Product actions,

Symmetric functions, and

QFT of partitions

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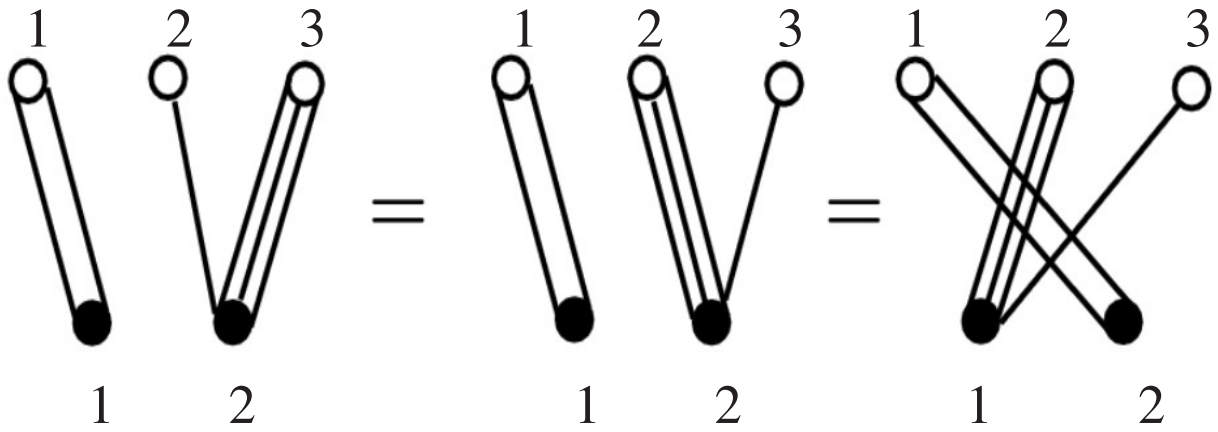
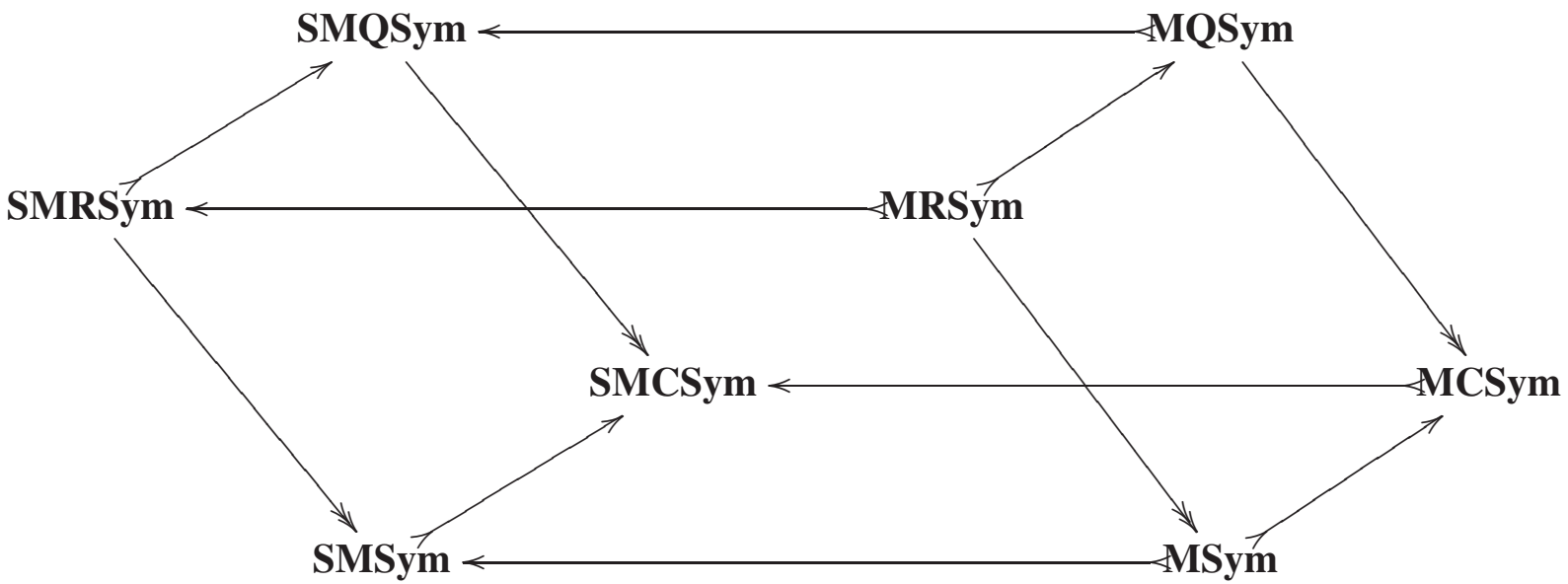
Feynman diagrams of QFTP



Equivalent labelled diagrams

Summing up

The following diagram commutes



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- Direct product actions and the cycle index algebra
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References

G.H.E. Duchamp, J.-G. Luque, K. Penson, CT, *SLC54A* (electronic)

G.H.E. Duchamp, J.-G. Luque, J.-C. Novelli, CT, F. Toumazet, Hopf algebras of diagrams, *Proc. FPSAC'07*

Direct product actions

Two pairs (G_1, X_1) and (G_2, X_2), G_i is a permutation group acting on X_i.

Intransitive action of $G_1 \times G_2$ on $X_1 \sqcup X_2$:

$$(\sigma_1, \sigma_2)x = \begin{cases} \sigma_1 x & \text{if } x \in X_1 \\ \sigma_2 x & \text{if } x \in X_2 \end{cases}$$

$(G_1, X_1) \dotplus (G_2, X_2) := (G_1 \times G_2, X_1 \sqcup X_2)$

Cartesian action of $G_1 \times G_2$ on $X_1 \times X_2$:

$$(\sigma_1, \sigma_2)(x_1, x_2) = (\sigma_1 x_1, \sigma_2 x_2)$$

$(G_1, X_1) \diagup\!\!\diagdown (G_2, X_2) := (G_1 \times G_2, X_1 \times X_2)$

Explicit realization

Denote

- by $\circ_N$ the natural action of $\mathfrak{S}_n$ on $\{0,\ldots,n-1\}$
- by $\circ_I$ the intransitive action of $\mathfrak{S}_n \times \mathfrak{S}_m$ on $\{0,\cdots,n+m-1\}$
- by $\circ_C$ the cartesian action of $\mathfrak{S}_n \times \mathfrak{S}_m$ on $\{0,\ldots,nm-1\}$

More precisely,

$$(\sigma_1,\sigma_2)\circ_I i = \begin{cases} \sigma_1 \circ_N i & \text{if } 0 \le i \le n-1 \\ \sigma_2 \circ_N (i-n)+n & \text{if } n \le i \le n+m-1 \end{cases}$$

and

$$(\sigma_1,\sigma_2)\circ_C (j+nk) = (\sigma_1 \circ_N j)+n(\sigma_2 \circ_N k)$$

for $0 \le i \le n+m-1$, $0 \le j \le n-1$ and $0 \le k \le m-1$

Intransitive action

Let the map $+\!\!\!\rightarrow : \mathfrak{S}_n \times \mathfrak{S}_m \rightarrow \mathfrak{S}_{n+m}$ be defined by

$$\sigma_1 +\!\!\!\rightarrow \sigma_2 = \sigma_1 \sigma_2[n]$$

$\sigma_1 = 1320 \in \mathfrak{S}_4$, $\sigma_2 = 534120 \in \mathfrak{S}_6$.

