sum_{m \in \mathcal{M}} \langle S|m \rangle m \ . \tag{4}$$

This series can be specialized, for all $z_0 \in V$ as

$$S(z_0) := \sum_{m \in \mathcal{M}} \Big(\langle S|m \rangle \Big)\Big|_{z=z_0} m \tag{5}$$

Series with variable coefficients (2).

These specializations are morphismes of $\mathbb{C}$-AAU

$$\epsilon_{z_0} \; : \; \mathcal{H}\langle\langle\mathcal{M}\rangle\rangle \to \mathbb{C}\langle\langle\mathcal{M}\rangle\rangle \tag{6}$$

Moreover, if d is a derivation of $\mathcal{H}$, its extension "coefficient by coefficient"

$$d(S) := \sum_{m\in\mathcal{M}} d(\langle S|m\rangle)m \tag{7}$$

is still a derivation of $\mathcal{H}\langle\langle\mathcal{M}\rangle\rangle$ (exercise).

Non commutative differential equations.

Now V is a connected and simply connected complex analytic variety (for example $\mathbb{C} - (]-\infty, 0[\cup]1, +\infty[)$ or the universal covering of $\mathbb{C} - \{0, 1\}$) ; $\mathcal{H}$ is the space of analytic functions on V. Let $\mathcal{M}$ be a locally finite monoid (we will not fix it too early in order to "double" Drinfel'd's equation). Let $\mathcal{H}\langle\langle\mathcal{M}\rangle\rangle$ be the algebra of fonctions on $\mathcal{M}$ with coefficients in $\mathcal{H}$. One will use the notations of Dirac-Schützenberger. Let M, be a multiplier in $\mathcal{H}_{\geq 1}\langle\langle\mathcal{M}\rangle\rangle$.

Non commutative differential equations (2).

We will consider three types of differential equations.

a) **Equation on the left**

$$\frac{d}{dz}S = MS \tag{8}$$

b) **Equation on the right**

$$\frac{d}{dz}S = SM \tag{9}$$

c) **Two-sided equation**

$$\frac{d}{dz}S = M_1 S + S M_2 \tag{10}$$

with $M, M_i \in \mathcal{H}_{\geq 1}\langle\langle \mathcal{M} \rangle\rangle$. We will give first the resolution of equations of type (10) because their properties specialize, with $M_2 = 0$ (resp. $M_1 = 0$) to the type (8) (resp. (9)).

Non commutative differential equations (3).

Theorem

In the preceding conditions.
i) Equation (10) has solutions all of the forme

$$S = H^* S_0 \tag{11}$$

where H is the operator $G \mapsto \int_{z_0}^{z} \Big(M_1(s)G(s) + G(s)M_2(s) \Big) ds$ and $S_0 = S(z_0)$ is a une constante series.
ii) Two solutions agree iff they agree at a point $z_0 \in V$.
iii) One supposes given a comultiplication Δ with constant i.e. a morphisme $\Delta : \mathcal{M} \to \mathbb{C}\langle \mathcal{M} \rangle \otimes \mathbb{C}\langle \mathcal{M} \rangle$ and that M_1, M_2 are primitive i.e.

$$\Delta(M_i) = M_i \otimes 1 + 1 \otimes M_i \ ; \ i = 1, 2 \ .$$

Then, if S is group-like at a point $z_0 \in V$ then S is group-like everywhere.

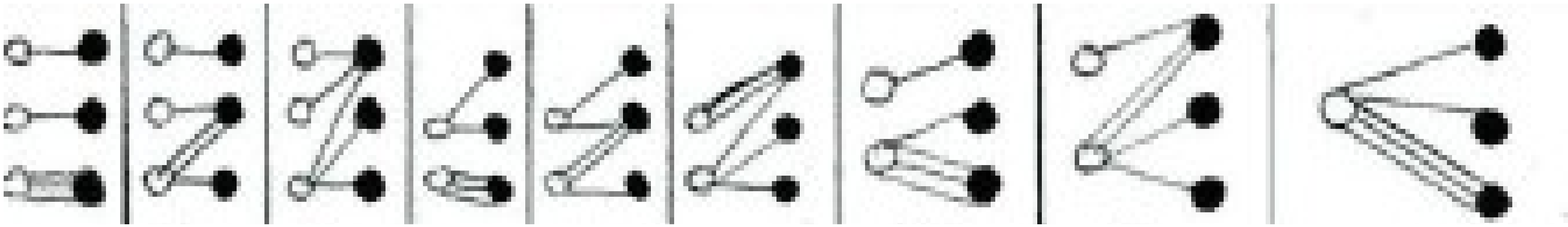
Theorem (2)

iv) Constant term of S does not depend on z and is that of S_0, in particular, if a solution is invertible at a point, it is so everywhere.
v) Let $S_i;\ i = 1, 2$ be invertible solutions of equations of type (8) (resp. (9)) with primitive multipliers. Let $\mathcal{F}$ be a un filter on V (ex. neibourhoods of 0, of 1, of infinity etc.). We moreover suppose that S_1, S_2 are asymtotically equivalent w.r.t. $\mathcal{F}$. Then if S_2 is de group-like, so is S_1.

About the LDIAG Hopf algebra

In a relatively recent paper Bender, Brody and Meister (*) introduce a special Field Theory described by a product formula (a kind of Hadamard product for two exponential generating functions - EGF) in the purpose of proving that any sequence of numbers could be described by a suitable set of rules applied to some type of Feynman graphs (see third Part of this talk). These graphs label monomials and are obtained in the case of special interest when the two EGF have a constant term equal to unity.

Bender, C.M, Brody, D.C. and Meister,
Quantum field theory of partitions, J. Math. Phys. Vol 40 (1999)



Some 5-line diagrams

If we write these functions as exponentials, we are led to witness a surprising interplay between the following aspects: **algebra** (of normal forms or of the exponential formula, Hopf structure), **geometry** (of one-parameter groups of transformations and their conjugates) and **analysis** (parametric Stieltjes moment problem and convolution of kernels).
Today, we will first focus on the algebra.
If time permits, we will touch on the other aspects.

Construction of the Hopf algebra LDIAG

How these diagrams arise and which data structures are around them

Let F, G be two EGFs.

$$F = \sum_{n\geq 0} a_n \frac{y^n}{n!};\; G = \sum_{m\geq 0} b_m \frac{y^m}{m!};\; \mathcal{H}(F,G) := \sum_{n\geq 0} a_n b_n \frac{y^n}{n!}$$

$$\mathcal{H}(F,G) = F(y\frac{d}{dx})G(x)|_{x=0}$$

Called « product formula » in the QFTP of Bender, Brody and Meister.

In case *F(0)=G(0)=1*, one can set

$$F(y) = exp(\sum_{n\geq 1} L_n \frac{y^n}{n!}) \quad G(x) = exp(\sum_{n\geq 1} V_m \frac{x^m}{m!})$$

and then,

$$\mathcal{H}(F,G) = F(y\frac{d}{dx})G(x)|_{x=0} =$$

$$\sum_{n\geq 0} \frac{y^n}{n!} \sum_{|\alpha|=|\beta|=n} numpart(\alpha)numpart(\beta)\mathbb{L}^{\alpha}\mathbb{V}^{\beta}$$

with $\alpha, \ \beta \in \mathbb{N}^{(\mathbb{N}^*)}$ multiindices

$$numpart(\alpha) = \frac{|\alpha|!}{(1!)^{a_1}(2!)^{a_2}\cdots(r!)^{a_r}(a_1)!(a_2)!\cdots(a_r)!}$$

We will adopt the notation

$$\alpha = 1^{a_1} 2^{a_2} \cdots r^{a_r}$$

for the *type* of a (set) partition which means that there are a_1 singletons a_2 pairs a_3 3-blocks a_4 4-blocks and so on.

