Draft of April 7, 2009

Heisenberg–Weyl Diagrams: Combinatorial Algebra and Decomposition

P. Blasiak,[1,*] G. H. E. Duchamp,[2,†] A. I. Solomon,[3,4,‡] A. Horzela,[1,§] and K. A. Penson[4,¶]
[1]*H. Niewodniczański Institute of Nuclear Physics, Polish Academy of Sciences ul. Eliasza-Radzikowskiego 152, PL 31342 Kraków, Poland*
[2]*Institut Galilée, LIPN, CNRS UMR 7030, 99 Av. J.-B. Clement, F-93430 Villetaneuse, France*
[3]*The Open University, Physics and Astronomy Department, Milton Keynes MK7 6AA, United Kingdom*
[4]*Laboratoire de Physique Théorique de la Matière Condensée, Université Pierre et Marie Curie, CNRS UMR 7600, Tour 24 - 2ième ét., 4 pl. Jussieu, F 75252 Paris Cedex 05, France*

......

PACS numbers: ...
Keywords: ...

I. INTRODUCTION

Algebraic approach to problems of Quantum Physics has been appreciated almost from the beginning of .... Presently we witness Combinatorial realm (Louck, 2008; **?**) Obscured from its

Philosophy of Hopf algebras and its use in physics: representation theory, decomposition of diagrams which is crucial for renormalization

In Section II we briefly recall the relation between the Heisenberg–Weyl algebra HW and the associated Lie algebra $\mathcal{L}_{HW}$ and its enveloping algebra $\mathcal{U}(\mathcal{L}_{HW})$ and set up the scene for ...

II. HEISENBERG–WEYL ALGEBRA

A common structure in theoretical description of physical phenomena is that of an *associative algebra with unit*. It is defined as a vector space $\mathcal{A}$ over a field $\mathbb{K}$ equipped with a bilinear multiplication law $\mathcal{A}\times\mathcal{A}\ni(x,y)\rightarrow x\,y\in\mathcal{A}$ which is associative and possesses a unit element I.

Lie algebra is a vector space $\mathcal{L}$ over a field $\mathbb{K}$ with a bilinear multiplication law $\mathcal{L}\times\mathcal{L}\ni(x,y)\rightarrow[x,y]\in\mathcal{L}$ which is antisymmetric $[x,y]=-[y,x]$ and satisfies the Jacobi identity: $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$. As evident from the axioms Lie algebras are not associative and lack the identity element. A standard remedy for these deficiencies consist in passing to its enveloping algebra $\mathcal{U}(\mathcal{L})$ which has a more familiar structure of unital associative algebra and at the same time captures all "‘relevant"’ properties of $\mathcal{L}$. The crucial step in this construction is Poincaré-Birkhoff-Witt theorem ...

$\mathcal{L}_{\mathcal{H}}=\mathbb{C}\,e\oplus\mathbb{C}\,b\oplus\mathbb{C}\,b^{\dagger}$ ...

$$\begin{tikzcd}
\mathcal{U}(\mathcal{L}_{HW}) \arrow[rr, two heads] & & HW \\
& \mathcal{L}_{HW} \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}$$

Reference to the PBW theorem, basis and structure constants in the HW algebra $b^{\dagger}\succ b\succ e$ ...

$$(b^{\dagger})^{p}b^{q}e^{r}\ (b^{\dagger})^{k}b^{l}e^{m}= \\ =\sum_{i}\binom{q}{i}\binom{k}{i}\ i!\ (b^{\dagger})^{p+k-i}\ b^{q+l-i}\ e^{r+l+i}\ . \quad (1)$$

(Blasiak *et al.*, 2007)(Blasiak *et al.*, 2008)