$$\sigma_1 +\!\!\!\rightarrow \sigma_2 = 1320978564,\ \sigma_2 +\!\!\!\rightarrow \sigma_1 = 5341207986$$

Proposition $(\sigma_1 +\!\!\!\rightarrow \sigma_2) \circ_N i = (\sigma_1, \sigma_2) \circ_I i$

Cartesian action

Let the map $\leftthreetimes: \mathfrak{S}_n \times \mathfrak{S}_m \to \mathfrak{S}_{nm}$ be defined by

$$\sigma_1 \leftthreetimes \sigma_2 = \prod_{i,j} c_i \leftthreetimes c'_j$$

where $\sigma_1 = c_1 \cdots c_k$ and $\sigma_2 = c'_1 \cdots c'_{k'}$ are the cycle decompositions of σ_1 and σ_2 and

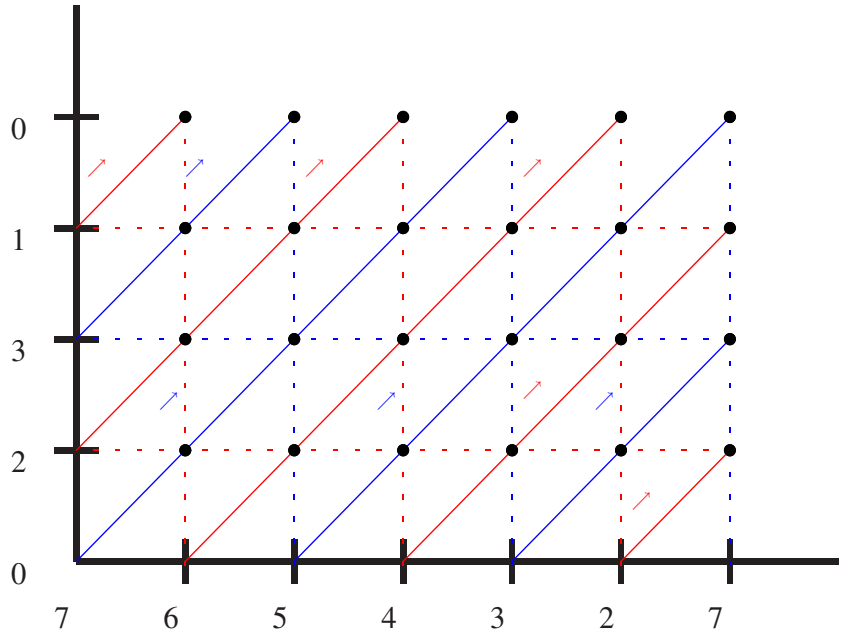
$$c \leftthreetimes c' = \prod_{s=0}^{l \wedge l' - 1} (\phi(s,0), \phi(s+1,1) \cdots, \phi(s + l \vee l' - 1, l \vee l' - 1)),$$

($\wedge := \gcd$, $\vee := \mathrm{lcm}$, $c = (i_0, \cdots, i_{l-1})$, $c' = (j_0, \cdots, j_{l'-1})$ are two cycles and $\phi(k,k') = i_{k \bmod l} + n j_{k' \bmod l'}$)

Proposition $(\sigma_1 \leftthreetimes \sigma_2) \circ_N i = (\sigma_1, \sigma_2) \circ_C i$

Example

$c_1 = (0,2,3,1)$, $c_2 = (7,6,5,4,3,2)$, $c_1 \diagup\!\!\diagdown c_2$:



$(28,26,23,17,12,10,31,25,20,18,15,9)$

$(30,27,21,16,14,11,29,24,22,19,13,8)$

Algebraic structure

Proposition *Associativity*

Let $\sigma_1 \in \mathfrak{S}_n$, $\sigma_2 \in \mathfrak{S}_m$ and $\sigma_3 \in \mathfrak{S}_p$ be 3 permutations

1. $\sigma_1 \twoheadrightarrow (\sigma_2 \twoheadrightarrow \sigma_3) = (\sigma_1 \twoheadrightarrow \sigma_2) \twoheadrightarrow \sigma_3$
2. $\sigma_1 \diagup\!\!\diagdown (\sigma_2 \diagup\!\!\diagdown \sigma_3) = (\sigma_1 \diagup\!\!\diagdown \sigma_2) \diagup\!\!\diagdown \sigma_3$

Proposition *Semi-distributivity*

$\sigma_1 \in \mathfrak{S}_n$, $\sigma_2 \in \mathfrak{S}_m$ and $\sigma_3 \in \mathfrak{S}_p$

$$\sigma_1 \diagup\!\!\diagdown (\sigma_2 \twoheadrightarrow \sigma_3) = (\sigma_1 \diagup\!\!\diagdown \sigma_2) \twoheadrightarrow (\sigma_1 \diagup\!\!\diagdown \sigma_3)$$

Cartesian product in *Sym*

$c_j(\sigma) :=$ number of cycles of σ of length j

$$\mathfrak{Z}(\sigma) = \prod_{j=0}^{\infty} p_j^{c_j(\sigma)}.$$

where $p_i = \sum_k x_k^i$ is a power sum symmetric function.

One defines

$$\prod_{1\le i\le\infty} p_i^{\alpha_i} \star \prod_{1\le j\le\infty} p_j^{\beta_j} = \prod_{1\le i,j\le\infty} p_{i\vee j}^{\alpha_i\beta_j(i\wedge j)}$$

(the sequences $(\alpha_i)_{i\ge1}$, $(\beta_j)_{j\ge1}$ have finite support)

A structure theorem

Proposition i) $\mathfrak{Z} : \bigoplus_{n\geq 0} \mathbb{Q}[\mathfrak{S}_n] \mapsto Sym$ is a homomorphism of 2-associative algebras sending the two laws $\mathrel{+\!\!\!\rightarrow}$; $\mathrel{\times\!\!\!\searrow}$ respectively to $\times$; $\star$. In particular,

$$\mathfrak{Z}(\sigma \mathrel{+\!\!\!\rightarrow} \tau) = \mathfrak{Z}(\sigma)\mathfrak{Z}(\tau); \quad \mathfrak{Z}(\sigma \mathrel{\times\!\!\!\searrow} \tau) = \mathfrak{Z}(\sigma) \star \mathfrak{Z}(\tau)$$

ii) $\star$ is commutative and distributive over $\times$.

