Hopf Algebras in General and in Combinatorial Physics: a practical introduction

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Abstract.

1. Introduction

Quantum Physics
Operator Algebras
Construction of new operator algebras
Need for $\oplus$ and $\otimes$
This very general setting specializes nicely in the combinatorial realm. This is what we call *combinatorial physics*.

2. Operators

2.1. Generalities

Throughout this text we will consider (linear) operators $\omega : V \longrightarrow V$, where V is a vector space over k (k, a field of scalars can be thought to be $\mathbb{R}$ or $\mathbb{C}$). The set of all (linear) operators $V \longrightarrow V$ is an algebra (see appendix) which will be denoted by $End_{\mathbb{K}}(V)$.

2.2. What is a representation

It is not rare in Physics that we consider, instead of a single operator, a set or a family of operators $(\omega_\alpha)_{\alpha\in A}$ and often the index set itself has a structure. As a family $(\omega_\alpha)_{\alpha\in A}$ is no more than a mapping [$\rho : A \mapsto End_{\mathbb{K}}(V)$] (see [4], Ch. II 3.4 remark), we will rather exhibit the mapping by considering it defined by a mapping . This will exactly be the concept of *representation* that we will illustrate by familiar examples.

- First case: A is a group

In this case, we ask that the action of the operators be compatible with the laws of group that is to say that, for all $\alpha, \beta \in A$,

$$\begin{cases} \rho(\alpha)\circ\rho(\beta) = \rho(\alpha.\beta) \\ \rho(\alpha^{-1}) = (\rho(\alpha))^{-1} \end{cases} \tag{1}$$

which is equivalent to

$$\begin{cases} \rho(\alpha)\circ\rho(\beta) = \rho(\alpha.\beta) \\ \rho(1_A) = 1_{End(V)} \end{cases} \tag{2}$$

note that, each of these conditions imply that ρ actually ranges in $Aut_{\mathbb{K}}(V)$, the set of one-to-one elements of $End_{\mathbb{K}}(V)$ [(called automorphisms)].

- Second case: A is a Lie algebra

In this case, one requires that

$$[\rho(\alpha), \rho(\beta)] = \rho(\alpha)\rho(\beta) - \rho(\beta)\rho(\alpha) = \rho([\alpha, \beta]). \tag{3}$$

We will see that these two types of actions (of a group or Lie algebra) can be unified through the concept of representation of an algebra (or which amounts to the same, of a module).

[The following sentence brakes the flow (might be put some earlier)]In the old books one finds the family-like notation, where $\rho(\alpha)$ is denoted, say, ω_α, but using arrows allows, as we will see more clearly below, for extensions and factorizations procedures. In the first case, one invokes the *group algebra* $\mathbb{C}[A]$ [Should be: $\mathbb{C}[G]$] (see appendix). In the case of a Lie algebra, one invokes the *enveloping algebra* $\mathcal{U}(A)$ [Should be: $\mathbb{C}[\mathfrak{G}]$ (also the letter $\mathfrak{G}$ is not readable - maybe could be changed)] (see appendix). [Reader is refered to appendix, but these notions are missing there] In both cases, the original representation ρ is extended to a representation of an *associative algebra with unit* (AAU) as follows:

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[d, "can"'] & End(V) \\ \mathbb{C}[G] \arrow[ur, "\hat\rho"'] & \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathfrak{G} \arrow[r, "\rho"] \arrow[d, "can"'] & End(V) \\ \mathcal{U}(\mathfrak{G}) \arrow[ur, "\hat\rho"'] & \end{tikzcd}$$

We have not, so far, defined what a representation of AAU was. Keeping the philosophy of (1) (or (2)) and (3), we can state the following definition:

Definition 2.1 *Let $(\mathcal{A}, +, \cdot)$ be an AAU. A collection of operators $\{\rho(\alpha)\}_{\alpha\in\mathcal{A}}$ in a vector space V is said to be a representation of $\mathcal{A}$ iff the mapping $\rho : \mathcal{A} \mapsto End(V)$ is compatible with the operations and units of A. This means that, identically (i. e. for all $\alpha, \beta \in \mathcal{A}$ and $\lambda \in \mathbb{K}$).*

$$\begin{cases} \rho(\alpha+\beta) = \rho(\alpha) + \rho(\beta), & \rho(\lambda\alpha) = \lambda\rho(\alpha), \\ \rho(\alpha\cdot\beta) = \rho(\alpha)\circ\rho(\beta), & \\ \rho(1_{\mathcal{A}}) = Id_V, & \end{cases} \tag{4}$$

[where $\circ$ denotes composition of operators.]

Remarks 2.2 *(i) It amounts to the same to say that the arrow $\rho : \mathcal{A} \mapsto End(V)$ from $\mathcal{A}$ to $End(V)$ is a morphism of algebras (with units).*
(ii) In this case, it is sometimes convenient to denote $\alpha.v$ the action of $\rho(\alpha)$ on v (i.e. the element $\rho(\alpha)[v]$) for $\alpha \in \mathcal{A}$ and $v \in V$.
(iii) It may happen (and this occurs many times) that a representation has relations that are not present in the original algebra. In this case the representation is said to be not faithful. [I think that more familiar definition of faithful representation is through $ker(\rho)$ - maybe it should be mentioned]

Example 2.3 *: Let $G = \{1, c, c^2\}$ be the cyclic group of order 3 (c is the cycle $1 \to 2 \to 3 \to 1$), G admits the plane representation by*

$$\rho(c) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \text{ (it is the matrix of 1/3 of the full turn).} \tag{5}$$

Thus,

$$\rho(c^2) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix} \quad \text{and, of course,} \quad \rho(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6}$$

The representation ρ *is faithful while its extension to the group algebra is not* , as seen from*:*

$$\rho(1 + c + c^2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ whereas } 1 + c + c^2 \neq 0 \text{ in } \mathbb{C}[G]. \tag{7}$$

Note that the situation is even worse for a Lie algebra, as $\mathcal{U}(\mathfrak{G})$ is infinite dimensional if only *$\mathfrak{G}$ is not zero.*

3. Operations on representations

Now, we would like to see the representations of AAU as building blocks to construct new ones. The elementary operations on vector spaces are:

- sums
- tensor products
- duals

Hence, important problem is given representations $\rho_i : \mathcal{A} \mapsto V_i$ on the building blocks V_i ; $i = 1,2$ how to naturally construct representations on $V_1 \oplus V_2$, $V_1 \otimes V_2$ and V_i^* $\rho_1 \oplus \rho_2 : \mathcal{A} \mapsto V_1 \otimes \mathbb{V}_2$
Sums will cause no problem as the sum $V_1 \oplus V_2$ of two vector spaces V_1 and V_2 amounts to taking their cartesian product $V_1 \oplus V_2 \cong V_1 \times V_2$. Then, if $\rho_i : \mathcal{A} \mapsto V_i$; $i = 1,2$ are two representations of $\mathcal{A}$ then the mapping $\rho_1 \oplus \rho_2 : \mathcal{A} \mapsto V_1 \otimes V_2$ such that

$$\rho_1 \oplus \rho_2(a)[(v_1, v_2)] = (\rho_1(a)[v_1], \rho_2(a)[v_2]) \tag{8}$$

which can be symbolically written

$$\rho_1 \oplus \rho_2 = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix}, \tag{9}$$

is representation of $\mathcal{A}$ in $\mapsto V_1 \oplus V_2$.
Dualization will be discussed later and solved by the existence of an antipode. Now, we start with the problem of constructing representations on tensor products. This will be solved by means of the notion of "scheme of actions" which is to be formalized, in our case, by the concept of comultiplication (or coproduct).

3.1. Arrows and addition or multiplication formulas

Let us give first some examples where the comultiplication naturally arises.
We begin with functions admitting an "addition formula" (see exercise (11.1) on representative functions) or "multiplication formula". This means functions such that for all x, y

$$f(x * y) = \sum_{i=1}^{n} f_i^{(1)}(x) f_i^{(2)}(y), \tag{10}$$

where $*$ is a certain (associative) operation on the defining set of f and $(f_i^{(1)}, f_i^{(2)})_{i=1}^{n}$ be two (finite) families of functions on the same set.
The first examples are taken in the function space $\mathbb{R}^{\mathbb{R}}$ (with $* = +$, the ordinary addition of real numbers). The following functions admit "addition formulas" which can be diagramized as follows.

Diagram	Addition formula
$\begin{tikzcd} \mathbb{R}\times\mathbb{R} \arrow[r, "+"] \arrow[dr, "\cos_1\cos_2-\sin_1\sin_2"'] & \mathbb{R} \arrow[d, "\cos"] \\ & \mathbb{R} \end{tikzcd}$	$\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$
$\begin{tikzcd} \mathbb{R}\times\mathbb{R} \arrow[r, "+"] \arrow[dr, "\sin_1\cos_2+\sin_2\cos_1"'] & \mathbb{R} \arrow[d, "\sin"] \\ & \mathbb{R} \end{tikzcd}$	$\sin(x+y)=\sin(x)\cos(y)+\sin(y)\cos(x)$
$\begin{tikzcd} \mathbb{R}\times\mathbb{R} \arrow[r, "+"] \arrow[dr, "\exp_1\exp_2"'] & \mathbb{R} \arrow[d, "\exp"] \\ & \mathbb{R} \end{tikzcd}$	$\exp(x+y)=\exp(y)\exp(x)$
$\begin{tikzcd} \mathbb{R}\times\mathbb{R} \arrow[r, "+"] \arrow[dr, "\sum_{j=0}^{n}\binom{n}{j}pr_1^{(j)}\ pr_2^{(n-j)}"'] & \mathbb{R} \arrow[d, "(\ )^n"] \\ & \mathbb{R} \end{tikzcd}$	$(x+y)^n=\sum_{j=0}^{n}\binom{n}{j}x^j\ y^{n-j}$

Another example can be given where the domain set (source) is $\mathbb{C}^{n\times n}$, the algebra of square $n\times n$ matrices with complex coefficients. Let $a_{ij}:\mathbb{C}^{n\times n}\longrightarrow\mathbb{C}$ be the linear form which "takes" the coefficient of address (i,j) (raw i and column j), that is to say $a_{ij}(M):=M[i,j]$. Then, the law of multiplication of matrices says that $MN[i,j]=\sum_{k=1}^{n}M[i,k]N[k,j]$, which can be represented in the style of Eq. (10) by

$$a_{ij}(MN)=\sum_{k=1}^{n}a_{ik}(M)a_{kj}(N). \tag{11}$$

$$\begin{tikzcd} \mathbb{C}^{n\times n}\times\mathbb{C}^{n\times n} \arrow[r, "product"] \arrow[dr, "\sum_{k=1}^{n}(a_{ik})_1(a_{kj})_2"'] & \mathbb{C}^{n\times n} \arrow[d, "a_{ij}"] \\ & \mathbb{C} \end{tikzcd}$$

[I've improved the above drawiang]

Remark 3.1 *Note that formula (11) holds when the definition set (source) is a (multiplicative) semigroup of matrices (for example, the semigroup of unipotent positive matrices).*

We go now to linear mappings that admit such "addition" or, rather, "multiplication" formula.

Derivations: Let $\mathcal{A}$ be an arbitrary algebra with law of multiplication:

$$\mathcal{A}\otimes\mathcal{A}\xrightarrow{\mu}\mathcal{A}. \tag{12}$$

A derivation of $\mathcal{A}$ is an operator $D : \mathcal{A} \longrightarrow \mathcal{A}$ which follows the Leibniz rule, that is for all $x, y \in \mathcal{A}$, one has

$$D(xy) = D(x)y + xD(y) \quad \text{(Leibniz rule)}. \tag{13}$$

In the spirit of what has been represented above one has

$$\begin{tikzcd} \mathcal{A}\otimes\mathcal{A} \arrow[r, "\mu"] \arrow[dr, "D_1Id_2+Id_1D_2"'] & \mathcal{A} \arrow[d, "D"] \\ & \mathcal{A} \end{tikzcd}$$

which (as we have linear spaces and mappings) can be better represented by

$$\begin{tikzcd} \mathcal{A}\otimes\mathcal{A} \arrow[r, "\mu"] \arrow[d, "D\otimes Id+Id\otimes D"'] & \mathcal{A} \arrow[d, "D"] \\ \mathcal{A}\otimes\mathcal{A} \arrow[r, "\mu"] & \mathcal{A} \end{tikzcd}$$