III. HEISENBERG–WEYL DIAGRAMS

A. Definitions

We start by recalling a few basic definitions from graph theory (Diestel, 2005; **?**). A *directed graph* is a collection of vertices V and edges E with the structure determined by two functions $h,t:E\longrightarrow V$ prescribing how the *head* and *tail* of an edge are attached to vertices. Here we shall consider graphs for which the functions are h and t are *partially* defined with the restriction that $h^{-1}\cup t^{-1}=E$, *i.e.* allow the edges to have one of the ends free (but not both). A *cycle* in a graph is a sequence of edges $e_1,e_2,...,e_n$ such that $h(e_k)=t(e_{k+1})$ for $k<n$ and $h(e_n)=t(e_1)$. Two graphs are said to be *equivalent* if they can be isomorphically transformed one into another, *i.e.* they have the same number of vertices and edges and there exist two isomorphisms $\alpha_E:E_1\longrightarrow E_2$ and $\alpha_V:V_1\longrightarrow V_2$ faithfully transferring structure of the graphs in the following sense $h_2\circ\alpha_E=\alpha_V\circ h_1$ and $t_2\circ\alpha_E=\alpha_V\circ t_1$. This equivalence relation defines *classes of graphs* for which specific realizations are immaterial and only relations between vertices and edges are relevant.

In this context we put the following formal definition:

*Electronic address: `pawel.blasiak@ifj.edu.pl`
†Electronic address: `ghed@lipn-univ.paris13.fr`
‡Electronic address: `a.i.solomon@open.ac.uk`
§Electronic address: `andrzej.horzela@ifj.edu.pl`
¶Electronic address: `penson@lptmc.jussieu.fr`

Definition 1 (Heisenberg–Weyl Diagrams)
A Heisenberg–Weyl diagram Γ is a class of partially defined directed graphs without cycles. It consists of three sorts of lines: the inner ones Γ^0 having both head and tail attached to vertices, the ingoing lines Γ^- with free tails, and the outgoing lines Γ^+ with free heads.

A typical modus operandi when working with classes is to invoke to its representatives. Following this practice we shall by default make all statements concerning Heisenberg–Weyl diagrams for its representatives assuming that they are class invariants which can be routinely checked in each case.

The formal Definition 1 gains intuitive picture in a graphical form illustrated in Fig. 1. We represent a diagram as a set of verices $\bullet$ connected by lines each carrying an arrow indicating the direction from the tail to the head. Lines having one of the ends not attached to a vertex will be marked with $\vartriangle$ or $\blacktriangle$ at the free head or tail respectively. We conventionally draw all ingoing lines at the bottom and the outgoing lines at the top with all arrows heading upwards which is always possible since the diagrams can not have cycles. It forms a picture of the Heisenberg-Weyl diagram as a sort of process or transformation with vertices playing the role of intermediate steps.

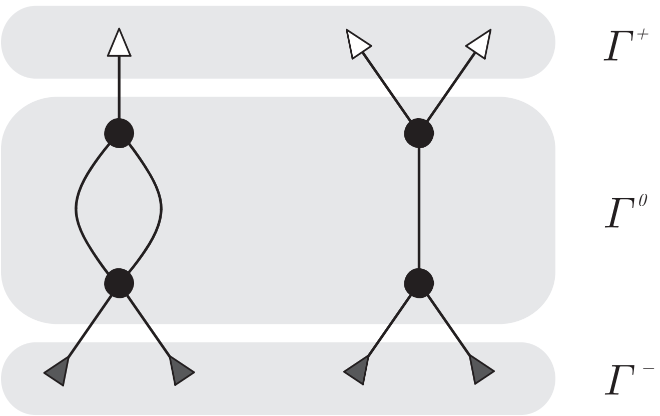

FIG. 1 An example of a Heisenberg–Weyl diagram with distinguished three characteristic sorts of lines: the inner ones $|\Gamma^0| = 3$, the ingoing lines $|\Gamma^-| = 4$ and outgoing lines $|\Gamma^+| = 3$.

An important characteristic of a diagram is the total number of its lines denoted by $|\Gamma|$. In the next sections we shall further specify this counting to the inner, the ingoing and the outgoing lines, denoting the result by $|\Gamma^0|$, $|\Gamma^-|$ and $|\Gamma^+|$ respectively.