Proposition $P := \{\prod_{i=1}^{\infty} p_i^{\alpha_i}\}_{(\alpha_i)_{i\geq 1} \in \mathbb{N}^{(\mathbb{N}^*)}}$ is closed under $\times$ and $\star$. More precisely: $(P, \times, \star)$ is isomorphic to a subsemiring of the $\mathbb{Z}$-algebra $\mathbb{Z}[\mathbb{N}^{(\mathfrak{p})}]$ of the monoid $(\mathbb{N}^{(\mathfrak{p})}, \sup)$ (where sup stands for the componentwise supremum and $\mathfrak{p}$:= the set of prime numbers)

Polyà's cycle index polynomial

$$Z(G) = \mathfrak{Z}\left(\frac{1}{|G|}\sum_{\sigma\in G}\sigma\right)$$

1. Symmetric group $\mathfrak{S}_n$: $Z(\mathfrak{S}_n) = h_n$.
2. Alternating group A_n : $Z(A_n) = h_n + e_n$.

Cycle index of products of symmetric groups

Since $\mathfrak{Z}$ is a homomorphism of 2-associative algebra,

$$Z(G_1 \mathbin{+\!\!\!\rightarrow} G_2) = Z(G_1)Z(G_2); \quad Z(G_1 \mathbin{\nearrow\!\!\!\!\searrow} G_2) = Z(G_1) \star Z(G_2)$$

1. Intransitive product of $\mathfrak{S}_n$ and $\mathfrak{S}_m$:

$$Z(\mathfrak{S}_n \mathbin{+\!\!\!\rightarrow} \mathfrak{S}_m) = h_n h_m.$$

2. Cartesian product of $\mathfrak{S}_n$ and $\mathfrak{S}_m$:

$$Z(\mathfrak{S}_n \mathbin{\nearrow\!\!\!\!\searrow} \mathfrak{S}_m) = h_n \star h_m = \sum_{\substack{|\lambda|=n,\\ |\rho|=m}} \frac{1}{z_\lambda z_\rho} \prod_{i,j} p_{\lambda_i \vee \rho_j}^{\lambda_i \wedge \rho_j},$$

where $z_\lambda = \prod i^{n_i} n_i!$ if n_i is the number of parts of λ equal to i.

Enumeration of QFTP Feynman diagrams

The *type* $t(f)$ of $f : X \to L = \{l_0 < l_1 \cdots < l_k \cdots\}$ is the vector $(i_0, \ldots, i_p, \ldots)$ where $i_k := \#f^{-1}(l_k)$.
The *shape* $s(f)$ of f is the partition obtained by sorting $t(f)$ in the decreasing order (and erasing the zeroes).

Let n:= # of WS and m:= # of BS of an unlabelled 2-coloured multigraph.
The multiedge set is encoded as a function e from $\{0, \ldots, n-1\} \times \{0, \ldots, m-1\}$ to $\mathbb{N}$.
The ith component of the type vector gives the number of (multi)edges with weight i.

Enumeration of QFTP Feynman diagrams (cont'd)

Let $d_I(n,m)$:= # of (2-coloured) multigraphs with type I
So $d_I(n,m)$= # of orbits with type I, for the action of $\mathfrak{S}_n \wr \mathfrak{S}_m$ on $\mathbb{N}^{\{0,\ldots,n-1\}\times\{0,\ldots,m-1\}}$.

Hence, the generating function of the shape is given by

$$g(n,m) := \sum_{\lambda} d_\lambda^s(n,m) m_\lambda = Z(\mathfrak{S}_n) \star Z(\mathfrak{S}_m)$$

Specializing $g(n,m)$ to the alphabet $\{y_0,\ldots,y_k,\ldots,\}$, the coefficient $d_I^t(n,m)$ of $\prod y_k^{i_k}$ in the expansion of $g(n,m)$ is equal to the number of multigraphs with type $I = (i_0,\ldots,i_k,\ldots)$,

$$g(n,m) = \sum_{I=(i_0,\ldots,i_p,\ldots)} d_I^t(n,m) \prod_{k=0}^{\infty} y_k^{i_k}.$$

Enumeration of QFTP Feynman diagrams (cont'd)

Generating functions of the type of QFTP Feynman diagrams:

$$F(n,m) := \sum_{I=(i_0,\ldots,i_p,\ldots)} f_I^t(n,m) \prod_{k=0}^{\infty} y_k^{i_k},$$

where $f_I^t(n,m)$:= # of Feynman diagrams of type I.
Observe that $F(n,m)$ is a symmetric function w.r.t. $\{y_1,\ldots,y_p,\ldots\}$ but not w.r.t. $\{y_0,\ldots,y_p,\ldots\}$.

1. $F(1,1) = y_1 + y_2$
2. $F(2,1) = F(1,2) = y_1^2 + y_1 y_2 + y_2^2$
3. $F(2,2) = y_0^2 y_1^2 + y_0^2 y_2^2 + y_0^2 y_1 y_2 + y_0 y_1^3 + 3 y_0 y_1^2 y_2 + 3 y_0 y_1 y_2^2 + y_0 y_2^3 + y_1^4 + y_1^3 y_2 + 3 y_1^2 y_2^2 + y_1 y_2^3 + y_2^4$

Enumeration of QFTP Feynman diagrams (cont'd)

Under this specialization,

$$F(2,2)+F(2,1)y_0^2+F(1,2)y_0^2+F(1,1)y_0^3+y_0^4 = 3m_{22}+m_4+3m_{211}+m_{31} = g(2,2).$$

More generally, the cycle index polynomial can be decomposed:

$$Z(\mathfrak{S}_n \wr \mathfrak{S}_m) = y_0^{nm} + \sum_{(1,1)\leq_{lex}(k,p)\leq_{lex}(n,m)} F(k,p)y_0^{nm-kp}.$$

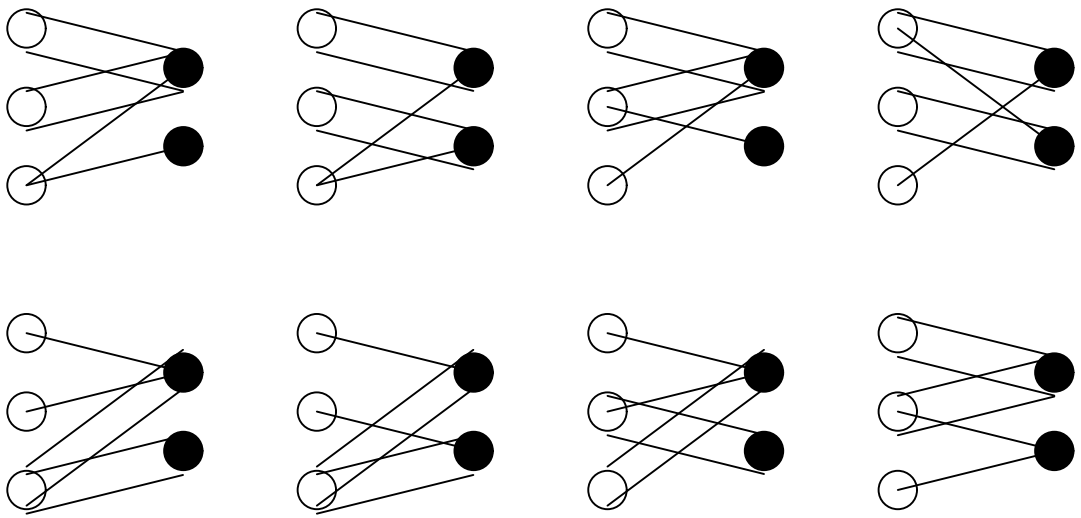
which yields an induction formula for the $F(n,m)$'s.

