# Hopf Algebras in General and in Combinatorial Physics: a practical introduction

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**Abstract.**

## 1. Introduction

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Quantum Theory seen in action is an interplay of mathematical ideas and physical concepts. From the present-day perspective its formalism and structure is founded on theory of Hilbert spaces [Ish95, Per02]. According to a few basic postulates physical notions of a system and apparatus, as well as transformations and measurements, are described in terms of linear operators. In this way algebra of operators constitutes the proper mathematical framework within which quantum theories are built. Structure of this algebra is determined by two operations, addition and multiplication of operators, which lie at the root of all fundamental aspects of Quantum Theory.
The formalism of quantum theory represents the physical concepts of states, observables and their transformations as objects in some Hilbert space $\mathcal{H}$ and subsequently provides a scheme for measurement predictions. Briefly, vectors in the Hilbert space

Version compilée le 24-01-2008 16:05

describe states of a system, and linear forms in $\mathcal{V}^*$ represent basic observables. Both concepts combine in the measurement process which provides probabilistic distribution of results and is given by the Born rule. Physical information about the system is gained by transforming the system and/or apparatus in various ways and performing measurements. Set of transformations usually possesses some structure – such as group, semi-group, Lie algebra, etc. – and in general can be handled within the structure of an algebra $\mathcal{A}$. Action of the algebra on the vector space of states $\mathcal{V}$ and observables $\mathcal{V}^*$ is simply its representation. Hence if an algebra is to describe physical transformations it has to have representations in all physically relevant systems. This requirement directly leads to the Hopf algebra structures in physics.

From the mathematical viewpoint structure of the theory, modulo details, seems to be clear. Physicists, however, need to have some additional properties and constructions to move freely in this playground. Here we will show how the structure of Hopf algebras enter into the game in the context of representations. The first issue at point is the construction of tensor product of vector spaces which is needed for description of composite systems. Suppose, we now how the algebra of some transformations acts on individual systems, i.e. we know representations of the algebra in each vector space $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively. Hence, the natural need for a canonical construction of induced representation of this algebra in $\mathcal{V}_1 \otimes \mathcal{V}_2$ which would describe its action on the composite system. Such scheme exists and is provided by the *co-product* in the algebra, i.e. a morphism $\Delta : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$. Physical plausibility of this construction requires equivalence of representations built on $(\mathcal{V}_1 \otimes \mathcal{V}_2) \otimes \mathcal{V}_3$ and $\mathcal{V}_1 \otimes (\mathcal{V}_2 \otimes \mathcal{V}_3)$ – since composition of three systems can not depend on the order in which it is done. This requirement forces the co-product to be *co-associative*. Another point is connected with the fact that from the physical point of view vector space $\mathbb{C}$ represents a trivial system having only one property – "being itself" – which can not change. Hence one should have a canonical representation of the algebra on a trivial system, denoted by $\epsilon : \mathcal{A} \longrightarrow \mathbb{C}$. Next, since composition of any system with a trivial can not introduce new quality representations on $\mathcal{V}$ ans $\mathcal{V} \otimes \mathbb{C}$ should be equivalent. This requirement imposes the condition on $\epsilon$ to be a *co-unit* in the algebra. In this way we motivate need for a structure of *bi-algebra* in physics. The concept of an antipode enters the game in the context of measurement. Measuring apparatus combined with a system is described in $\mathcal{V}^* \times \mathcal{V}$ and measurement predictions are given through the canonical pairing $c : \mathcal{V}^* \times \mathcal{V} \longrightarrow \mathbb{C}$. Observables, described in the dual space $\mathcal{V}^*$, also can be transformed and representations of appropriate algebras are given with the help of an anti-morphism $\alpha : \mathcal{A} \longrightarrow \mathcal{A}$. Physics requires that transformation preformed on the system and apparatus simultaneously should not change the measurement results, hence the pairing should trivially transform under the action of the bi-algebra. We thus obtain the condition on $\alpha$ to be an *antipode* the last condition for a Hopf Algebra.

The conception of many Hopf algebras is motivated by various theories (physical or close to physics) like renormalization [?, ?, ?, ?, ?], non-commutative geometry [?, ?], physical chemistry [?, ?], computer science [?], algebraic combinatorics [?, ?, ?, ?], algebra [**?**, **?**, **?**].

## References


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