B. Diagram composition

A *matching* m of two sets A and B is a choice of pairs $c_i = (a_i, b_i) \in A \times B$ all having different components, *i.e.* for any two pairs $c_1, c_2 \in m$ one requires that $a_1 \neq a_2$ as well as $b_1 \neq b_2$. We shall denote the collection of all possible matchings by $A \bowtie B$, and its restriction to matchings comprising i pairs only by $A \overset{i}{\bowtie} B$. It is straightforward to check by exact enumeration the formula $|A \overset{i}{\bowtie} B| = \binom{|A|}{i}\binom{|B|}{i}\, i!$ which is valid for any i with the convention $\binom{n}{k} = 0$ for $n < k$.

The concept of diagram composition suggests itself, as:

Definition 2 (Diagram composition)
Consider two diagrams Γ_1 and Γ_2 and a matching m between the free lines going out from the first one Γ_1^+ and the free lines going into the second one Γ_2^-. The composite diagram, denoted as $\Gamma_2 \overset{m}{\lhd\!\!\lhd} \Gamma_1$, is constructed by joining the lines coupled by the matching $m \in \Gamma_2^- \bowtie \Gamma_1^+$.

This descriptive definition can be formalized in the following way. We take two disjoint graphs $V_{\Gamma_2} \cap V_{\Gamma_1} = \emptyset$ and $E_{\Gamma_2} \cap E_{\Gamma_1} = \emptyset$ and construct the composite graph $\Gamma_2 \overset{m}{\lhd\!\!\lhd} \Gamma_1$ which consists of vertices $V_{\Gamma_2 \overset{m}{\lhd\!\!\lhd} \Gamma_1} = V_{\Gamma_2} \cup V_{\Gamma_1}$ and edges $E_{\Gamma_2 \overset{m}{\lhd\!\!\lhd} \Gamma_1} = E_{\Gamma_2} \cup E_{\Gamma_1} \cup m - (\,pr_{V_{\Gamma_2}}(m) \cup pr_{V_{\Gamma_1}}(m)\,)$. The head and tail functions then unambiguously extend on $E_{\Gamma_2} \cup E_{\Gamma_1} - (\,pr_{V_{\Gamma_2}}(m) \cup pr_{V_{\Gamma_1}}(m)\,)$ and for $e = (e_{\Gamma_2}, e_{\Gamma_1}) \in m$ one puts $h_{\Gamma_2 \overset{m}{\lhd\!\!\lhd} \Gamma_1}(e) = h_{\Gamma_2}(e_{\Gamma_2})$ and $t_{\Gamma_2 \overset{m}{\lhd\!\!\lhd} \Gamma_1}(e) = t_{\Gamma_1}(e_{\Gamma_1})$. As expected, the resulting graph does not contain cycles. Finally, we note that since we are working with classes of graphs choice of the disjoint graphs is always possible, and definition of diagram composition with reference to its representatives proves to be class invariant.

Definition 2 can be intuitively seen as if diagrams were put one over another with some of the lines going out from the lower one plugged into some of the lines going into to the upper one in accordance with a given matching $m \in \Gamma_2^- \bowtie \Gamma_1^+$ (for illustration see Fig. 2). Observe that in general two graphs can be composed in many ways, *i.e.* as many as there are possible matchings (elements in $\Gamma_2^- \bowtie \Gamma_1^+$). In Section III.C we shall exploit all these possible compositions to endow the diagrams with the structure of an algebra. Note also that that the above construction depends on the order in which diagrams are composed and the reverse order yields different results.