For example,

$$F(3,2) = Z(\mathfrak{S}_3 \wr \mathfrak{S}_2) - F(3,1)y_0^3 - F(2,2)y_0^2 - F(2,1)y_0^4 - F(1,2)y_0^4 - F(1,1)y_0^5 - y_0^6.$$

It suffices to compute $F(3,1)$ to enumerate Feynman diagrams whose edges are weighted by 0, 1 or 2.

$$\begin{aligned} F(3,2) \quad = \quad & y_2^6 + y_2^5 y_1 + 3y_2^4 y_1 + 3y_2^4 y_1 y_0 + 2y_2^4 y_0^2 + 3y_2^3 y_1^3 + 6y_2^3 y_1^2 y_0 \\ & + 5y_2^3 y_1 y_0^2 + y_2^3 y_0^3 + 3y_2^2 y_1^4 + 3y_2^2 y_1^3 y_0 + 8y_2^2 y_1^2 y_0^2 \\ & + 3y_2^2 y_1 y_0^3 + y_2 y_1^5 + 3y_2 y_1^4 y_0 + 5y_2 y_1^3 y_0^2 + 3y_2 y_1^2 y_0^3 + y_1^6 \\ & + y_1^5 y_0 + y_1^3 y_0^3 + 2y_1^2 y_0^4. \end{aligned}$$



Free quasi-symmetric cycle index algebra

The algebra **FQSym** is defined by one of its bases, indexed by $\mathfrak{S}$:

$$\mathbf{F}_\sigma = \sum_{Std(w)=\sigma^{-1}} w \in \mathbb{Z}\langle\langle A \rangle\rangle$$

One has

$$\mathbf{F}^\sigma \mathbf{F}^\tau = \mathbf{F}^{\sigma \leftarrow\!\!\cdot\, \tau}.$$

$$\begin{array}{rccc} \underline{\mathfrak{Z}} : \left(\bigoplus \mathbb{Q}[\mathfrak{S}_n], \leftarrow\!\!\cdot, +\right) & \rightarrow & (\mathbf{FQSym}, \times, +) \\ \sigma & \mapsto & \mathbf{F}^\sigma. \end{array}$$

The $\star$ product on **FQSym** defined by $\mathbf{F}^\sigma \star \mathbf{F}^\tau := \mathbf{F}^{\sigma \barwedge \tau}$.

$\underline{\mathfrak{Z}}$ is a homomorphism of 2-associative algebras.

$\star$ is distributive over the sum and semi-distributive over the shifted concatenation

(Polyà's) free quasi-symmetric cycle index polynomial

$$\underline{Z}(G) := \underline{\mathfrak{Z}}\left(\frac{1}{|G|}\sum_{\sigma\in G}\sigma\right).$$

Proposition

Let $G_1, G_2 \in \mathfrak{S}_{sg}$ be two permutation groups,

1. $\underline{Z}(G_1 \leftarrow\!\!\cdot\, G_2) = \underline{Z}(G_1)\underline{Z}(G_2)$.
2. $\underline{Z}(G_1 \diagup\!\!\!\diagdown G_2) = \underline{Z}(G_1) \star \underline{Z}(G_2)$.

$$\begin{array}{ccc} \mathfrak{S}_{sg} & \xrightarrow{\underline{Z}} & \mathbf{FQSym} \\ Z\downarrow & {}^{z}\swarrow & \uparrow \underline{\mathfrak{Z}} \\ Sym & \xleftarrow[\mathfrak{Z}]{} & \bigoplus \mathbb{Q}[\mathfrak{S}_n] \end{array}$$

Examples

1. Free quasi-symmetric cycle index of $\mathfrak{S}_n$:

$$\mathbf{H}_n := \underline{Z}(\mathfrak{S}_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} F^{\sigma},$$

   analogous to $z(\mathbf{H_n}) = Z(\mathfrak{S}_n) = h_n$.

2. Free cycle index polynomial of the alternating groups:

$$\mathbf{E}_n := \underline{Z}(A_n) - \underline{Z}(\mathfrak{S}_n),$$

   analogous to elementary symmetric functions:

$$z(\mathbf{E_n}) = Z(A_n) - Z(\mathfrak{S}_n) = e_n.$$

From diagrams to packed matrices

The diagrams are in one-to-one correspondence with the classes of packed matrices under the permutations of lines and columns

$$\begin{array}{ccc} \text{Packed matrices} & \xrightarrow{Class} & \text{Classes of packed matrices} \\ \downarrow & & \downarrow \\ \text{Labelled diagrams} & \longrightarrow & \text{Diagrams} \end{array} \qquad (1)$$

The structure of Hopf algebra on diagrams can be pulled back to labelled diagrams.

Set and integer packed matrices

lldiag $\leftrightarrow$ *set packed matrices* :

$k \in m_{i,j} = k$ iff edge k connects WS i to BS j

$$\begin{pmatrix} \{3\} & \{6\} & \{2\} \\ \emptyset & \{1,5\} & \{4\} \end{pmatrix}$$

ldiag $\leftrightarrow$ *integer packed matrices* :

$m_{i,j} =$ # of edges connecting WS i to BS j

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

Set packed matrices $\leftrightarrow$ Ordered pairs of set compositions of $[1,n]$:

$$\begin{pmatrix} \{3\} & \{6\} & \{2\} \\ \emptyset & \{1,5\} & \{4\} \end{pmatrix} \leftrightarrow [\{2,3,6\},\{1,4,5\}],[\{3\},\{1,5,6\},\{2,4\}]$$

SMQSym: Set matrix quasi-symmetric functions

$$\mathbf{SMQ}_P\mathbf{SMQ}_Q := \sum_{R \in \underline{\sqcup\!\sqcup}(P,Q)} \mathbf{SMQ}_R$$

where $\underline{\sqcup\!\sqcup}(P,Q)$ is the *augmented shuffle* of P and Q

For example,

$$\mathbf{SMQ}_{\begin{pmatrix}\{3,4\} & \{1\}\\ \{2\} & \emptyset\end{pmatrix}}\mathbf{SMQ}_{\begin{pmatrix}\{2,3,4\} & \{1\}\end{pmatrix}} = \mathbf{SMQ}_{\begin{pmatrix}\{3,4\} & \{1\} & \emptyset & \emptyset\\ \{2\} & \emptyset & \emptyset & \emptyset\\ \emptyset & \emptyset & \{6,7,8\} & \{5\}\end{pmatrix}} + \mathbf{SMQ}_{\begin{pmatrix}\{3,4\} & \{1\} & \emptyset & \emptyset\\ \{2\} & \emptyset & \{6,7,8\} & \{5\}\end{pmatrix}}$$
$$+\mathbf{SMQ}_{\begin{pmatrix}\{3,4\} & \{1\} & \emptyset & \emptyset\\ \emptyset & \emptyset & \{6,7,8\} & \{5\}\\ \{2\} & \emptyset & \emptyset & \emptyset\end{pmatrix}} + \mathbf{SMQ}_{\begin{pmatrix}\{3,4\} & \{1\} & \{6,7,8\} & \{5\}\\ \{2\} & \emptyset & \emptyset & \emptyset\end{pmatrix}} + \mathbf{SMQ}_{\begin{pmatrix}\emptyset & \emptyset & \{6,7,8\} & \{5\}\\ \{3,4\} & \{1\} & \emptyset & \emptyset\\ \{2\} & \emptyset & \emptyset & \emptyset\end{pmatrix}}.$$

Coproduct in **SMQSym**

$$\Delta\mathbf{SMQ}_A = \sum_{A=\binom{A_1}{A_2}} \mathbf{SMQ}_{\mathrm{std}(A_1)} \otimes \mathbf{SMQ}_{\mathrm{std}(A_2)},$$

where $\mathrm{std}(A)$ denotes the standardized of the matrix A, that is the matrix obtained by the substitution $a_i \mapsto i$, where $a_1 < \cdots < a_n$ are the integers appearing in A. For example,

$$\Delta\mathbf{SMQ}_{\left(\begin{smallmatrix}\{2,4\} & \{1\}\\ \{6\} & \{3,5\}\end{smallmatrix}\right)} = \mathbf{SMQ}_{\left(\begin{smallmatrix}\{2,4\} & \{1\}\\ \{6\} & \{3,5\}\end{smallmatrix}\right)} \otimes 1 + \mathbf{SMQ}_{\left(\begin{smallmatrix}\{2,3\} & \{1\}\end{smallmatrix}\right)} \otimes \mathbf{SMQ}_{\left(\begin{smallmatrix}\{3\} & \{1,2\}\end{smallmatrix}\right)} + 1 \otimes \mathbf{SMQ}_{\left(\begin{smallmatrix}\{2,4\} & \{1\}\\ \{6\} & \{3,5\}\end{smallmatrix}\right)}$$