Automorphisms: An automorphism of $\mathcal{A}$ is an invertible linear mapping $g : \mathcal{A} \longrightarrow \mathcal{A}$ such that for all $x, y \in \mathcal{A}$, one has

$$g(xy) = g(x)g(y), \tag{14}$$

which, in the spirit of what precedes can be represented by

$$\begin{tikzcd} \mathcal{A}\otimes\mathcal{A} \arrow[r, "\mu"] \arrow[d, "g\otimes g"'] & \mathcal{A} \arrow[d, "g"] \\ \mathcal{A}\otimes\mathcal{A} \arrow[r, "\mu"] & \mathcal{A} \end{tikzcd}$$

Now, remark that, classically, `group representations act as` automorphisms and `representations of Lie algebras act as` derivations. This provides immediately a scheme for constructing tensor products of two representations.

Tensor product of two representations (groups and Lie algebras): First, take two representations of a group G, $\rho_i : G \longrightarrow End(V_i)$, $i = 1, 2$. The action of $g \in G$ on the tensor space $V_1 \otimes V_2$ is given by

$$g(v_1 \otimes v_2) = g(v_1) \otimes g(v_2). \tag{15}$$

This means that the "tensor product" of the two (group) representations $\rho_i,\ i = 1, 2$ is given by the following data

- Space : $V_1 \otimes V_2$
- Action : $\rho_1 \boxtimes \rho_2 : g \to \rho_1(g) \otimes \rho_2(g)$

Likewise, if we have two representations $\rho_i : \mathfrak{G} \longrightarrow End(V_i)$, $i = 1, 2$ of the Lie algebra $\mathfrak{G}$ the action of $g \in \mathfrak{G}$ on a tensor product $V_1 \otimes V_2$ is given by

$$g(v_1 \otimes v_2) = g(v_1) \otimes v_2 + v_1 \otimes g(v_2). \tag{16}$$

Again, the "tensor product" of the two (Lie algebra) representations $\rho_i,\ i = 1, 2$ is given by the following data

- Space : $V_1 \otimes V_2$
- Action : $\rho_1 \boxtimes \rho_2 : g \to \rho_1(g) \otimes Id_{V_2}(g) + Id_{V_1}(g) \otimes \rho_2(g)$

Roughly speaking, in the first case g acts by $g \otimes g$ and in the second one by $g \otimes 1 + 1 \otimes g$.
In view of two above cases it is convenient to construct linear mappings:

$$\mathcal{A} \xrightarrow{\Delta} \mathcal{A} \otimes \mathcal{A} \ , \tag{17}$$

such that $\rho_1 \boxtimes \rho_2 = (\rho_1 \otimes \rho_2) \circ \Delta$.
In the first case ($\mathcal{A} = \mathbb{C}[G]$) one gets

$$\Delta\left(\sum_{g\in G} \alpha_g g\right) = \sum_{g\in G} \alpha_g g \otimes g. \tag{18}$$

In the second case, one has first to construct the comultiplication on the monomials $g_1 ... g_n; g_i \in \mathfrak{G})$ as they span ($\mathcal{A} = \mathcal{U}(\mathfrak{G})$). Then, using the rule $\Delta(g) = g \otimes 1 + 1 \otimes g$ (for $g \in \mathfrak{G}$) and the fact that Δ is supposed to be a morphism for the multiplication (the justification of this rests on the fact that the constructed action must be a representation see below around formula (35) and exercise (11.2)), one has

$$\begin{aligned}\Delta(g_1 ... g_n) &= (g_1 \otimes 1 + 1 \otimes g_1)(g_2 \otimes 1 + 1 \otimes g_2)...(g_n \otimes 1 + 1 \otimes g_n) \\ &= \sum_{I+J=[1...n]} g[I] \otimes g[J].\end{aligned} \tag{19}$$

Where, for $I = \{i_1, i_2, \cdots, i_k\}$ $(1 \le i_1 < i_2 < \cdots < i_k \le n)$, $g[I]$ stands for $g_{i_1} g_{i_2} \cdots g_{i_k}$
In each case (group algebra and envelopping algebra). One again gets a mapping $\Delta : \mathcal{A} \mapsto \mathcal{A} \otimes \mathcal{A}$ which will be expressed by

$$\Delta(a) = \sum_{i=1}^{n} a_i^{(1)} \otimes a_i^{(2)} \tag{20}$$

which is rephrased compactly by

$$\Delta(a) = \sum_{(1)(2)} a_{(1)} \otimes a_{(2)}. \tag{21}$$

The action of $a \in \mathcal{A}$ on a tensor $v_1 \otimes v_2$ is then, in these two cases, given by

$$a.(v_1 \otimes v_2) = \sum_{i=1}^{n} a_i^{(1)}.v_1 \otimes a_i^{(2)}.v_2 = \sum_{(1)(2)} a_{(1)}.v_1 \otimes a_{(2)}.v_2. \tag{22}$$

[I've lowered dots]
One can easily check that $a.b.(v_1 \otimes v_2) = (ab).(v_1 \otimes v_2)$.
Expression (21) is very convenient for proofs and computations and known as Sweedler's notation.

Remarks 3.2 *i) In every case, we have extracted the "scheme of action" for building the tensor product of two representations. This scheme (a linear mapping $\Delta : \mathcal{A} \mapsto \mathcal{A} \otimes \mathcal{A}$) is* independent *of the considered representations and, in each case,*

$$\rho_1 \boxtimes \rho_2 = (\rho_1 \otimes \rho_2) \circ \Delta \tag{23}$$

ii) [I'd skip point ii)] *For readers that are unfamiliar (and untrusting) Sweedler's notation, one can speak the language of "structure constants". Let $\Delta : \mathcal{C} \mapsto \mathcal{C} \otimes \mathcal{C}$ be a comultiplication and $(b_i)_{i\in I}$ a (linear) basis of $\mathcal{C}$. One has*

$$\Delta(b_i) = \sum_{j,k\in I} \lambda_i^{j,k}\ b_j \otimes b_k\ . \tag{24}$$

the family $(\lambda_i^{j,k})_{i,j,k\in I}$ is called the "structure constants" of the comultiplication Δ. Note the duality with the notion the "structure constants" of a multiplication
$\mu : \mathcal{A} \otimes \mathcal{A} \mapsto \mathcal{A}$: if $(b_i)_{i\in I}$ is a (linear) basis of $\mathcal{A}$, one has

$$\mu(b_i \otimes b_j) = \sum_{k\in I} \lambda_{i,j}^{k}\ b_k\ . \tag{25}$$

For necessary and sufficient conditions for a family to be structure constants see exercise (11.7).