We conclude by two simple remarks concerning compositions of two diagrams Γ_2 and Γ_1 constructed by joining exactly i lines. Firstly, the these diagrams can be enumerated explicitly by the formula

$$|\Gamma_2^- \overset{i}{\bowtie} \Gamma_1^+| = \binom{|\Gamma_2^-|}{i}\binom{|\Gamma_2^+|}{i}\; i!\;. \tag{2}$$

Secondly, the number of ingoing, outgoing and inner lines in the composed diagram does not depend on the choice of matching $m \in \Gamma_2^- \overset{i}{\bowtie} \Gamma_1^+$ and reads respectively

$$\begin{aligned} |(\Gamma_2 \overset{m}{\lhd\!\!\lhd} \Gamma_1)^+| &= |\Gamma_2^-| + |\Gamma_1^+| - i\;, \\ |(\Gamma_2 \overset{m}{\lhd\!\!\lhd} \Gamma_1)^-| &= |\Gamma_2^+| + |\Gamma_1^-| - i\;, \\ |(\Gamma_2 \overset{m}{\lhd\!\!\lhd} \Gamma_1)^0| &= |\Gamma_2^0| + |\Gamma_1^0| + i\;. \end{aligned} \tag{3}$$

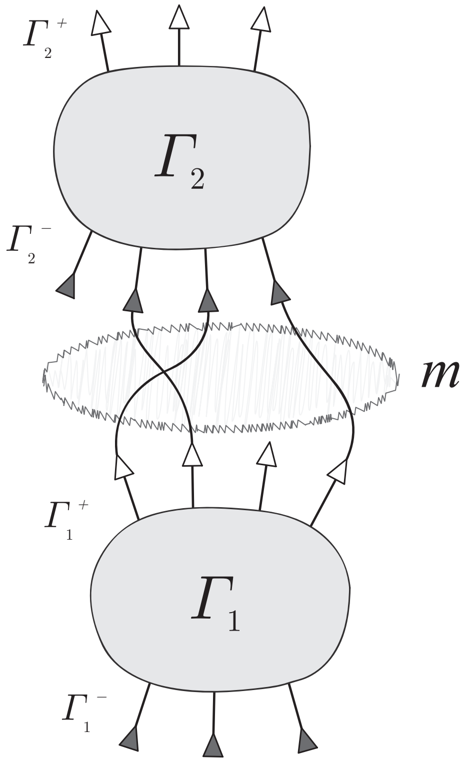

FIG. 2 Composition of two diagrams $\Gamma_2 \overset{m}{\lhd\!\!\lhd} \Gamma_1$ according to the matching $m \in \Gamma_2^- \bowtie \Gamma_1^+$.

C. Algebra of Heisenberg–Weyl Diagrams

Here, we show that the Heisenberg–Weyl diagrams come along with a natural structure of an algebra which is based on diagram composition. Moreover, it will occur to be a combinatorial refinement of the familiar algebras $\mathcal{U}(\mathcal{L}_{HW})$ and HW.

Algebra requires two operations, addition and multiplication, which are constructed as follows. We define $\mathcal{G}$ as the vector space (over $\mathbb{K}$) generated by the basis set composed of all HW diagrams, *i.e.*

$$\mathcal{G} = \Big\{ \sum\nolimits_i \alpha_i\ \Gamma_i :\ \alpha_i \in \mathbb{K},\ \Gamma_i\ -\ HW \text{ diagram} \Big\}. \quad (4)$$

Addition in $\mathcal{G}$ has the usual form

$$\sum\nolimits_i \alpha_i\ \Gamma_i + \sum\nolimits_i \beta_i\ \Gamma_i = \sum\nolimits_i (\alpha_i + \beta_i)\ \Gamma_i. \quad (5)$$

Nontrivial part in the definition of algebra $\mathcal{G}$ concerns multiplication, which by bilinearity

$$\sum\nolimits_i \alpha_i\ \Gamma_i * \sum\nolimits_j \beta_j\ \Gamma_j = \sum\nolimits_{i,j} \alpha_i \beta_j\ \Gamma_i * \Gamma_j, \quad (6)$$

comes down to determining it on the basis set of HW diagrams. Recalling the notions of Section III.B we define product of two diagrams Γ_2 and Γ_1 as the sum of the composed diagrams over all possible matchings, *i.e.*

$$\Gamma_2 * \Gamma_1 = \sum_{m \in \Gamma_2^- \bowtie \Gamma_1^+} \Gamma_2 \overset{m}{\lhd\!\!\lhd} \Gamma_1\ . \quad (7)$$

Note that all terms in the sum are distinct with coefficients equal to one. Such defined multiplication is noncommutative and possesses unit element which is the void graph Ø (no vertices, no lines). Moreover the following theorem holds (for the proof of associativity see Appendix A):

Theorem 1 (Algebra of Diagrams)
Heisenberg–Weyl diagrams form a (noncommutative) associative algebra with unit $(\mathcal{G}, +, *, \emptyset)$.