The number of set partitions of type α as above is well known (see Comtet for example)

$$numpart(\alpha) = \frac{|\alpha|!}{(1!)^{a_1}(2!)^{a_2}\cdots(r!)^{a_r}(a_1)!(a_2)!\cdots(a_r)!}$$

Then, with

$$F(y) = exp(\sum_{n\geq 1} L_n \frac{y^n}{n!}) \quad G(x) = exp(\sum_{n\geq 1} V_m \frac{x^m}{m!})$$

$$F(y) = exp(\sum_{n\geq 1} L_n \frac{y^n}{n!}) \quad G(x) = exp(\sum_{n\geq 1} V_m \frac{x^m}{m!})$$

one has

$$\mathcal{H}(F,G) = F(y\frac{d}{dx})G(x)|_{x=0} =$$

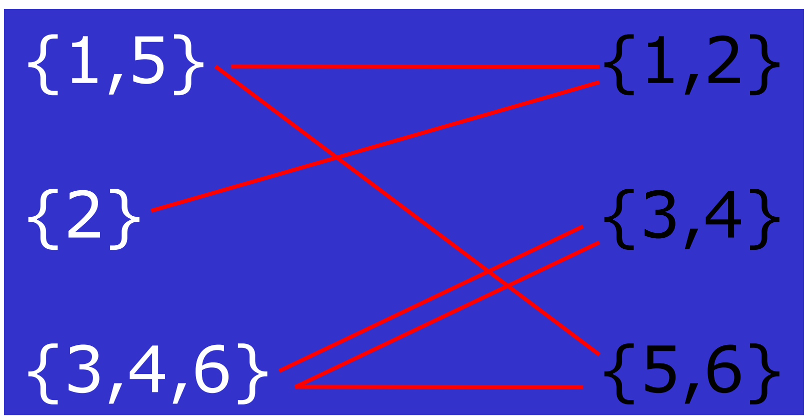
$$\sum_{n\geq 0} \frac{y^n}{n!} \sum_{|\alpha|=|\beta|=n} numpart(\alpha)numpart(\beta)\mathbb{L}^\alpha\mathbb{V}^\beta$$

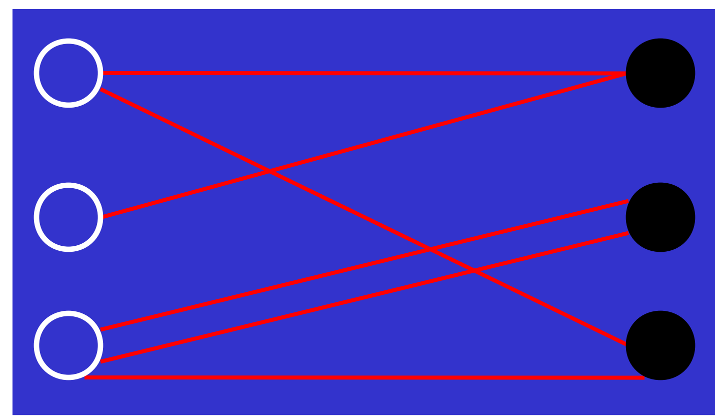
Now, one can count in another way the term *numpart(α)numpart(β)*. Remarking that this is the number of pairs of set partitions (P1,P2) with type(P1)=α, type(P2)=β. But every pair of partitions (P1,P2) has an intersection matrix ...

	{1,5}	{2}	{3,4,6}
{1,2}	1	1	0
{3,4}	0	0	2
{5,6}	1	0	1

Classes of packed matrices see NCSF VI (GD, Hivert, and Thibon)

Feynman-type diagram (Bender & al.)





Now the product formula for EGFs reads

$$\mathcal{H}(F,G) = \sum_{d \ FB-diagram} \frac{y^{|d|}}{|d|!} \mathbb{L}^{\alpha(d)} \mathbb{V}^{\beta(d)}$$

$$\mathcal{H}(F,G) = \sum_{d \in \mathbf{diag}} \frac{y^{|d|}}{|d|!} mult(d) \mathbb{L}^{\alpha(d)} \mathbb{V}^{\beta(d)}$$

The main interest of these new forms is that we can impose rules on the counted graphs and we can call these (and their relatives) graphs : Feynman-Bender Diagrams of this theory (here, the simplified model of Quantum Field Theory of Partitions).

One has now 3 types of diagrams :

- the diagrams with labelled edges (from 1 to |d|). Their set is denoted (see above) FB-diagrams.

- the unlabelled diagrams (where permutations of black and white spots are allowed). Their set is denoted (see above) **diag.**

- the diagrams, as drawn, with black (resp. white) spots ordered i.e. labelled. Their set is denoted **ldiag.**