Biword realization of **SMQSym**

A noncommutative bi-word is a word over an alphabet of bi-letters $\left\langle \begin{smallmatrix} a_i \\ \mathbf{b}_j \end{smallmatrix} \right\rangle$ with $i, j \in \mathbb{N}$ where a_i and $\mathbf{b}_j$ are letters of two distinct ordered alphabets A and $\mathbf{B}$. One denotes by $\mathbb{C}\left\langle \begin{smallmatrix} A \\ \mathbf{B} \end{smallmatrix} \right\rangle$ the algebra of the (potentially infinite) polynomials over the bi-letters $\left\langle \begin{smallmatrix} a \\ \mathbf{b} \end{smallmatrix} \right\rangle$ for the product $\star$ defined by

$$\left\langle \begin{smallmatrix} u_1 \\ \mathbf{v}_1 \end{smallmatrix} \right\rangle \star \left\langle \begin{smallmatrix} u_2 \\ \mathbf{v}_2 \end{smallmatrix} \right\rangle = \left\langle \begin{smallmatrix} u_1 . u_2 \\ \mathbf{v}_1 . \mathbf{v}_2 [\max(\mathbf{v}_1)] \end{smallmatrix} \right\rangle,$$

For example,

$$\left\langle \begin{smallmatrix} 142 \\ 236 \end{smallmatrix} \right\rangle \star \left\langle \begin{smallmatrix} 24 \\ 31 \end{smallmatrix} \right\rangle = \left\langle \begin{smallmatrix} 14224 \\ 23697 \end{smallmatrix} \right\rangle$$

Lemma The product $\star$ is associative.

Biword realization of **SMQSym** (cont'd)

$$\text{bi-word}\begin{pmatrix} \{2,4\} & \{1\} \\ \{6\} & \{3,5\} \end{pmatrix} = \left\langle \begin{matrix} a_1a_1a_2a_1a_2a_2 \\ \mathbf{b}_2\mathbf{b}_1\mathbf{b}_2\mathbf{b}_1\mathbf{b}_2\mathbf{b}_1 \end{matrix} \right\rangle \text{ (bi-packed bi-word)}$$

Theorem

- The algebra **SMQSym** can be realized on bi-words by

$$\mathbf{SMQ}_{\left\langle \begin{smallmatrix} u \\ \mathbf{v} \end{smallmatrix} \right\rangle} := \sum_{\text{bipacked}\left\langle \begin{smallmatrix} u' \\ \mathbf{v}' \end{smallmatrix} \right\rangle = \left\langle \begin{smallmatrix} u \\ \mathbf{v} \end{smallmatrix} \right\rangle} \left\langle \begin{matrix} u' \\ \mathbf{v}' \end{matrix} \right\rangle .$$

- **SMQSym** is a Hopf algebra.

The coproduct of **SMQSym** is given by the usual trick of noncommutative symmetric functions, considering the alphabet A as an ordered sum of two mutually commuting alphabets $A' \hat{+} A''$

SMQSym and word quasi-symmetric functions

Theorem **SMQSym** is Hopf isomorphic to the graded endomorphisms of **WQSym**:

$$\mathrm{End}_{gr}\mathbf{WQSym} = \bigoplus_{n\in\mathbb{N}} \mathbf{WQSym}_n \otimes \mathbf{WQSym}_n^*$$

through the Hopf homomorphism

$$\phi\left(\mathbf{SMQ}_{\left\langle \begin{smallmatrix} u \\ \mathbf{v} \end{smallmatrix} \right\rangle}\right) = \mathbf{WQ}_u \otimes \mathbf{F}^{\mathbf{v}}.$$

where

$$\mathbf{WQSym}(A) := \mathbb{C}\langle A\rangle^{\mathfrak{S}(A)_{QS}}.$$

A is an infinite alphabet, $\mathbf{WQSym}(A)$ is a graded Hopf algebra whose basis is indexed by set compositions (or *packed words*)

SMQSym and word quasi-symmetric functions (cont'd)

WQSym is generated by the polynomials

$$\mathbf{WQ}_u := \sum_{\mathrm{pack}(w)=u} w,$$

$$\mathbf{WQ}_u \mathbf{WQ}_v = \sum_{w \in u \star_W v} \mathbf{WQ}_w,$$

where

$$u \star_W v = \sum_{\substack{w_1, w_2; w = w_1 w_2 \text{ packed} \\ \mathrm{pack}(w_1)=u,\ \mathrm{pack}(w_2)=v}} w.$$

$$\Delta \mathbf{WQ}_w(A) = \sum_{u,v; w \in u \Cup_W v} \mathbf{WQ}_u \otimes \mathbf{WQ}_v$$

where $u \Cup_W v$ denotes the *packed shifted shuffle*, *i.e.* the shuffle of u and $v' = v[max(u)]$ ($v'_i = v_i + max(u)$)

Set matrix half-symmetric functions: The Hopf algebra **SMRSym**

Let **SMRSym** be the subalgebra of **SMQSym** generated by the polynomials $\mathbf{SMR}_{\pi_1,\Pi_2}$ indexed by a set partition π_1 and a set composition Π_2 and defined by

$$\mathbf{SMR}_{(\pi_1,\Pi_2)} := \sum_{\mathrm{sp}(\Pi_1)=\pi_1} \mathbf{SMQ}_{\Pi_1,\Pi_2}$$

$$\begin{aligned}\mathbf{SMR}_{\{\{13\},\{4\},\{2\}\},[\{123\},\{4\}]} = \; & \mathbf{SMQ}_{\begin{pmatrix}\{1,3\} & \emptyset\\ \emptyset & \{4\}\\ \{2\} & \emptyset\end{pmatrix}} + \mathbf{SMQ}_{\begin{pmatrix}\{1,3\} & \emptyset\\ \{2\} & \emptyset\\ \emptyset & \{4\}\end{pmatrix}} + \mathbf{SMQ}_{\begin{pmatrix}\emptyset & \{4\}\\ \{1,3\} & \emptyset\\ \{2\} & \emptyset\end{pmatrix}}\\ & + \mathbf{SMQ}_{\begin{pmatrix}\emptyset & \{4\}\\ \{2\} & \emptyset\\ \{1,3\} & \emptyset\end{pmatrix}} + \mathbf{SMQ}_{\begin{pmatrix}\{2\} & \emptyset\\ \emptyset & \{4\}\\ \{1,3\} & \emptyset\end{pmatrix}} + \mathbf{SMQ}_{\begin{pmatrix}\{2\} & \emptyset\\ \{1,3\} & \emptyset\\ \emptyset & \{4\}\end{pmatrix}}\end{aligned}$$

The Hopf algebra **SMRSym** (cont'd)

Note that a pair constituted by a set partition and a set composition is equivalent to a set packed matrix, considered up to a permutation of its rows. Hence, the realization on bi-words follows:

$$\mathbf{SMR}_{\{\{13\},\{4\},\{2\}\},[\{123\},\{4\}]} = \sum_{\substack{i_1,i_2,i_3 \text{ distinct} \\ j_1<j_2}} \left\langle \begin{matrix} a_{i_1}\, a_{i_2}\, a_{i_1}\, a_{i_3} \\ \mathbf{b}_{j_1}\, \mathbf{b}_{j_1}\, \mathbf{b}_{j_1}\, \mathbf{b}_{j_2} \end{matrix} \right\rangle$$

Proposition

- **SMRSym** is isomorphic to $\bigoplus \mathbf{WSym}_n \otimes \mathbf{WQSym}_n^*$.
- **SMRSym** is a co-commutative Hopf subalgebra of **SMQSym**.