Our objective, now, is to clarify the relation of the algebra of the Heisenberg–Weyl diagrams $\mathcal{G}$ to the physically relevant algebras $\mathcal{U}(\mathcal{L}_{HW})$ and HW. We shall construct forgetful functors giving a simple combinatorial prescription how to descend from $\mathcal{G}$ to the later two.

Let us define the morphism $\varphi : \mathcal{G} \longrightarrow \mathcal{U}(\mathcal{L}_{HW})$ on the basis of HW diagrams as follows

$$\varphi(\Gamma) \overset{df}{=} (b^\dagger)^{|\Gamma^+|}\ b^{|\Gamma^-|}\ e^{|\Gamma^0|}\ , \quad (8)$$

and by linearity extend it to the whole algebra. On can intuitively understand this definition by seeing the diagrams as if they were carrying auxiliary labels $b^\dagger$, b and e attached to all the outgoing, ingoing and inner lines respectively. Then the morphism of Eq. (8) just neglects structure of the graph and pays attention to the lines only by counting them according to labels. Clearly, φ is onto and it can be proved to be a genuine algebra morphism, *i.e.* preserves addition and multiplication in $\mathcal{G}$ (for the proof see Appendix B).

Similarly, we define the morphism $\bar{\varphi} : \mathcal{G} \longrightarrow HW$ as

$$\bar{\varphi}(\Gamma) \overset{df}{=} (a^\dagger)^{|\Gamma^+|}\ a^{|\Gamma^-|}\ , \quad (9)$$

which differs from φ by additionally ignoring all inner lines in the diagrams. It can be expressed as $\bar{\varphi} = \varphi \circ \varepsilon$ and hence fulfills all properties of an algebra morphism.

We recap the above results in the following theorem:

Theorem 2 (Forgetful Functors)
The mappings $\varphi : \mathcal{G} \longrightarrow \mathcal{U}(\mathcal{L}_{HW})$ *and* $\bar{\varphi} : \mathcal{G} \longrightarrow HW$ *defined in Eqs. (8) and (9) are suriective algebra morphisms, and the following diagram commutes:*

$$\begin{tikzcd} & \mathcal{G} \arrow[dl, "\varphi"'] \arrow[dr, "\bar{\varphi}"] & \\ \mathcal{U}(\mathcal{L}_{HW}) \arrow[rr, two heads, "\varepsilon"] & & HW \end{tikzcd} \quad (10)$$

In this way, the algebra of Heisenberg–Weyl diagrams $\mathcal{G}$ can be seen as a lifting of the algebras $\mathcal{U}(\mathcal{L}_{HW})$ and HW, and the latter two can be recovered by applying appropriate forgetful functors φ and $\bar{\varphi}$. In other words, $\mathcal{G}$ is a sort of fine graining of the abstract algebras $\mathcal{U}(\mathcal{L}_{HW})$ and HW which gain a concrete combinatorial interpretation in terms the richer structure of diagrams.

IV. HOPF ALGEBRA AND DIAGRAM DECOMPOSITION

A. Basic notions

Gentle intro and basic notions of the Hopf algebra. Maybe a la G.-C. Rota and section coefficients. (Joni and Rota, 1979)

Suppose that we are given a class of objects $\mathcal{C}$ and a rule of their decomposition, *i.e.* a procedure how to cut up objects $i \in \mathcal{C}$ into ordered pairs of pieces from the same class $j, k \in \mathcal{C}$, in short denoted by $i \longrightarrow (j, k)$. Then the *section coefficients* $(i|j, k) \in \mathbb{N}$ are defined to count the number of ways in which this cutting can be done. The idea is quite general at this point as the result obviously depends on the choice of the rule. One usually supplements this concept with further conditions.