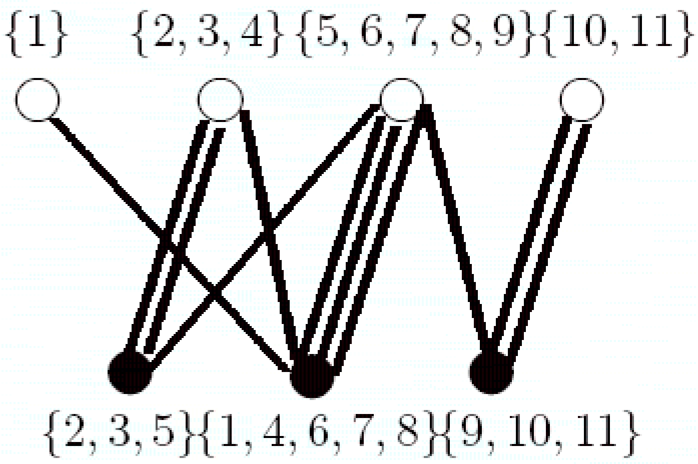
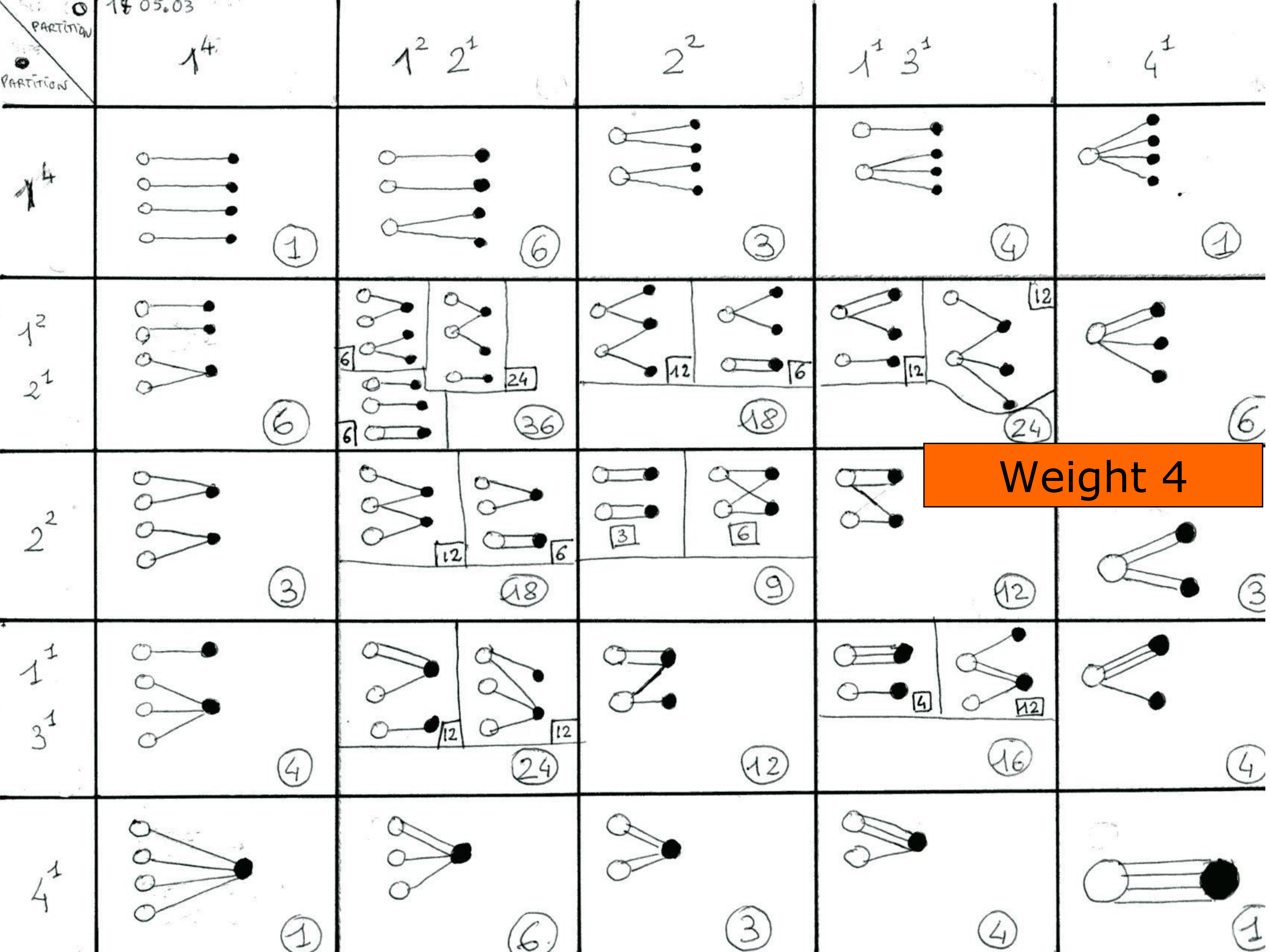
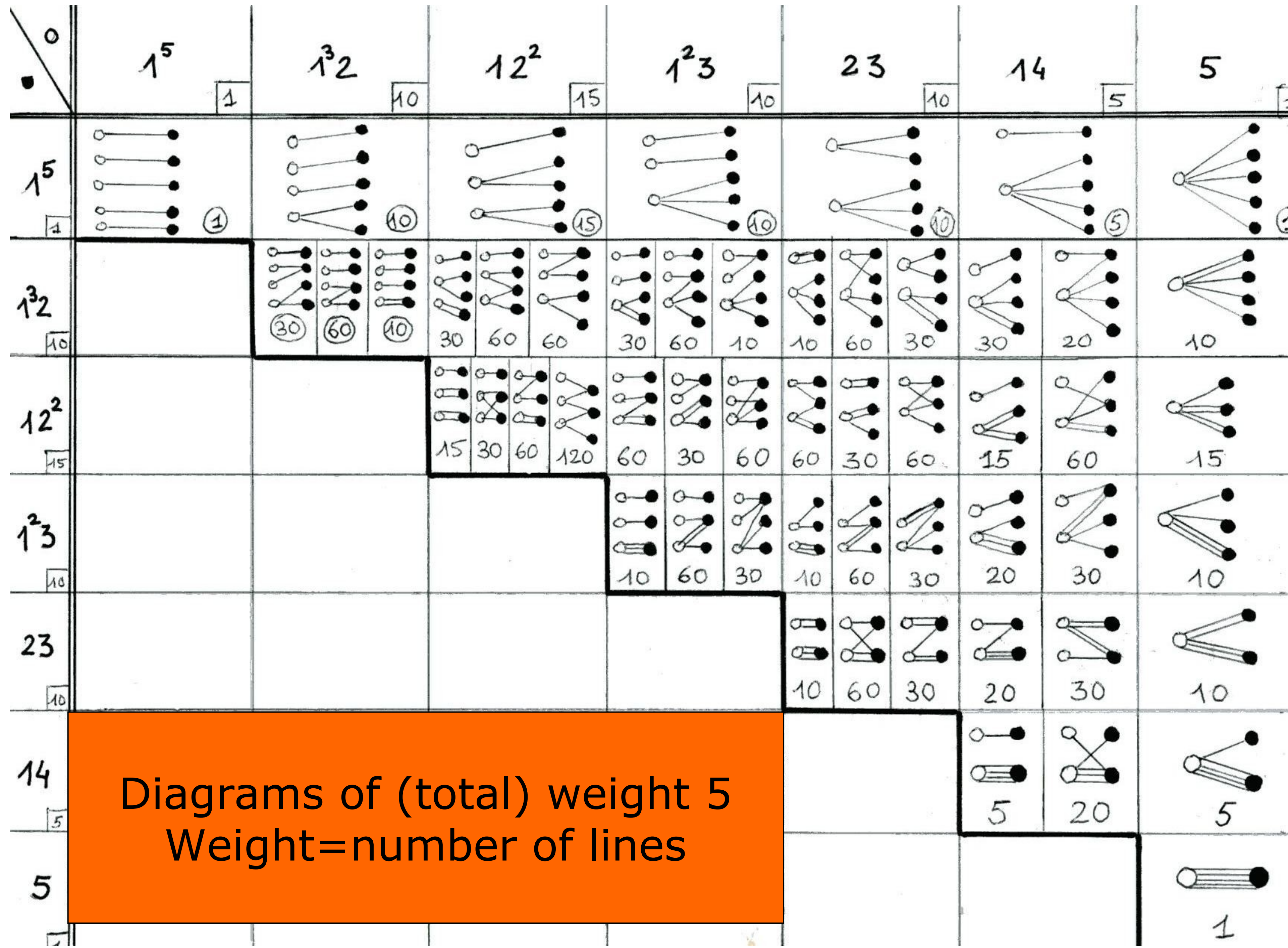


Fig 1. — *Diagram from P_1, P_2 (set partitions of $[1 \cdots 11]$).*
$P_1 = \{\{2,3,5\},\{1,4,6,7,8\},\{9,10,11\}\}$ and $P_2 = \{\{1\},\{2,3,4\},\{5,6,7,8,9\},\{10,11\}\}$ (respectively black spots for P_1 and white spots for P_2).
The incidence matrix corresponding to the diagram (as drawn) or these partitions is $\begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$. But, due to the fact that the defining partitions are unordered, one can permute the spots (black and white, between themselves) and, so, the lines and columns of this matrix can be permuted. the diagram could be represented by the matrix $\begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 3 & 1 \end{pmatrix}$ as well.