The Hopf algebra **SMCSym**

Forgetting about the order of the columns instead of the rows leads to another Hopf algebra, **SMCSym**.
It is the quotient of **SMQSym** by the polynomials

$$\mathbf{SMQ}_{\Pi_1,\Pi_2} - \mathbf{SMQ}_{\Pi_1,\Pi_2'}$$

where $\mathrm{sp}(\Pi_2) = \mathrm{sp}(\Pi_2')$. The algebra **SMCSym** is spanned by the $\mathbf{SMC}_{\Pi_1,\pi_2}$ where Π_1 is a set composition and π_2 is a set partition.

Proposition

- **SMCSym** is isomorphic to $\bigoplus \mathbf{WQSym}_n \otimes \mathbf{WSym}_n^*$,
- **SMCSym** is a Hopf algebra.

SMRSym and **SMCSym** have the same Hilbert series, given by the product of ordered Bell numbers by unordered Bell numbers.

Set matrix symmetric functions: The Hopf algebra **SMSym**

SMSym is the subalgebra of **SMCSym** generated by the

$$\mathbf{SM}_{\pi_1,\pi_2} = \sum_{\mathrm{sp}(\Pi_1)=\pi_1} \mathbf{SMR}_{\Pi_1,\pi_2}$$

For example,

$$\mathbf{SM}_{\{\{1,3\},\{4\},\{2\}\},\{\{1,2,3\},\{4\}\}} = \sum_{\substack{i_1,i_2,i_3 \text{ distinct} \\ j_1<j_2}} \left\langle \begin{matrix} a_{i_1} a_{i_3} a_{i_1} a_{i_2} \\ \mathbf{b}_{j_1} \mathbf{b}_{j_1} \mathbf{b}_{j_1} \mathbf{b}_{j_2} \end{matrix} \right\rangle$$

Theorem

- **SMSym** is a co-commutative Hopf subalgebra of **SMCSym**.
- **SMSym** is isomorphic to the quotient of **SMRSym** by the ideal generated by the polynomials $\mathbf{SMR}_{\pi_1,\Pi_2} - \mathbf{SMR}_{\pi_1,\Pi_2'}$ with $\mathrm{sp}(\Pi_2) = \mathrm{sp}(\Pi_2')$.

Hopf algebras of packed integer matrices

Let SA be the set of packed set matrices s.t. the coefficients read by column from top to bottom give the 1 to n in the usual order:

$$\begin{pmatrix} \{1,2\} & \emptyset \\ \emptyset & \{4\} \\ \{3\} & \emptyset \end{pmatrix} \qquad \begin{pmatrix} \{1,3\} & \emptyset \\ \emptyset & \{4\} \\ \{2\} & \emptyset \end{pmatrix} \tag{2}$$

Let us consider the subalgebra $\mathbf{MQSym}'$ of **SMQSym** generated by the elements of SA. For example,

$$\mathbf{MQ}_{\left(\begin{smallmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \end{smallmatrix}\right)} = \sum_{\substack{i_1<i_2 \\ j_1<j_2<j_3}} \left\langle \begin{matrix} a_{i_1} a_{i_1} a_{i_1} a_{i_2} a_{i_2} a_{i_2} a_{i_2} a_{i_2} a_{i_2} a_{i_2} \\ \mathbf{b}_{j_1} \mathbf{b}_{j_1} \mathbf{b}_{j_3} \mathbf{b}_{j_2} \mathbf{b}_{j_2} \mathbf{b}_{j_2} \mathbf{b}_{j_3} \mathbf{b}_{j_3} \mathbf{b}_{j_3} \mathbf{b}_{j_3} \end{matrix} \right\rangle \tag{3}$$

Theorem $\mathbf{MQSym}'$ is Hopf isomorphic to **MQSym**.

Matrix half-symmetric functions

One reproduces the same construction as for set packed matrices, and defines three algebras **MRSym** (resp. **MCSym**, **MSym**)

Let **MRSym** be the subalgebra of **MQSym** generated by the polynomials

$$\mathbf{MR}_A := \sum \mathbf{MQ}_B$$

where B is obtained from A by a permutation of its rows.

As for **SMRSym**, the realization of **MRSym** on bi-words is automatic.

$$\mathbf{MR}_{\begin{pmatrix}2&1&0\\0&3&4\end{pmatrix}} = \mathbf{MQ}_{\begin{pmatrix}2&1&0\\0&3&4\end{pmatrix}} + \mathbf{MQ}_{\begin{pmatrix}0&3&4\\2&1&0\end{pmatrix}} = \sum_{\substack{i_1\neq i_2\\ j_1<j_2<j_3}} \left\langle \begin{matrix} a_{i_1}\,a_{i_1}\,a_{i_1}\,a_{i_2}\,a_{i_2}\,a_{i_2}\,a_{i_2}\,a_{i_2}\,a_{i_2}\,a_{i_2} \\ \mathbf{b}_{j_1}\mathbf{b}_{j_1}\mathbf{b}_{j_3}\mathbf{b}_{j_2}\mathbf{b}_{j_2}\mathbf{b}_{j_2}\mathbf{b}_{j_3}\mathbf{b}_{j_3}\mathbf{b}_{j_3}\mathbf{b}_{j_3} \end{matrix} \right\rangle$$

Matrix half-symmetric functions (cont'd)

Theorem

1. **MRSym** is a co-commutative Hopf subalgebra of **MQSym**,
2. **MRSym** is also the subalgebra of **SMRSym** generated by the elements $\mathbf{SMR}_A$ where A is any matrix s.t. each element of the set composition of its columns is an interval of $[1, n]$.
3. In terms of the realization, it amounts to taking the commutative image of all bi-words.

The Hopf algebra **MCSym**

One constructs the algebra **MCSym** as the quotient of **MQSym** by the ideal generated by the polynomials $\mathbf{MQ}_A - \mathbf{MQ}_B$ where B can be obtained from A by a permutation of its columns.

Theorem

1. **MCSym** is a commutative Hopf algebra,
2. **MCSym** is Hopf isomorphic to the subalgebra of **SMCSym** generated by the elements $\mathbf{SMC}_A$, s.t. each element of the set partition of its columns is an interval of $[1,n]$.

Dimensions of **MRSym** and **MCSym**

$\dim \mathbf{MRSym}_n = \dim \mathbf{MCSym}_n$ = # of packed matrices s.t. the sum of the entries is equal to n, considered up a permutation of the rows.

Let $\mathrm{PMuR}(p,q,n)$:= # of such $p \times q$ matrices.

$$\dim \mathbf{MRSym}_n = \sum_{1 \le p,q \le n} \mathrm{PMuR}(p,q,n)$$

The integers $\mathrm{PMuR}(p,q,n)$ can be computed through the induction

$$\mathrm{PMuR}(p,q,n) = \mathrm{MuR}(p,q,n) - \sum_{1 \le k,l \le p,q} \binom{q}{l} \mathrm{PMuR}(k,l,n),$$

where $\mathrm{MuR}(p,q,n)$ is the number of $p \times q$ matrices up to a permutations of the rows with the sums of the entries equal to n.