The concept of decomposition naturally extends to extended to splitting of an object into three pieces $i \longrightarrow (j, p, q)$ with the related *multisection coefficients* $(i|j, p, q)$. An obvious way to carry out the multiple splitting is by applying the same procedure repeatedly, *i.e.* decomposing one of the components obtained in the previous steps. However, following this prescription one naturally requires that the result do not depend on the choice of the component it is applied to. In other words, we expect to obtain the same number of elements when cutting $i \longrightarrow (j, k)$ and then cutting the second component $k \longrightarrow (p, q)$, as in the case when cutting $i \longrightarrow (s, q)$ and then cutting the first component $s \longrightarrow (j, p)$. It is a sort of associativity condition for decomposition, which in explicit form reads

$$\sum_k (i|j,k)(k|p,q) = \sum_s (i|s,q)(s|j,p). \tag{11}$$

Often, in a class there exists a sort of a (unique) void element $\emptyset$ such that any object $i \in \mathcal{C}$ decomposes into a pair containing $\emptyset$ in a unique way, *i.e.* $i \longrightarrow (\emptyset, i)$ and $i \longrightarrow (i, \emptyset)$. Then we have $(i|\emptyset, j) = (i|j, \emptyset) = \delta_{i,j}$, and one canonically defines the function $\varepsilon : \mathcal{C} \longrightarrow \mathbb{K}$ by setting

$$\varepsilon(j) = \begin{cases} 1 & \text{if} \quad j = \emptyset\ , \\ 0 & \text{otherwise}\ . \end{cases} \tag{12}$$

Clearly, for this choice of ϵ the following general conditions hold

$$\sum_j (i|j,k)\,\varepsilon(j) \;=\; \delta_{i,k}\ , \tag{13}$$

$$\sum_k (i|j,k)\;\varepsilon(k) \;=\; \delta_{i,j}\ , \tag{14}$$

which are required to pass to the realm of coalgebras. In our case, however, the simplest definition of Eq. (12) shall be completely sufficient.

B. Hopf algebra of $\mathcal{U}(\mathcal{L}_{HW})$

C. Hopf algebra of Heisenberg–Weyl diagrams

We define the comultipication in $\mathcal{G}$ on the basis elements as

$$\Delta(\Gamma) = \sum_{R+Y=V_\Gamma} \Gamma|_R \otimes \Gamma|_Y \tag{15}$$

Theorem 3 (...)
...

V. CONCLUSIONS

Possible referees: Severini, Louck, Vourdas, Katriel, Burdik, ...

Acknowledgments

......

Appendix A: Associativity of multiplication in $\mathcal{G}$

We shall prove associativity of multiplication defined in Eq. (7). From bilinearity we only need to check it for the basis elements, *i.e.*

$$\Gamma_3 * (\Gamma_2 * \Gamma_1) = (\Gamma_3 * \Gamma_2) * \Gamma_1\ . \tag{A1}$$

Written explicitly, the left and right hand sides of this equation have the form

$$\Gamma_3 * (\Gamma_2 * \Gamma_1) = \sum_{m'} \sum_{m_{21}} \Gamma_3 \overset{m'}{\lhd} (\Gamma_2 \overset{m_{21}}{\lhd} \Gamma_1) \tag{A2}$$

where $m' \in \Gamma_3 \bowtie (\Gamma_2 \overset{m_{21}}{\lhd} \Gamma_1)^+$ and $m_{21} \in \Gamma_2^- \bowtie \Gamma_1^+$, whilst

$$(\Gamma_3 * \Gamma_2) * \Gamma_1 = \sum_{m_{32}} \sum_{m''} (\Gamma_3 \overset{m_{32}}{\lhd} \Gamma_2) \overset{m''}{\lhd} \Gamma_1 \tag{A3}$$

where $m_{32} \in \Gamma_3^- \bowtie \Gamma_2^+$ and $m'' \in (\Gamma_3 \overset{m_{32}}{\lhd} \Gamma_2)^- \bowtie \Gamma_1^+$.