PARTITION
PARTITION
1^4
$1^2 2^1$
2^2
$1^1 3^1$
4^1
1
6
3
4
1
6
6
24
6
36
12
6
18
12
12
24
6
3
12
6
18
3
6
9
12
3
4
12
12
24
12
4
12
16
4
1
6
3
4
1
Weight 4


Diagrams of (total) weight 5
Weight=number of lines

Hopf algebra structure

First step: Define the spaces

$Diag = \oplus_{d \in \text{diagrams}} \mathbf{C}\, d$ $\quad$ $LDiag = \oplus_{d \in \text{labelled diagrams}} \mathbf{C}\, d$

(functions with finite supports on the set of diagrams).

At this stage, we have a natural arrow *LDiag* → *Diag*.

Second step: The product on *Ldiag* is just the concatenation of diagrams

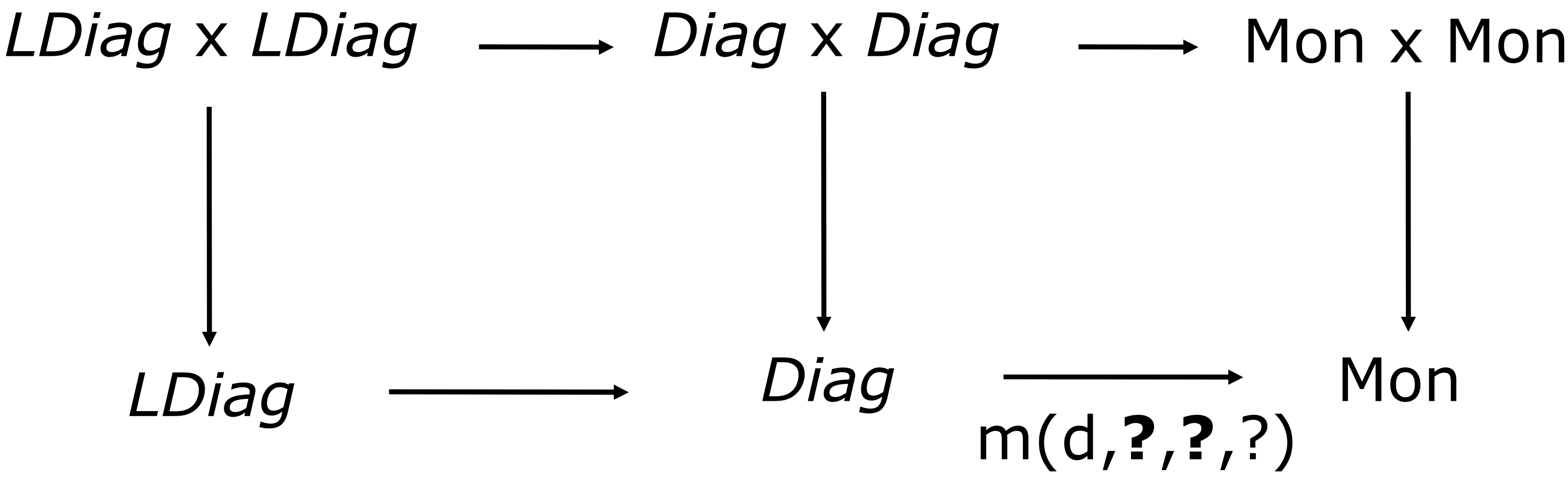
$$d_1 \pm d_2 = d_1\, d_2$$

And, setting $m(d,\mathbf{L},\mathbf{V},z)=\mathbf{L}^{\alpha(d)}\mathbf{V}^{\beta(d)}\, z^{|d|}$

one gets

$$m(d_1 * d_2,\mathbf{L},\mathbf{V},z)= m(d_1,\mathbf{L},\mathbf{V},z)m(d_2,\mathbf{L},\mathbf{V},z)$$

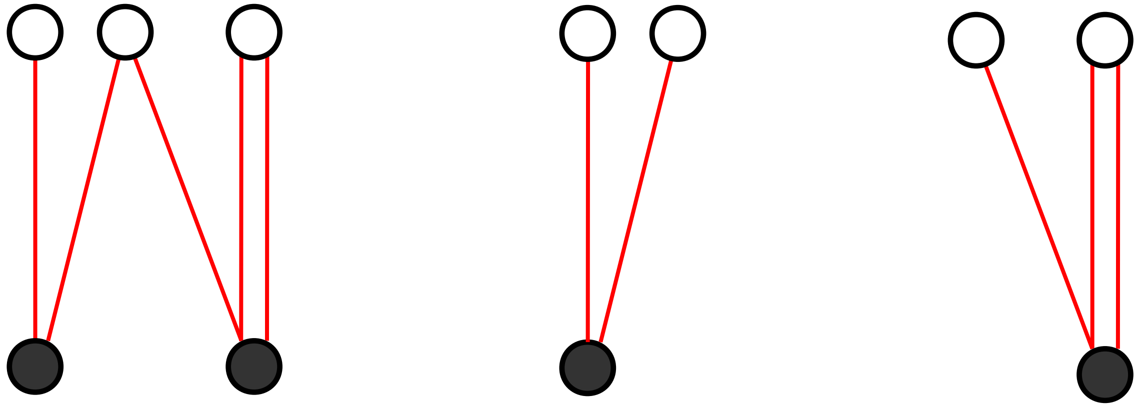
This product is associative with unit (the empty diagram). It is compatible with the arrow *LDiag* → *Diag* and so defines the product on *Diag* which,
in turn, is compatible with the product of monomials.



The coproduct needs to be compatible with m(d,**?**,**?**,?).
One has two symmetric possibilities (black spots and white spots).
The « black spots co-product » reads

$$\Delta_{BS}(d)=\sum d_I \otimes d_J$$

the sum being taken over all the decompositions, (I,J) of the Black Spots of d into two subsets.
For example, with the following diagrams d, d_1 and d_2



one has $\Delta_{BS}(d)=d\otimes\emptyset + \emptyset\otimes d + d_1\otimes d_2 + d_2\otimes d_1$

If we concentrate on the multiplicative structure of *Ldiag,* we remark that the objects are in one-to-one correspondence with the so-called packed matrices of NCSFVI (Hopf algebra MQSym), but the product of MQSym is given (w.r.t. a certain basis **MS**) according to the following example

$$\mathbf{MS}\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{MS}_{[3\ 1]} =$$

$$\mathbf{MS}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix} + \mathbf{MS}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 1 \end{bmatrix} + \mathbf{MS}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \mathbf{MS}\begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \mathbf{MS}\begin{bmatrix} 0 & 0 & 3 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