Then, the general construction for tensor products goes as follows.

Definition 3.3 *: Let $\mathcal{A}$ be a vector space, a comultiplication Δ on $\mathcal{A}$ is a linear mapping*

$$\mathcal{A} \xrightarrow{\Delta} \mathcal{A} \otimes \mathcal{A}.$$

Such a pair (vector space, comultiplication) without any prescription about the linear mapping "comultiplication" is called a coalgebra.

Now, imitating (23), if $\mathcal{A}$ is an algebra and ρ_1, ρ_2 are representations of $\mathcal{A}$ in V_1, V_2, for each $a \in \mathcal{A}$, we can construct an action of a on $V_1 \otimes V_2$ by

$$V_1 \otimes V_2 \overset{(\rho_1\otimes\rho_2)\circ\Delta(a)}{\longrightarrow} V_1 \otimes V_2. \tag{26}$$

This means that if $\Delta(a) = \sum_{(1)(2)} a_{(1)} \otimes a_{(2)}$ then group representations act as

$$a.(v_1 \otimes v_2) = \sum_{(1)(2)} a_{(1)}.v_1 \otimes a_{(2)}.v_2 = \sum_{(1)(2)} \rho_1(a_{(1)})[v_1] \otimes \rho_2(a_{(2)})[v_2]. \tag{27}$$

[I've lowered dots]
But, at this stage, it is just an action and not (necessarily) a representation of $\mathcal{A}$. We will see further on the requirements on Δ for the construction of tensor product to be reasonable (i. e. compatible with the usual tensor properties).
For the moment let us take a pause and consider some well known examples of comultiplications.

4. Combinatorics of some comultiplications [I would make it a subsection]

The first type of comultiplication is given by duality. This means by a formula of type

$$\langle \Delta(x) | y \otimes z \rangle^{\otimes 2} = \langle x | y * z \rangle \tag{28}$$

for a certain law of algebra $V\otimes V\overset{*}{\mapsto} V$, where $\langle\ |\ \rangle$ is a non degenerate scalar product in V and $\langle\ |\ \rangle^{\otimes 2}$ stands for its extension to $V\otimes V$. In the case of words $*$ being the concatenation and $\langle\ |\ \rangle$ being given by $\langle u|v\rangle=\delta_{u,v}$ the comultiplication Δ_{Cauchy}, dual to the concatenation, is given on a word w by

$$\Delta(w)=\sum_{uv=w} u\otimes v. \tag{29}$$

In the same spirit, one can define a comultiplication on the algebra of a finite group by

$$\Delta(g)=\sum_{g_1g_2=g} g_1\otimes g_2. \tag{30}$$

The second example is given by multiplication law of elementary comultiplications, that is, if for each letter x one has $\Delta(x)=x\otimes 1+1\otimes x$, then

$$\begin{aligned}\Delta(w)&=\Delta(a_1...a_n)=\Delta(a_1)\Delta(a_2)...\Delta(a_n)\\ &=\sum_{I+J=[1...n]} w[I]\otimes w[J]\end{aligned} \tag{31}$$

where $w[\{i_1,i_2,\cdots i_k\}]=a_{i_1}a_{i_2}\cdots a_{i_k}$ (for $1\le i_1<i_2<\cdots<i_k\le n$). This comultiplication is dual to (33) below for $q=0$ (shuffle product).
Another example is a deformation (perturbation for small q) of the preceding. With $\Delta(a)=a\otimes 1+1\otimes a+qa\otimes a$, one has

$$\begin{aligned}\Delta(w)&=\Delta(a_1...a_n)=\Delta(a_1)\Delta(a_2)...\Delta(a_n)\\ &=\sum_{I\cup J=[1...n]} q^{|I\cap J|}w[I]\otimes w[J].\end{aligned} \tag{32}$$

Note that this comultiplication is dual (in the sense of (28)) of the q-infiltration product given by the recursive formula (for general q and with 1_{A^*} as the empty word)

$$\begin{aligned}&w\uparrow 1_{A^*}=1_{A^*}\uparrow w=w\\ &au\uparrow bv=a(u\uparrow bv)+b(au\uparrow v)+q\delta_{a,b}(u\uparrow v)\end{aligned} \tag{33}$$

this product is an interpolation between shuffle ($q=0$) and (classical) infiltration ($q=1$) [12].

4.1. Requirements for a reasonable construction of tensor products

We have so far constructed an action of $\mathcal{A}$ on tensors, but nothing indicates that this is a representation (see exercise (11.2)). So, the following question is natural.

Q.1.) If $\mathcal{A}$ is an algebra and $\Delta:\mathcal{A}\mapsto\mathcal{A}\otimes\mathcal{A}$, what do we require on Δ if we want the construction above to be a representation of $\mathcal{A}$ on tensor products ?

For $a,b\in\mathcal{A}$, ρ_i representations of $\mathcal{A}$ in V_i, and $v_i\in V_i$ for $i=1,2$, we must have the following identity:

$$a.(b.v_1\otimes v_2)=(ab).v_1\otimes v_2 \implies \Delta(ab).v_1\otimes v_2=\Delta(a).(\Delta(b).v_1\otimes v_2). \tag{34}$$

[I've added before last $\Delta(b)$ and added $\Longrightarrow$ (reads better) One can prove that, if this is true identically for all $a, b \in \mathcal{A}$ and all pairs of representations (see exercise (11.2)), one has

$$\Delta(ab) = \Delta(a)\Delta(b). \tag{35}$$

and, of course, if the latter holds, (34) is true.
This can be rephrased by saying that Δ is morphism $\mathcal{A} \mapsto \mathcal{A} \otimes \mathcal{A}$.