Let us take a look at the double sums in in the above equations, indexed by (m', m_{21}) and (m_{32}, m'') respectively, and observe that there exists a one-to-one correspondence between theirs elements. We construct it by fine graining of the matchings, see Fig. 3, and define the following two mappings. The first one as

$$(m', m_{21}) \;\longrightarrow\; (m_{32}, m'')\ , \tag{A4}$$

where $m_{32} = m' \cap (\Gamma_3^- \times \Gamma_2^+)$ and $m'' = m_{21} \cup (m' \cap (\Gamma_3^- \times \Gamma_1^+))$, and similarly the second one

$$(m_{32}, m'') \;\longrightarrow\; (m', m_{21})\ , \tag{A5}$$

where $m' = m_{32} \cup (m'' \cap (\Gamma_3^- \times \Gamma_1^+))$ and $m_{21} = m'' \cap (\Gamma_2^- \times \Gamma_1^+)$. Clearly, the mappings are inverses one of another, assuring that the correspondence between elements of the double sums in Eqs. (A2) and (A3) is one-to-one. Moreover, the summands that are mapped one onto another are equal, *i.e.* the corresponding diagrams $\Gamma_3 \overset{m'}{\triangleleft\!\triangleleft} (\Gamma_2 \overset{m_{21}}{\triangleleft\!\triangleleft} \Gamma_1)$ and $(\Gamma_3 \overset{m_{32}}{\triangleleft\!\triangleleft} \Gamma_2) \overset{m''}{\triangleleft\!\triangleleft} \Gamma_1$ are exactly the same. This ends the proof by showing equality of the right hand sides of Eqs. (A2) and (A3).

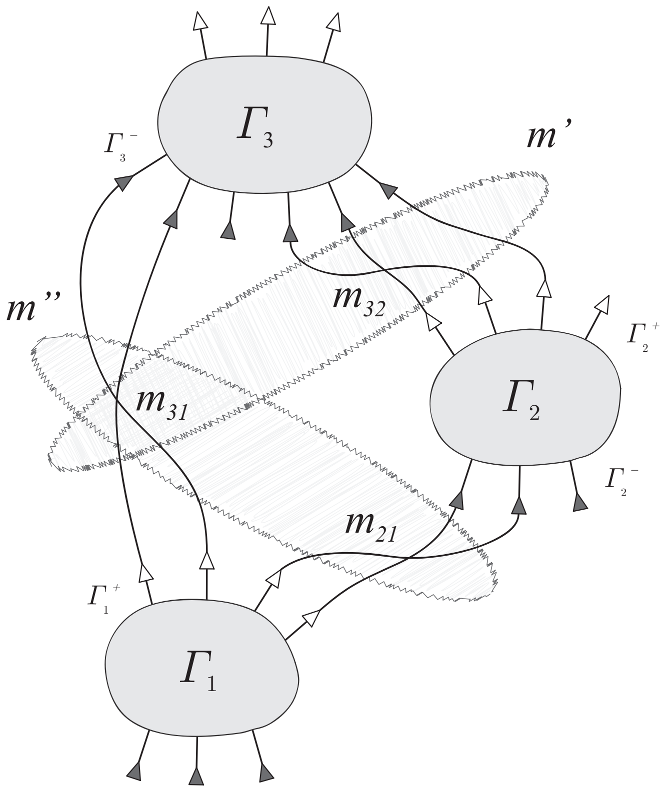

FIG. 3 Fine graining of the matchings $m' \in \Gamma_3^- \bowtie (\Gamma_2 \overset{m_{21}}{\triangleleft\!\triangleleft} \Gamma_1)^+$ and $m'' \in (\Gamma_3 \overset{m_{32}}{\triangleleft\!\triangleleft} \Gamma_2)^- \bowtie \Gamma_1^+$ used in the proof of associativity of multiplication.