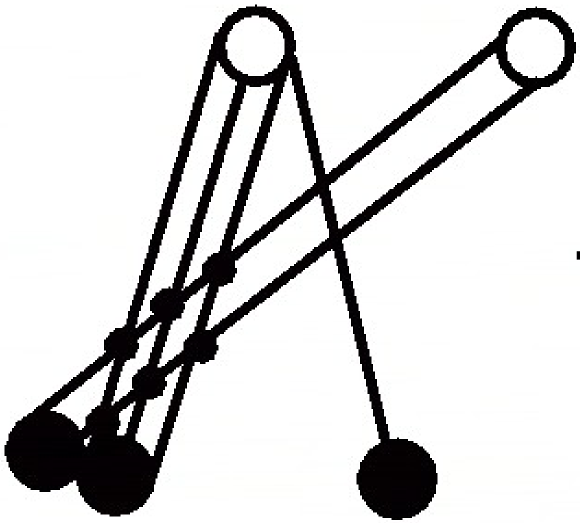
It is possible to (re)connect these Hopf algebras to MQSym and others of interest for physicists, by deforming the product with two parameters.
The double deformation goes as follows

- Concatenate the diagrams

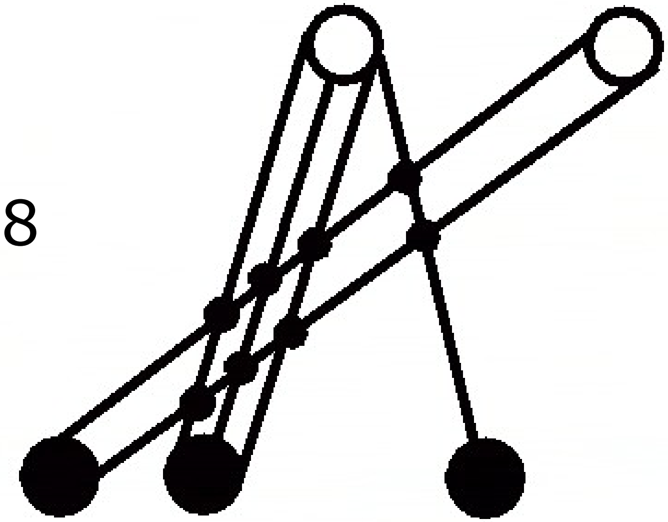
- Develop according to the rules :
  - Every crossing "pays" a q_c
  - Every node-stacking "pays" a q_s

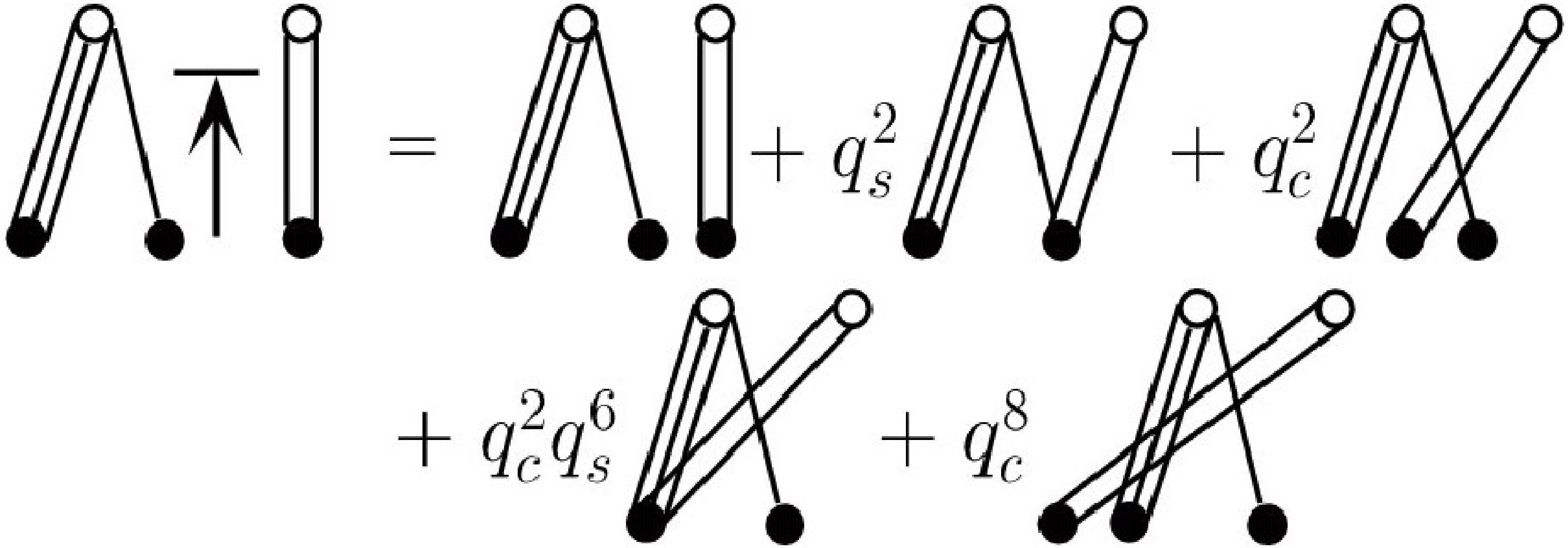
In the expansion, the weights are given by the intersection numbers.