Now, one would like to keep the compatibility with the associativity of tensor products, this means that if we want to tensor $u \otimes v$ with w it must give the same actions as tensoring u with $v \otimes w$. This means that we have to address the following question

Q.2.) If $\mathcal{A}$ is an algebra and $\Delta : \mathcal{A} \mapsto \mathcal{A} \otimes \mathcal{A}$ a morphism of algebras, what do we require on Δ if we want the construction above to be associative ?

More precisely, for three representations $\rho_i,\ i = 1, 2, 3$ of $\mathcal{A}$, we want

$$\rho_1 \boxtimes (\rho_2 \boxtimes \rho_3) = (\rho_1 \boxtimes \rho_2) \boxtimes \rho_3 \tag{36}$$

up to the identifications $(u \otimes v) \otimes w = u \otimes (v \otimes w) = u \otimes v \otimes w$ (if one is not satisfied with this identification, see exercise (11.3)).
Let us compute (up to the identification above)

$$a.[(u \otimes v) \otimes w] = \Delta(a).(u \otimes v) \otimes w = \Big((\Delta \otimes Id) \circ \Delta(a)\Big)\Big(u \otimes v \otimes w\Big) \tag{37}$$

on the other hand

$$a.[u \otimes (v \otimes w)] = \Delta(a).u \otimes (v \otimes w) = \Big((Id \otimes \Delta) \circ \Delta(a)\Big)\Big(u \otimes v \otimes w\Big). \tag{38}$$

Again, one can prove (see exercise (11.3)) that this holds identically (i. e. for every $a \in \mathcal{A}$ and triple of representations) iff $(Id \otimes \Delta) \circ \Delta = (\Delta \otimes Id) \circ \Delta$, *i.e.*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathcal{A} \otimes \mathcal{A} \arrow[d, "Id\otimes\Delta"] \\ \mathcal{A} \otimes \mathcal{A} \arrow[r, "\Delta\otimes Id"] & \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \end{tikzcd} \tag{39}$$

Remark 4.1 *Property (39) is called co-associativity as, if one reverses the arrows and replaces Δ by μ, the multiplication in an algebra, the diagram expresses associativity (see also exercise (11.4) on duals of co-algebras).*

$$\begin{tikzcd} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \arrow[r, "\mu\otimes Id"] \arrow[d, "Id\otimes\mu"'] & \mathcal{A} \otimes \mathcal{A} \arrow[d, "\mu"] \\ \mathcal{A} \otimes \mathcal{A} \arrow[r, "\mu"] & \mathcal{A} \end{tikzcd}$$

But the tensor product is not only associative, it has a "neutral" which is "tensoring by the field of scalars". This derives from the fact that the canonical mappings

$$V \otimes_{\mathbb{C}} \mathbb{C} \xrightarrow{can_r} V \xleftarrow{can_l} \mathbb{C} \otimes_{\mathbb{C}} V. \tag{40}$$

`[ we had field` k `everywer efore and now we insist on` $\mathbb{C}$ `- is it necessary? ]` are isomorphisms and we must ask that they be as well compatible with the procedure defined in (27).
This can be summarized by the following question.

Q.3.) If $\mathcal{A}$ is an algebra and $\Delta : \mathcal{A} \mapsto \mathcal{A} \otimes \mathcal{A}$ a co-associative morphism of algebras, what do we require on Δ if we want the construction above to admit "tensoring by the field of scalars" as neutral ?

More precisely, we must have a representation of $\mathcal{A}$ in $\mathbb{C}$ (which means a morphism of algebras $\mathcal{A} \xrightarrow{\epsilon} \mathbb{C}$) such that for a representations ρ of $\mathcal{A}$, we want

$$\rho \boxtimes \epsilon = \epsilon \boxtimes \rho = \rho \tag{41}$$

up to the identification $u \otimes 1 = 1 \otimes u = u$ through the isomorphisms (40) (if one is not satisfied with this identification, see exercise (11.5)).
`Hence, for all` $a \in \mathcal{A}$ `and` ρ `representation on` V `we should have:`

$$can_r(a.(v \otimes 1)) = a.v, \tag{42}$$

`and`

$$\begin{aligned} a.(v \otimes 1) &= \Big(\sum_{(1)(2)} \rho(a_{(1)}) \otimes \epsilon(a_{(2)}) \Big)[v \otimes 1] \\ &= (\rho \otimes Id_{\mathbb{C}}) \circ \Big(\sum_{(1)(2)} a_{(1)} \otimes \epsilon(a_{(2)}) \Big)[v \otimes 1] \\ &= (\rho \otimes Id_{\mathbb{C}}) \circ (Id \otimes \epsilon) \circ \Delta(a)[v \otimes 1], \end{aligned} \tag{43}$$

$$a.v \;\; = \rho(a)[v] = can_r(\rho(a)[v] \otimes 1) = can_r(\rho \otimes Id(a)[v \otimes 1]). \tag{44}$$

`Similar computations should be made on the left. [ For me I had to make this calculation longer - maybe also helpful for reader (decide) ]`
This means that one should require that

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\Delta"] \arrow[d, "Id"'] & \mathcal{A} \otimes \mathcal{A} \arrow[d, "Id\otimes\epsilon"] \\ \mathcal{A} & \mathcal{A} \otimes \mathbb{C} \arrow[l, "can_r"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \arrow[r, "\Delta"] \arrow[d, "Id"'] & \mathcal{A} \otimes \mathcal{A} \arrow[d, "\epsilon\otimes Id"] \\ \mathcal{A} & \mathbb{C} \otimes \mathcal{A} \arrow[l, "can_r"'] \end{tikzcd}$$

Such a mapping $\epsilon : \mathcal{A} \longrightarrow \mathbb{C}$ is called a co-unit

Remark 4.2 *Again, one can prove (see excercice (11.4) for details) that*

$$\epsilon \textit{ is a counit for } (\mathcal{A}, \Delta) \iff \epsilon \textit{ is a counit for } (\mathcal{A}, *_\Delta) \tag{45}$$

5. Bialgebras

Motivated by the preceding discussion, we will call bialgebra an algebra (associative with unit) endowed with a comultiplication (co-associative with counit) which allows for the two tensor properties of associativity and unit (see discussion above). More precisely

Definition 5.1 *: $(\mathcal{A}, \cdot, 1_{\mathcal{A}}, \Delta, \epsilon)$ is said to be a bialgebra iff*

(1) $(\mathcal{A}, \cdot, 1_{\mathcal{A}})$ is an AAU,

(2) $(\mathcal{A}, \Delta, \epsilon)$ is a coalgebra coassociative with counit,

(3) Δ is a morphism of AAU and ϵ is a morphism of AAU.