Dimensions of **MRSym** and **MCSym** (cont'd)

One gets

$$\dim \mathbf{MRSym}_n = \sum_{i=1}^{n+1} (-1)^{n-i} T_{n+1,i+1} \mathrm{MuR}(n,i,n),$$

where $T_{n,k} = \sum_{j=0}^{n-k} (-1)^{n-k-j} \binom{j+k-1}{j}$ is known as the number of minimal covers of an unlabelled n-set that cover k points of that set uniquely

The generating series of the $T_{n,k}$ is

$$\sum_{i,j} T_{i,j} x^i y^j = \frac{1-x}{(1+x)(1-x-xy)}.$$

Dimensions of **MRSym** and **MCSym** (cont'd)

The integer $\mathrm{MuR}(n,i,n)$, computed via the Pólya enumeration theorem, is the coefficient of x^n in the cycle index $Z(G_{n,i})$, evaluated over the alphabet $1+x+\cdots+x^n+\cdots$, of the subgroup $G_{n,i}$ of $\mathfrak{S}_{in}$ generated by the permutations $\sigma.\sigma[n].\sigma[2n].\ \cdots\ .\sigma[in]$ for $\sigma \in \mathfrak{S}_n$ (here . denotes the concatenation).

This coefficient is also the number $N_{n,i}$ of partitions of n objects with i colors whose generating series is

$$\sum_n N_{n,i} x^n = \prod_k \left(\frac{1}{1-x^k} \right)^{\binom{i+k}{i}}.$$

Dimensions of **MRSym** and **MCSym** (cont'd)

Proposition

$$\dim \mathbf{MRSym}_n = \sum_{i=1}^{n+1} (-1)^{n-i} T_{n+1,i+1} N_{n,i}.$$

The first values are

$$\begin{aligned}\mathrm{Hilb}(\mathbf{MRSym}) = \mathrm{Hilb}(\mathbf{MCSym}) = 1 + t + 4t^2 + 16t^3 + 76t^4 + 400t^5 + 2356t^6 \\ + 15200t^7 + 106644t^8 \\ + 806320t^9 + 6526580t^{10} + \cdots\end{aligned}$$

Matrix symmetric functions

The algebra **MSym** of *matrix symmetric functions* is the subalgebra of **MCSym** generated by

$$\sum_{\sigma \in \mathfrak{S}_n} \mathbf{MR}_B,$$

where B is obtained from A by a permutation of its rows.

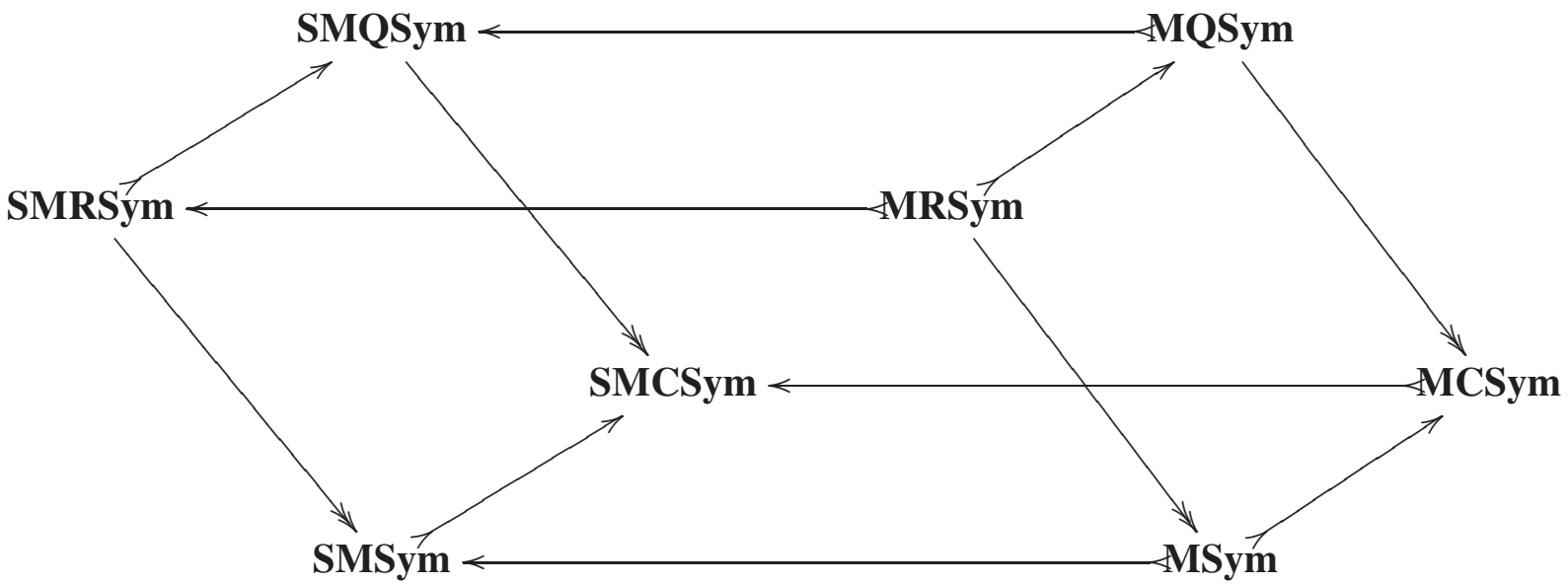
Theorem

1. **MSym** is a commutative and co-commutative Hopf subalgebra of **MCSym**.
2. **MSym** is Hopf isomorphic to the subalgebra of **SMSym** generated by the elements $\mathbf{SM}_A$, s.t. each element of the set partition of its columns is an interval of $[1,n]$.
3. **MSym** is isomorphic to the quotient of **MRSym** by the ideal

generated by the polynomials $\mathbf{MR}_A - \mathbf{MR}_B$ where B can be obtained from A by a permutation of its columns.

Summing up

The following diagram commutes



Tridendriform structure over **SMQSym**

One defines three products over bi-words:

1. $\left\langle \begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix} \right\rangle \prec \left\langle \begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix} \right\rangle =$
   $\left\langle \begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix} \right\rangle \star \left\langle \begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix} \right\rangle$ if $\max(u_1) > \max(u_2)$, and 0 otherwise.
2. $\left\langle \begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix} \right\rangle \circ \left\langle \begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix} \right\rangle =$
   $\left\langle \begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix} \right\rangle \star \left\langle \begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix} \right\rangle$ if $\max(u_1) = \max(u_2)$, and 0 otherwise.
3. $\left\langle \begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix} \right\rangle \succ \left\langle \begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix} \right\rangle =$
   $\left\langle \begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix} \right\rangle \star \left\langle \begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix} \right\rangle$ if $\max(u_1) < \max(u_2)$, and 0 otherwise.

Proposition The algebra of bi-words endowed with the three product rules $\prec, \circ$, and $\succ$ is a tridendriform algebra. **SMQSym** is stable by the three products, thus it is a tridendriform algebra.