Appendix B: Morphism φ

We shall prove that the mapping $\varphi : \mathcal{G} \longrightarrow \mathcal{U}(\mathcal{L}_{HW})$ defined in Eq. (8) preserves multiplication in $\mathcal{G}$. From linearity it is enough to check for the basis elements that $\varphi(\Gamma_2 * \Gamma_1) = \varphi(\Gamma_2)\varphi(\Gamma_1)$, which is shown in the following sequence of equalities:

$$\varphi(\Gamma_2 * \Gamma_1) \overset{(7)}{=} \sum_{m \in \Gamma_2^- \bowtie \Gamma_1^+} \varphi(\Gamma_2 \overset{m}{\triangleleft\!\triangleleft} \Gamma_1) = \sum_i \sum_{m \in \Gamma_2 \overset{i}{\bowtie} \Gamma_1} \varphi(\Gamma_2 \overset{m}{\triangleleft\!\triangleleft} \Gamma_1) \tag{B1}$$

$$\overset{(3)}{=} \sum_i \sum_{m \in \Gamma_2^- \overset{i}{\bowtie} \Gamma_1^+} (b^\dagger)^{|\Gamma_2^-|+|\Gamma_1^+|-i}\, b^{|\Gamma_2^+|+|\Gamma_1^-|-i}\, e^{|\Gamma_2^0|+|\Gamma_1^0|+i} \tag{B2}$$

$$= \sum_i (b^\dagger)^{|\Gamma_2^-|+|\Gamma_1^+|-i}\, b^{|\Gamma_2^+|+|\Gamma_1^-|-i}\, e^{|\Gamma_2^0|+|\Gamma_1^0|+i} \sum_{m \in \Gamma_2^- \overset{i}{\bowtie} \Gamma_1^+} 1 \tag{B3}$$

$$\overset{(2)}{=} \sum_i \binom{|\Gamma_2^-|}{i} \binom{|\Gamma_1^+|}{i}\, i!\, (b^\dagger)^{|\Gamma_2^-|+|\Gamma_1^+|-i}\, b^{|\Gamma_2^+|+|\Gamma_1^-|-i}\, e^{|\Gamma_2^0|+|\Gamma_1^0|+i} \tag{B4}$$

$$\overset{(1)}{=} \left((b^\dagger)^{|\Gamma_2^+|}\, b^{|\Gamma_2^-|}\, e^{|\Gamma_2^0|}\right) \left((b^\dagger)^{|\Gamma_1^+|}\, b^{|\Gamma_1^-|}\, e^{|\Gamma_1^0|}\right) = \varphi(\Gamma_2)\varphi(\Gamma_1)\ . \tag{B5}$$

In the above derivation the main trick in Eq. (B1) consists in splitting of the set of diagram matchings into disjoint subsets according to the number of connected lines, *i.e.* $\Gamma_2^- \bowtie \Gamma_1^+ = \bigcup_i \Gamma_2^- \overset{i}{\bowtie} \Gamma_1^+$. Then upon observation that the summands in Eq. (B3) do not depend on $m \in \Gamma_2^- \overset{i}{\bowtie} \Gamma_1^+$ we may execute explicitly one of the sums counting elements in $\Gamma_2^- \overset{i}{\bowtie} \Gamma_1^+$ with the help of Eq. (2).

References

Blasiak, P., G. H. E. Duchamp, A. Horzela, K. A. Penson, and A. I. Solomon, 2008, J. Phys. A: Math. Theor. **41**, 415204 (8pp).

Blasiak, P., A. Horzela, K. A. Penson, A. I. Solomon, and G. H. E. Duchamp, 2007, Am. J. Phys. **75**, 639, arXiv:0704.3116 [quant-ph].

Diestel, R., 2005, *Graph Theory* (Springer-Verlag, Heidelberg).

Joni, S. A., and G. C. Rota, 1979, Stud. Appl. Math. **61**, 93.

Louck, J. D., 2008, *Unitary Symmetry And Combinatorics* (World Scientific, Singapore).