The name bialgebra comes from the fact that the space $\mathcal{A}$ is endowed with two structures (one of AAU and one of co-AAU) with a certain compatibility between the two.

5.1. Examples of bialgebras

Free algebra (word version : noncommutative polynomials)

Let A be an alphabet (a set of variables) and A^* be the free monoid constructed on A (see Basic Structures (12.2)). For any field of scalars k (one can think first of $k = \mathbb{R}$ or $\mathbb{C}$), we call algebra of noncommutaive polynomials $k\langle A\rangle$, the algebra $k[A^*]$ of the free monoid A^* constructed on A. This is the set of functions $f : A^* \mapsto k$ with finite support endowed with the convolution product

$$f * g(w) = \sum_{uv=w} f(u)g(v) \tag{46}$$

Each word $w \in A^*$ is identified with its characteristic function (i.e. the Dirac function with value 1 at w and 0 elsewhere). These functions form a basis of $k\langle A\rangle$ and then, every $f \in k\langle A\rangle$ can be written uniquely as a finite sum $f = \sum f(w)w$.
The inclusion mapping $A \hookrightarrow k\langle A\rangle$ will be denoted here by can_A. [Most of the above is in appendix.]
Comultiplications The free algebra $k\langle A\rangle$ admits many comultiplications (even with the two requirements to be a morphism and coassociative). As A^* is a basis of $k\langle A\rangle$, it is sufficient to define it on the words (if we require Δ to be a morphism it is enough to define it on letters).
Example 1. — The first example is the dual of the Cauchy (or convolution) product

$$\Delta(w) = \sum_{uv=w} u \otimes v \tag{47}$$

is not a morphism as

$$\Delta(ab) = ab \otimes 1 + a \otimes b + 1 \otimes ab$$

and

$$\Delta(a)\Delta(b) = ab \otimes 1 + a \otimes b + b \otimes a + 1 \otimes ab$$

but, it can be checked that it is coassociative (see also exercice (11.4) for a lightspeed proof of this fact).

Example 2. — Second example is given, on the alphabet $A = \{a, b\}$ by

$$\Delta(a) = a \otimes b;\ \ \Delta(b) = b \otimes a$$

then $\Delta(w) = w \otimes \bar{w}$ where $\bar{w}$ stands for the word w with a (resp. b) changed in b (resp. a). This comultiplication is a morphism but not coassociative as

$$(I \otimes \Delta) \circ \Delta(a) = a \otimes b \otimes a\ ;\ \ (\Delta \otimes I) \circ \Delta(a) = a \otimes a \otimes b$$

Example 3. — The third example is given on the letters by

$$\Delta(a) = a \otimes 1 + 1 \otimes a + qa \otimes a$$

where $q \in k$. One can prove that

$$\Delta(w) = \Delta(a_1 ... a_n) = \Delta(a_1)\Delta(a_2)...\Delta(a_n) = \sum_{I \cup J = [1..|w|]} q^{|I \cap J|} w[I] \otimes w[J] \tag{48}$$

this comultiplication is coassociative.

For $q = 0$, one gets a comultiplication given on the letters by $\Delta_s(a) = a \otimes 1 + 1 \otimes a$. For every polynomial $P \in k\langle A \rangle$, set $\epsilon(P) = P(1_{A^*})$ (the constant term). Then $(k\langle A \rangle, *, \Delta_s, \epsilon)$ is a bialgebra.

One has also another bialgebra structure with, for all $a \in A$

$$\Delta_h(a) = a \otimes a\ ;\ \epsilon_{aug}(a) = 1 \tag{49}$$

this bialgebra $(k\langle A \rangle, *, \Delta_h, \epsilon_{aug})$ is a substructure of the bialgebra of the free group.

Algebra of polynomials (commutative polynomials)

We continue with the same alphabet A, but this time take as algebra $k[A]$. The construction is similar but the monomials, instead of words, are all the commutative products of letters i.e. $a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n}$ with n arbitrary and $\alpha_i \in \mathbb{N}$. Denoting $A^{\oplus}$ the monoid of these monomials (comprising, as neutral, the empty one) and with $\Delta_s(a) = a \otimes 1 + 1 \otimes a$, $\epsilon(P) = P(1_{A^{\oplus}})$, one can again check that $(k[A], *, \Delta_s, \epsilon)$ is a bialgebra.

Algebra of partially commutative polynomials

[For me it is introduced too quick and sketchily.]

For the detailed construction of partially commutative monoid, the reader is refered to [9, 13]. These monoids generalize both the free and free commutative monoids. To a given graph (non-oriented and without loop) $\vartheta \subset A \times A$, one can asssociate the monoid presented by generators and relations

$$M(A, \vartheta) = \langle A; (xy = yx)_{(x,y) \in \vartheta} \rangle_{\mathbf{Mon}} \ . \tag{50}$$

It is exactly the monoid obtained as a quotient structure of the free monoid (A^*) by the smallest equivalence compatible with products (congruence) which contains the pairs $(xy, yx)_{(x,y) \in \vartheta}$. A geometric model of this monoid using pieces was developped by X. Viennot [29] where pieces are located on "positions" (drawn on a plane) two pieces "commute" iff they are on positions which do not intersect (see fig 1 below).