Tridendriform structures (cont'd)

More precisely, one has:

1. $$\mathbf{SMQ}_{\left\langle \begin{smallmatrix} u \\ \mathbf{v} \end{smallmatrix} \right\rangle} \prec \mathbf{SMQ}_{\left\langle \begin{smallmatrix} u' \\ \mathbf{v}' \end{smallmatrix} \right\rangle} = \sum_{\substack{w = x.y \in u \star_W u' \\ |x| = |u|;\ \max(y) < \max(x)}} \mathbf{SMQ}_{\left\langle \begin{smallmatrix} w \\ \mathbf{v}\mathbf{v}'[\max(v)] \end{smallmatrix} \right\rangle}$$

2. $$\mathbf{SMQ}_{\left\langle \begin{smallmatrix} u \\ \mathbf{v} \end{smallmatrix} \right\rangle} \circ \mathbf{SMQ}_{\left\langle \begin{smallmatrix} u' \\ \mathbf{v}' \end{smallmatrix} \right\rangle} = \sum_{\substack{w = x.y \in u \star_W u' \\ |x| = |u|;\ \max(y) = \max(x)}} \mathbf{SMQ}_{\left\langle \begin{smallmatrix} w \\ \mathbf{v}\mathbf{v}'[\max(v)] \end{smallmatrix} \right\rangle}$$

3. $$\mathbf{SMQ}_{\left\langle \begin{smallmatrix} u \\ \mathbf{v} \end{smallmatrix} \right\rangle} \succ \mathbf{SMQ}_{\left\langle \begin{smallmatrix} u' \\ \mathbf{v}' \end{smallmatrix} \right\rangle} = \sum_{\substack{w = x.y \in u \star_W u' \\ |x| = |u|;\ \max(y) > \max(x)}} \mathbf{SMQ}_{\left\langle \begin{smallmatrix} w \\ \mathbf{v}\mathbf{v}'[\max(v)] \end{smallmatrix} \right\rangle}.$$

Corollary **SMCSym**, **MQSym** and **MCSym** are tridendriform algebras.

Bidendriform structures

One defines two product rules $\ll = \prec$ and $\gg = \circ + \succ$ on bi-words. One then splits the non-trivial parts of the coproduct of the **SMQ** of **SMQSym**:

1. $$\Delta_{\ll}(\mathbf{SMQ}_A) = \sum_{\substack{A=\binom{B}{C},\ A\neq B,C \\ \max(B)=\max(A)}} \mathbf{SMQ}_{\mathrm{std}(B)} \otimes \mathbf{SMQ}_{\mathrm{std}(C)}.$$

2. $$\Delta_{\gg}(\mathbf{SMQ}_A) = \sum_{\substack{A=\binom{B}{C},\ A\neq B,C \\ \max(C)=\max(A)}} \mathbf{SMQ}_{\mathrm{std}(B)} \otimes \mathbf{SMQ}_{\mathrm{std}(C)}.$$

Theorem **SMQSym** is a bidendriform bialgebra.

Bidendriform structures (cont'd)

Recall that **SMCSym** is the quotient of **SMQSym** by the ideal generated by $\mathbf{SMQ}_A - \mathbf{SMQ}_B$ where A and B are the same matrices up to a permutation of their columns; the row containing the maximal element is the same for any element of a given class, so that the left coproduct and the right coproduct are compatible with the quotient. Moreover, the left and right coproduct are internal within **MQSym**.

Corollary **SMCSym** and **MQSym** are bidendriform sub-bialgebras of **SMQSym**.

Bi-words realization of $LDiag(q_c, q_s)$

Consider bi-words with parameter-commuting bi-letters depending on the biletters:

$$\left\langle \begin{matrix} yx \\ zt \end{matrix} \right\rangle = q_c \left\langle \begin{matrix} xy \\ tz \end{matrix} \right\rangle \text{ if } y > x,$$
$$\left\langle \begin{matrix} xx \\ zt \end{matrix} \right\rangle = q_s \left\langle \begin{matrix} xx \\ tz \end{matrix} \right\rangle \text{ if } z < t.$$

Let us now define the realization as a sum of bi-words of a packed integer matrix with p rows and q columns:

$$\mathbf{LD}_M := \sum_{\substack{i_1<\cdots<i_p \\ j_1<\cdots<j_q}} \prod_{a=1}^{n} \prod_{b=p}^{1} \left\langle \begin{matrix} i_a \\ j_b \end{matrix} \right\rangle^{M_{ij}}. \tag{4}$$

Bi-words realization of $LDiag(q_c, q_s)$ (cont'd)

For example,

$$\mathbf{LD}_{\binom{3\ 5}{1\ 3}} := \sum_{\substack{i_1<i_2 \\ j_1<j_2}} \left\langle \begin{array}{c} i_1^5\, i_1^3\, i_2^3\, i_2 \\ j_2^5\, j_1^3\, j_2^3\, j_1 \end{array} \right\rangle .$$

Theorem

- The **LD** generate a Hopf algebra,
- The specialization $q_s = q_c = 1$ gives back **MQSym**.

Bi-words realization of $LDiag(q_c, q_s)$ (cont'd)

$$\mathbf{LD}_{\binom{2\ 0}{1\ 4}} \star \mathbf{LD}_{(1)} = \mathbf{LD}_{\binom{2\ 0\ 0}{1\ 4\ 0}{0\ 0\ 1}} + q_s^5\,\mathbf{LD}_{\binom{2\ 0\ 0}{1\ 4\ 1}} + q_c^5\,\mathbf{LD}_{\binom{2\ 0\ 0}{0\ 0\ 1}{1\ 4\ 0}} + q_c^5 q_s^2\,\mathbf{LD}_{\binom{2\ 0\ 1}{1\ 4\ 0}} + q_c^7\,\mathbf{LD}_{\binom{0\ 0\ 1}{2\ 0\ 0}{1\ 4\ 0}}$$

since

$$\sum_{\substack{i_1<i_2\\ j_1<j_2}} \left\langle \begin{array}{c} i_1\, i_1\, i_2\, i_2\, i_2\, i_2\, i_2 \\ j_1\, j_1\, j_2\, j_2\, j_2\, j_2\, j_1 \end{array} \right\rangle \star \sum \left\langle \begin{array}{c} i \\ j \end{array} \right\rangle = \sum_{\substack{i_1<i_2<i_3\\ j_1<j_2<j_3}} \left\langle \begin{array}{c} i_1^2\, i_2^4\, i_2\, i_3 \\ j_1^2\, j_2^4\, j_1\, j_3 \end{array} \right\rangle + \sum_{\substack{i_1<i_2(=i_3)\\ j_1<j_2<j_3}} q_s^5 \left\langle \begin{array}{c} i_1^2\, i_2\, i_2^4 i_2 \\ j_1^2\, j_3\, j_2^4\, j_1 \end{array} \right\rangle$$

$$+ \sum_{\substack{i_1<i_3<i_2\\ j_1<j_2<j_3}} q_c^5 \left\langle \begin{array}{c} i_1^2\, i_3\, i_2^4\, i_2 \\ j_1^2\, j_3\, j_2^4\, j_1 \end{array} \right\rangle + \sum_{\substack{i_1(=i_3)<i_2\\ j_1<j_2<j_3}} q_s^2 q_c^5 \left\langle \begin{array}{c} i_1\, i_1^2\, i_2^4\, i_2 \\ j_3\, j_1^2\, j_2^4\, j_1 \end{array} \right\rangle$$

$$+ \sum_{\substack{i_3<i_1<i_2\\ j_1<j_2<j_3}} q_c^7 \left\langle \begin{array}{c} i_3\, i_1^2\, i_2^4\, i_2 \\ j_3\, j_1^2\, j_2^4\, j_1 \end{array} \right\rangle .$$