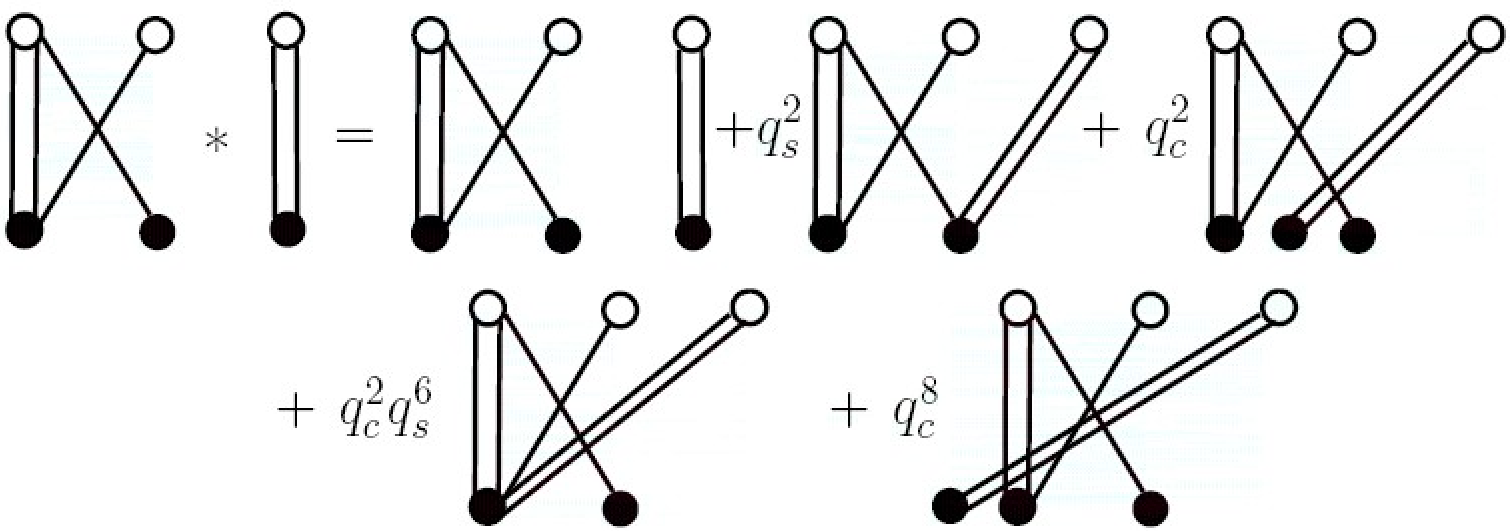
$q_c^2 q_s^6$



$+ q_c^8$




=
+ q_s^2
+ q_c^2
+ q_c^2 q_s^6
+ q_c^8


*
=
+q_s^2
+ q_c^2
+ q_c^2 q_s^6
+ q_c^8

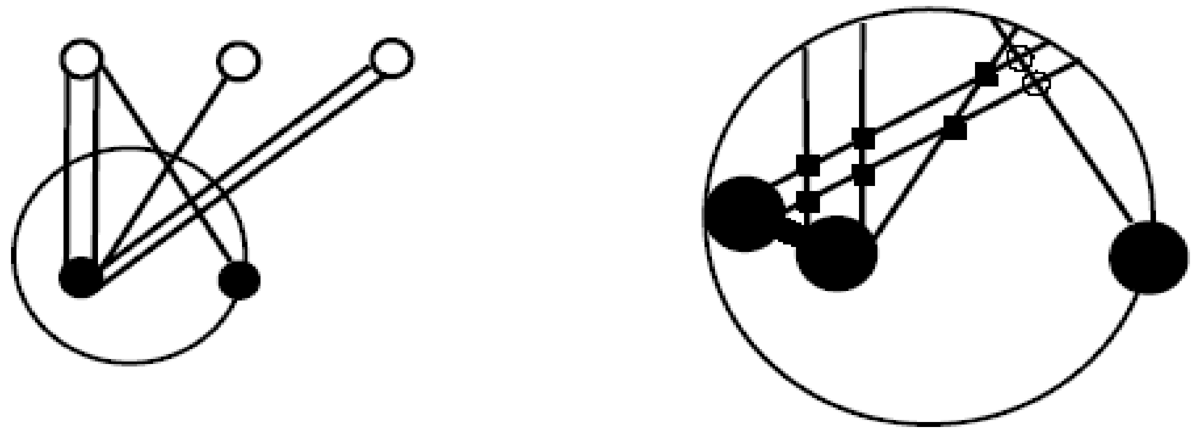


Fig 6. — *Detail of the fourth monomial (with coefficient $q_c^2 q_s^6$), crossings (circles) and superposings (black squares) are counted the same way but with a different variable.*

We could check that this law is associative (now three independent proofs). For example, direct computation reads

$$(au \uparrow bv) \uparrow cw = \left(a(u \uparrow bv) + q^{|u||b|} t^{|a||b|} \begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v) + q^{|au||b|} b(au \uparrow v)\right) \uparrow cw$$

$$\left[a((u \uparrow bv) \uparrow cw) + q^{(|u|+|bv|)|c|} t^{|a||c|} \begin{bmatrix} c \\ a \end{bmatrix} ((u \uparrow bv) \uparrow w) + q^{(|au|+|bv|)|c|} c(a(u \uparrow bv) \uparrow w)\right]$$

$$\left[q^{|u||b|} t^{|a||b|} \begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v \uparrow cw) + q^{|u||b|+(|u|+|v|)|c|} t^{|a||b|} t^{(|a|+|b|)|c|} \begin{bmatrix} c \\ b \\ a \end{bmatrix} (u \uparrow v \uparrow w)\right.$$

$$\left. q^{|u||b|+(|au|+|bv|)|c|} t^{|a||b|} c\left(\left(\begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v)\right) \uparrow w\right)\right]$$

$$\left[q^{|au||b|} b((au \uparrow v) \uparrow cw) + q^{|au||b|+(|au|+|v|)|c|} t^{|b||c|} \begin{bmatrix} c \\ b \end{bmatrix} (au \uparrow v \uparrow w) + q^{|au||b|+(|au|+|bv|)|c|} c(b(au \uparrow v) \uparrow w\right.$$

$$au \uparrow (bv \uparrow cw) = au \uparrow \left(b(v \uparrow cw) + q^{|v||c|} t^{|b||c|} \begin{bmatrix} c \\ b \end{bmatrix} (v \uparrow w) + q^{|bv||c|} c(bv \uparrow w)\right) =$$

$$\left[a(u \uparrow b(v \uparrow cw)) + q^{|u||b|} t^{|a||b|} \begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v \uparrow cw) + q^{|au||b|} b(au \uparrow v \uparrow cw)\right] +$$

$$\left[q^{|v||c|} t^{|b||c|} a(u \uparrow \begin{bmatrix} c \\ b \end{bmatrix} (v \uparrow w)) + q^{|v||c|+|u|(|c|+|b|)} t^{|b||c|+|a|(|b|+|c|)} \begin{bmatrix} c \\ b \\ a \end{bmatrix} (u \uparrow v \uparrow w) + \right.$$

$$\left. q^{|v||c|+|au|(|b|+|c|)} t^{|b||c|} \begin{bmatrix} c \\ b \end{bmatrix} (au \uparrow v \uparrow w)\right] +$$

$$\left[q^{|bv||c|} a(u \uparrow c(bv \uparrow w)) + q^{(|u|+|bv|)|c|} t^{|a||c|} \begin{bmatrix} c \\ a \end{bmatrix} (u \uparrow bv \uparrow w) + q^{(|au|+|bv|)|c|} c(au \uparrow bv \uparrow w)\right] \quad (3)$$

dans la deuxième expression, on regroupe les trois termes de tête des crochets et on trouve

$$a(u \uparrow b(v \uparrow cw)) + q^{|v||c|} t^{|b||c|} a(u \uparrow \begin{bmatrix} c \\ b \end{bmatrix} (v \uparrow w)) + q^{|bv||c|} a(u \uparrow c(bv \uparrow w)) = a(u \uparrow bv \uparrow cw) \quad (4)$$

dans la première expression, on regroupe les trois termes de queue des crochets et on trouve

$$q^{(|au|+|bv|)|c|} c(a(u \uparrow bv) \uparrow w) + q^{|u||b|+(|au|+|bv|)|c|} t^{|a||b|} c((\begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v)) \uparrow w) +$$

$$q^{|au||b|+(|au|+|bv|)|c|} c(b(au \uparrow v) \uparrow w) = q^{(|au|+|bv|)|c|} c(au \uparrow bv \uparrow w) \quad (5)$$

$$au \uparrow (bv \uparrow cw) = au \uparrow \left(b(v \uparrow cw) + q^{|v||c|}t^{|b||c|} \begin{bmatrix} c \\ b \end{bmatrix} (v \uparrow w) + q^{|bv||c|}c(bv \uparrow w)\right) =$$

$$\left[a(u \uparrow b(v \uparrow cw)) + q^{|u||b|}t^{|a||b|} \begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v \uparrow cw) + q^{|au||b|}b(au \uparrow v \uparrow cw)\right] +$$

$$\left[q^{|v||c|}t^{|b||c|}a(u \uparrow \begin{bmatrix} c \\ b \end{bmatrix} (v \uparrow w)) + q^{|v||c|+|u|(|c|+|b|)}t^{|b||c|+|a|(|b|+|c|)} \begin{bmatrix} c \\ b \\ a \end{bmatrix} (u \uparrow v \uparrow w) +\right.$$

$$\left. q^{|v||c|+|au|(|b|+|c|)}t^{|b||c|} \begin{bmatrix} c \\ b \end{bmatrix} (au \uparrow v \uparrow w)\right] +$$

$$\left[q^{|bv||c|}a(u \uparrow c(bv \uparrow w)) + q^{(|u|+|bv|)|c|}t^{|a||c|} \begin{bmatrix} c \\ a \end{bmatrix} (u \uparrow bv \uparrow w) + q^{(|au|+|bv|)|c|}c(au \uparrow bv \uparrow w)\right] \quad (3)$$