`[All this part is a bit to formal and dry (at least for me). Some simple illustration would be in order. I recall that once you gave us a very nice and simple example which you called boxed numbers, i.e. free commutative algebra of natural numbers (generated by primes). It is a Hopf algebra I guess. For me it was a very nice and simple example - hence maybe it is worth adding.]`

The partially commutative algebra $k\langle A, \vartheta\rangle$ is the algebra $k[M(A, \vartheta)]$. Again, one can check that $(k\langle A, \vartheta\rangle, *, \Delta_s, \epsilon)$ (constructed as above) is a bialgebra.

Algebra of a group

Let G be a group. The algebra under consideration is $k[G]$. We define, for $g \in G$, $\Delta(g) = g \otimes g$ and $\epsilon(g) = 1$, then, one can check that $(k[G], ., \Delta, \epsilon)$ is a bialgebra.

6. The problem of duals

Each vector space V comes with its dual

$$V^* = Hom_{\mathbb{C}}(V, \mathbb{C}). \tag{51}$$

They are in duality by

$$\langle p, \psi\rangle = p(\psi) \tag{52}$$

[As I recall this duality is only for continuous homomorphism. Is it relevant?] Now, if one has representation (on the left) of $\mathcal{A}$ on V, one gets a representation on the right on V^* by

$$\langle p.a, \psi\rangle = \langle p, a.\psi\rangle \tag{53}$$

If we want to have the action of $\mathcal{A}$ on the right again, one should use an anti-morphism $\alpha : \mathcal{A} \longrightarrow \mathcal{A}$ that is $\alpha \in End_k(\mathcal{A})$ such that, for all $x, y \in \mathcal{A}$

$$\alpha(xy) = \alpha(y)\alpha(x). \tag{54}$$

In the case of groups $g \longrightarrow g^{-1}$ does the job, in the case of Lie algebra $g \longrightarrow -g$ (extended by reverse products to the enveloping algebra) works.
The concept that unifies the two preceding examples is that of the antipode. It is a linear mapping $\alpha : \mathcal{A} \longrightarrow \mathcal{A}$ such that for all $a \in \mathcal{A}$

$$\sum_{(1)(2)} \alpha(a_{(1)})a_{(2)} = \sum_{(1)(2)} a_{(1)}\alpha(a_{(2)}) = 1_{\mathcal{A}}\epsilon(a). \tag{55}$$

One can prove (see exercise (11.8)), that this means that α is the inverse of $Id_{\mathcal{A}}$ for a certain product of algebra (AAU) on $End_k(\mathcal{A})$ and this implies (see exercise (11.8)) that

(i) If α exists (as a solution of Eq.(55)), it is unique.

(ii) If α exists, it is an antimorphism.

Definition 6.1 *: (Hopf Algebra) $(\mathcal{A}, \cdot, 1_{\mathcal{A}}, \Delta, \epsilon, S)$ is said to be a Hopf algebra iff*

(1) $\mathcal{B} = (\mathcal{A}, \cdot, 1_{\mathcal{A}}, \Delta, \epsilon)$ is a bialgebra,

(2) S is an antipode (then unique) for $\mathcal{B}$

In many combinatorial cases (see exercise on local finiteness (11.6)), one can compute the antipode by

$$\alpha(d) = \sum_{k=0}^{\infty} (-1)^{k+1} (I^+)^{(*k)}(d)). \tag{56}$$

where I^+ is the projection on $\mathcal{B}^+ = ker(\epsilon)$ such that $I^+(1_{\mathcal{B}}) = 0$.

`[This is a very concise and sketch if compared with the previous part on bi-algebra. I understood why we need an antimorphism - but there might be many. I do not understand what is so special about this additional requirement Eq.(55). I feel that it is because of this convolution introduced in the exercises but I have no feeling what that could be useful for and what could be physical need for that very specific property. Could you please explain the motivation in a simple way?]`

7. Hopf algebra and partition functions

7.1. Partition Function Integrand

Consider the Partition Function Z of a Quantum Statistical Mechanical System

$$Z = Trexp(-\beta H) \tag{57}$$

whose hamiltonian is H ($\beta \equiv 1/kT$, k = Boltzmann's constant T = absolute temperature). We may evaluate the trace over any complete set of states; we choose the (over-)complete set of coherent states

$$|z\rangle = e^{-|z|^2/2} \sum_n (z^n/\sqrt{n!}) a^{+n} |0\rangle \tag{58}$$

where a^+ is the boson creation operator satisfying $[a, a^+] = 1$ and for which the completeness or resolution of unity† property is

$$\frac{1}{\pi} \int dz |z\rangle\langle z| = I \equiv \int d(z) |z\rangle\langle z|. \tag{59}$$

† Sometimes physicists write d^2z to emphasize that the integral is two dimensional (over $\mathbb{R}$) but here, the l. h. s. of (59) is the integration of the operator valued function $z \to |z\rangle\langle z|$ - see [7] Chap. III Paragraph 3 - w.r.t. the Haar mesure of $\mathbb{C}$ which is dz.

The simplest, and generic, example is the free singleboson hamiltonian $H = \epsilon a^+ a$ for which the appropriate trace calculation is

$$Z = \frac{1}{\pi}\int dz \rangle z|exp(-\beta a^+ a)|z\langle = \\ = \frac{1}{\pi}\int dz\langle z| : exp(a^+ a(e^{-\beta\epsilon} - 1) : |z\rangle \ . \tag{60}$$

Here we have used the following wellknown relation for the forgetful normal ordering operator $: f(a, a^+) :$ which means "normally order the creation and annihilation operators in f forgetting the com mutation relation $[a, a^+] = 1$" ‡. We may write the Partition Function in general as

$$Z(x) = \int F(x, z) d(z) \tag{61}$$

thereby defining the Partition Function Integrand (PFI) $F(x, z)$. We have explicitly written the dependence on $x = -\beta$, the inverse tem perature, and ϵ, the energy scale in the hamiltonian.