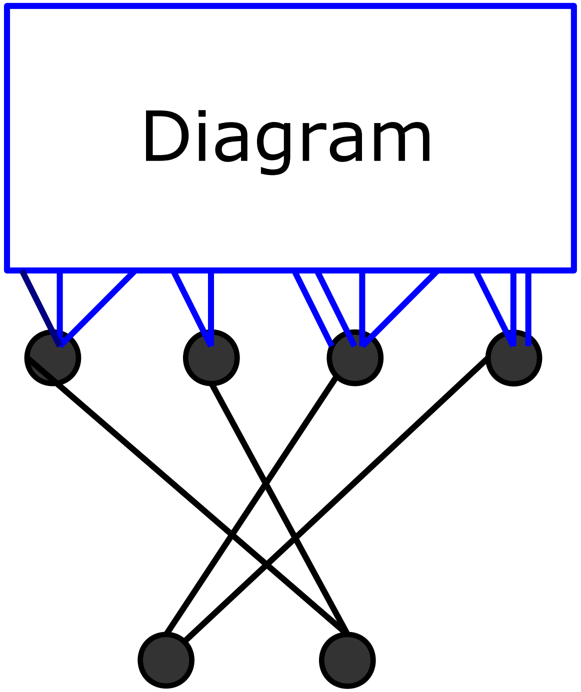
dans la deuxième expression, on regroupe les trois termes de tête des crochets et on trouve

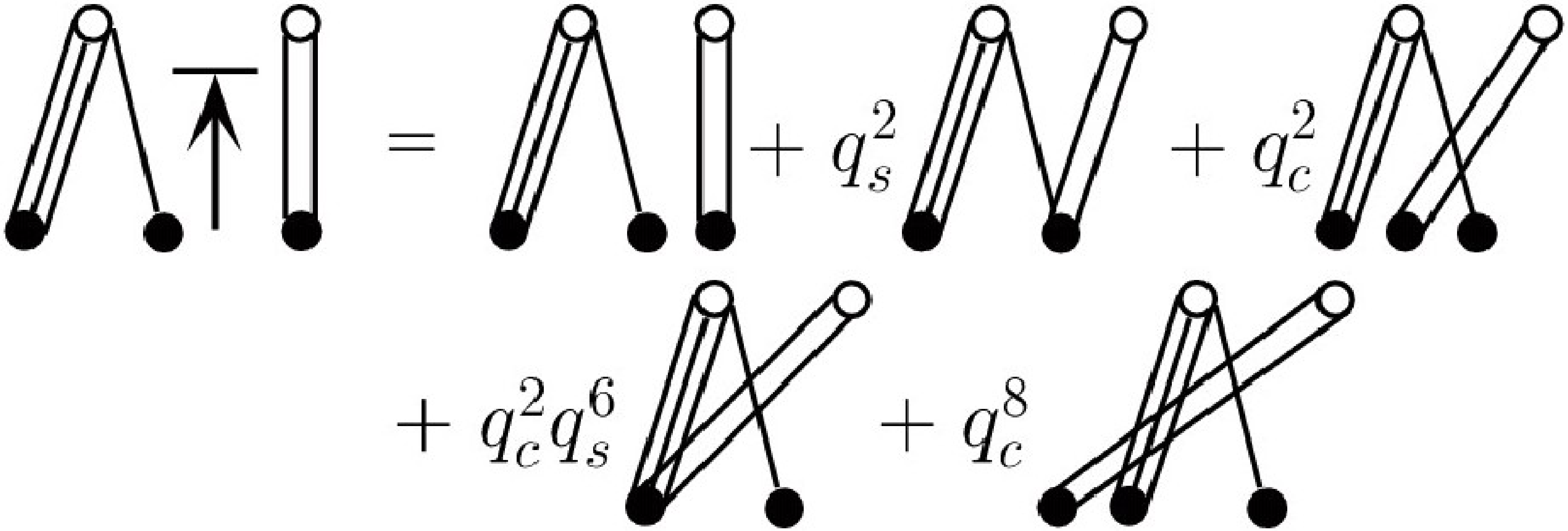
$$a(u \uparrow b(v \uparrow cw)) + q^{|v||c|}t^{|b||c|}a(u \uparrow \begin{bmatrix} c \\ b \end{bmatrix} (v \uparrow w)) + q^{|bv||c|}a(u \uparrow c(bv \uparrow w)) = a(u \uparrow bv \uparrow cw) \quad (4)$$

dans la première expression, on regroupe les trois termes de queue des crochets et on trouve

$$q^{(|au|+|bv|)|c|}c(a(u \uparrow bv) \uparrow w) + q^{|u||b|+(|au|+|bv|)|c|}t^{|a||b|}c((\begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v)) \uparrow w) +$$
$$q^{|au||b|+(|au|+|bv|)|c|}c(b(au \uparrow v) \uparrow w) = q^{(|au|+|bv|)|c|}c(au \uparrow bv \uparrow w) \quad (5)$$

This amounts to use a monoidal action with two parameters. Associativity provides an identity in an algebra which acts on a diagram as the algebra of the sum of symmetric semigroups. Here, it is the symmetric semigroup which acts on the black spots





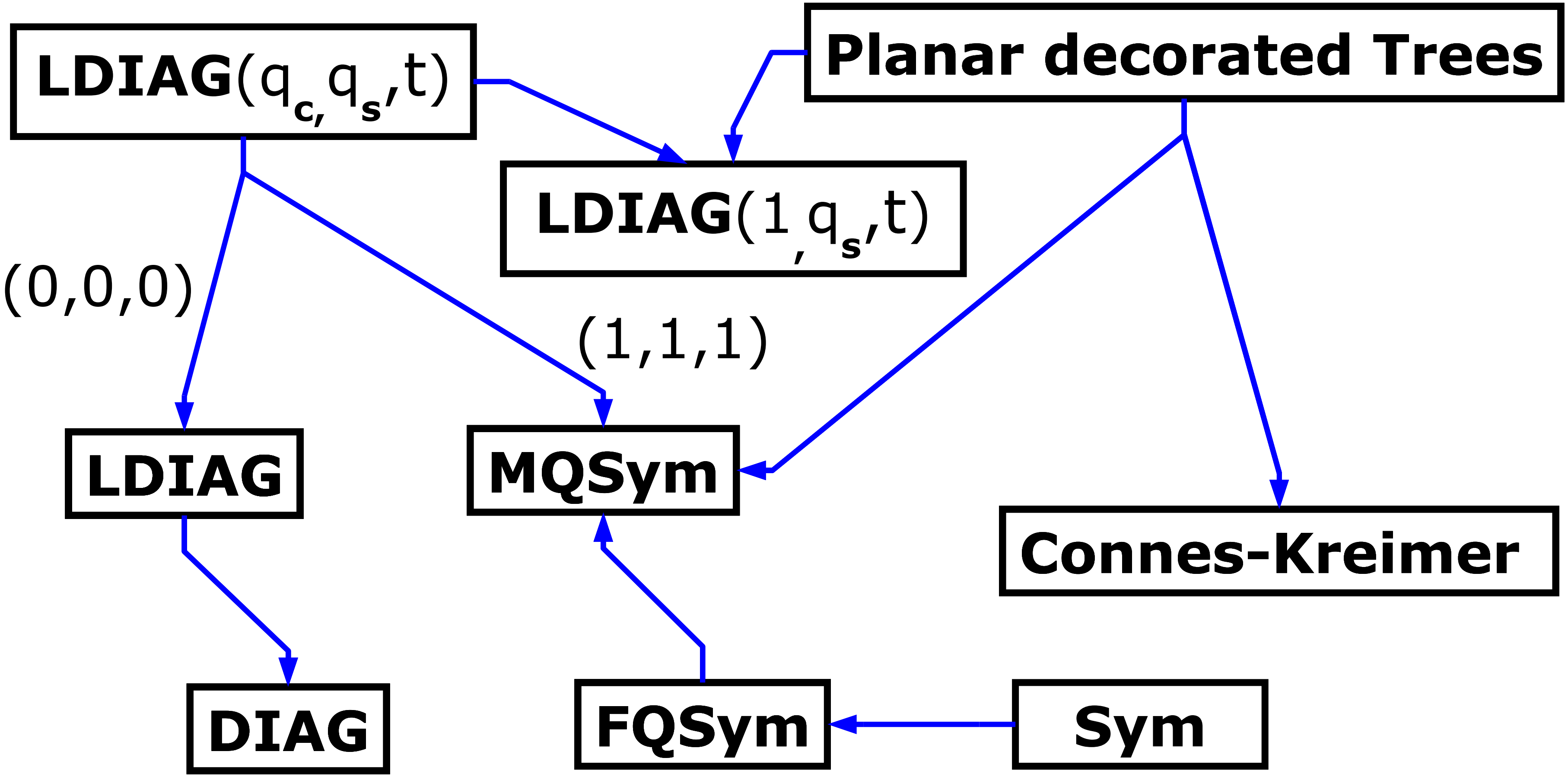
The labelled diagrams are in one to one correspondence with the packed matrices of MQSym and we can see easily that the product of the latter is obtained for

$$q_c = 1 = q_s$$

Hopf interpolation : One can see that the more intertwined the diagrams are the fewer connected components they have. This is the main argument to prove that LDIAG($q_{c,}$ q_s) is free on indecomposable diagrams. Therefore one can define a coproduct on these generators by

$$\Delta_t = (1\text{-}t)\Delta_{BS} + t\ \Delta_{MQSym}$$

this is LDIAG($q_{c,}$ q_s,t) (note that, here, t belongs to {0,1})



Notes :

i) The arrow *Planar Dec. Trees* → *LDIAG(1,q_s,t)* is due to L. Foissy

ii) **LDIAG**, through a noncommutative alphabetic realization shows to be a bidendriform algebra (FPSAC07 paper by ParisXIII & Monge).

In order to have an algebraic expression of the product, we use a process similar to the shuffle product. We must define local partial degrees.
For a black spot with label « i », we denote by BS(d,i) its degree (number of adjacents edges).
Then, for d_1 (resp. d_2) with p (resp. q) black spots, the product reads

$$[d_1|d_2]_{L(q_c,q_s)} = \sum_{f\in Shs(p,q)} \Big(\prod_{\substack{i<j \\ f(i)>f(j)}} q_c^{BS(d,i).BS(d,j)} \Big) \Big(\prod_{\substack{i<j \\ f(i)=f(j)}} q_s^{BS(d,i).BS(d,j)} \Big) [d_1|d_2]_L.f$$

where Shs(p,q) is the set of mappings f in SSG_{[1..p+q]} with image of type [1...m] and such that

$$f(1) < f(2) < \cdots < f(p)\ ;\ f(p+1) < f(p+2) < \cdots < f(p+q)\ .$$

This condition, similar to that of the shuffle product, guarantees that the black spots of the diagrams are kept in order during the process of shuffling with superposition (hence the name Shs).

The graphic and symmetric-semigroup-indexed description of the deformed law neither give immediately a recursive definition nor an explanation of ``why'' the law is associative. We will, on our way to understand this (as well as the different natures of its parameters), proceed in three steps:

a) code the diagrams by words of monomials
b) present the law as a shifted law
c) give a recursive definition of the (non-shifted) law.

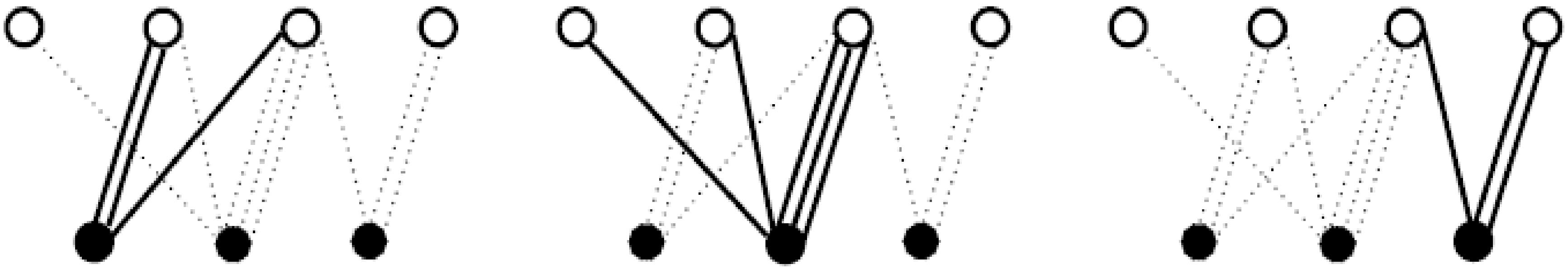


Fig 3. — *Coding the diagram of fig 2 by a word of monomials. The code here is* $[x_2^2x_3\ ,\ x_1x_2x_3^3\ ,\ x_3x_4^2]$

Then, one has only to find the law « without shifting » it reads

$$\begin{cases} 1_{(\mathfrak{MON}^+(X))^*} \uparrow w &= w \uparrow 1_{(\mathfrak{MON}^+(X))^*} = w \\ au \uparrow bv &= a(u \uparrow bv) + q_c^{|au||b|} b(au \uparrow v) + q_c^{|u||b|} q_s^{|a||b|} (a \cdot b)(u \uparrow v) \end{cases}$$

It is now easy to interpret the crossing parameter as a deformation of the tensor structure and the superposing parameter as a perturbation of the ordinary coproduct.

Conclusion

There are arrows from LDIAG to polyzetas. Tuning the parameters differently, one obtains the shuffle and stuffle products. Other features (like coloured polyzetas) can be obtained with a slightly modified setting

Thank you for your attention !