7.2. Combinatorial aspects: Bell numbers

The generic freeboson example Eq. (61) above may be rewritten to show the connection with certain wellknown combinatorial numbers. Writing $y = |z|^2$ and $x = -\beta\epsilon$, Eq.(**??**) becomes

$$Z = \int_0^\infty dy \ exp(y(e^x - 1)) \ . \tag{62}$$

This is an integral over the classical exponential generating function for the Bell polynomials

$$exp(y(e^x - 1)) = \sum_{n=0}^{\infty} B_n(y)\frac{x^n}{n!} \tag{63}$$

where the Bell number is $B_n(1) = B(n)$, the number of ways of putting n different objects into n identical containers (some may be left empty). Related to the Bell numbers are the Stirling numbers of the second kind $S(n, k)$, which are defined as the number of ways of putting n different objects into k identical containers, leaving none empty. From the definition we have $B(n) = \sum_{k=1}^{n} S(n, k)$ (for $n \neq 0$). The foregoing gives a combinatorial interpretation of the partition function integrand $F(x, y)$ as the exponential generating function of the Bell polynomials.

7.3. Graphs

We now give a graphical representation of the Bell numbers. Consider labelled lines which emanate from a white dot, the origin, and finish on a black dot, the vertex. We shall allow only one line from each white dot but impose no limit on the number of

‡ Of course, this procedure may alter the value of the operator to which it is applied.

lines ending on a black dot. Clearly this simulates the definition of $S(n,k)$ and $B(n)$, with the white dots playing the role of the distinguishable objects, whence the lines are labelled, and the black dots that of the indistinguishable containers. The identification of the graphs for 1,2 and 3 lines is given in Figure 1. We have concentrated on the Bell number sequence and its associated graphs since, as we shall show, there is a sense in which this sequence of graphs is generic. That is, we can represent any combinatorial sequence by the same sequence of graphs as in the Fig. 1

with suitable vertex multipliers (denoted by the V terms in the same figure). Consider a general partition function

$$Z = Tr exp(-\beta H) \tag{64}$$

where the Hamiltonian is given by $H = \epsilon w(a, a^{+})$, with w a string (= sum of products of positive powers) of boson creation and annihilation operators. The partition function integrand F for which we seek to give a graphical expansion, is

$$Z(x) = \int F(x,z)d(z) \tag{65}$$

where

$$\begin{aligned} F(x,z) &= \langle z|exp(xw)|z\rangle = \qquad (x = -\beta\epsilon) \\ &= \sum_{n=0}^{\infty} \langle z|w^n|z\rangle \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} W_n(z) \frac{x^n}{n!} \\ &= exp\Big(\sum_{n=1}^{\infty} V_n(z) \frac{x^n}{n!}\Big) \end{aligned} \tag{66}$$

with obvious definitions of W_n and V_n. The sequences $\{W_n\}$ and $\{V_n\}$ may each be recursively obtained from the other. This relates the sequence of multipliers $\{V_n\}$ of Figure 1 to the Hamiltonian of Eq. (64). The lower limit 1 in the V_n summation is a consequence of the normalization of the coherent state $|z\rangle$.

7.4. The Hopf Algebra $\mathcal{L}_{\mathrm{Bell}}$

We briefly describe the Hopf algebra $\mathcal{L}_{\mathrm{Bell}}$ which the diagrams of Figure 1 define.
1. Each distinct diagram is an individual basis element of $\mathcal{L}_{\mathrm{Bell}}$; thus the dimension is infinite. (Visualise each diagram in a "box".) The sum of two diagrams is simply the two boxes containing the diagrams. Scalar multiples are formal; for example, they may be provided by the V coefficients. Precisely, as a vector space, $\mathcal{L}_{\mathrm{Bell}}$ is the space freely generated by the digrams of Fig 1 (see APPENDIX : Function Spaces).
2. The identity element e is the empty diagram (an empty box).
3. Multiplication is the juxtaposition of two diagrams within the same "box". $\mathcal{L}_{\mathrm{Bell}}$ is generated by the connected diagrams; this is a consequence of the Connected Graph

Theorem [28]. Since we have not here specified an order for the juxtaposition, multipli cation is commutative.
4. The comultiplication $\Delta : \mathcal{L}_{\text{Bell}} \mapsto \mathcal{L}_{\text{Bell}} \otimes \mathcal{L}_{\text{Bell}}$ is defined by

$$\begin{aligned}
&\Delta(e) = e \otimes e \text{ (unit } e\text{, the empty box)}\\
&\Delta(x) = x \otimes e + e \otimes x \text{ (generator } x\text{)}\\
&\Delta(AB) = \Delta(A)\Delta(B) \text{ otherwise}\\
&\text{so that } \Delta \text{ is an algebra homomorphism.}
\end{aligned} \tag{67}$$

8. The case of two modes

Let us consider an hamiltonian on two modes $H(a, a^+, b, b^+)$, with

$$\begin{aligned}
&[a, a^+] = 1\\
&[b, b^+] = 1\\
&[a^{\epsilon_1}, b^{\epsilon_2}] = 0,\ \epsilon_i \text{ being } + \text{ or empty (4 relations)}
\end{aligned} \tag{68}$$

Since all the $a's$ commute with all the $b's$, one can express H as

$$H(a, a^+, b, b^+) = H_1(a, a^+) + H_2(b, b^+). \tag{69}$$

[Be cautious: it factorizes is only in specific cases, e.g. $H = ab$ do not factorize into a sum.]
Suppose that, $exp(\lambda H_1)$ and $exp(\lambda H_2)$ ($\lambda = -\beta$) are solved i.e. that we have expressions

$$F(\lambda) = exp(\lambda H_1) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} H_1^{(n)}(a, a^+)\ ;\ G(\lambda) = exp(\lambda H_2) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} H_2^{(n)}(b, b^+) \tag{70}$$

It is not difficult to check that

$$exp(\lambda H) = exp(\lambda H_1 + H_2) = F(\lambda \frac{d}{dx}) G(x)\Big|_{x=0} \tag{71}$$

This leads us to define, in general, the "Hadamard exponential product". Let

$$F(z) = \sum_{n\geq 0} a_n \frac{z^n}{n!},\ \ G(z) = \sum_{n\geq 0} b_n \frac{z^n}{n!} \tag{72}$$

and define their product (the "Hadamard exponential product") by

$$\mathcal{H}(F, G) := \sum_{n\geq 0} a_n b_n \frac{z^n}{n!} = \mathcal{H}(F, G) = F\left(z \frac{d}{dx}\right) G(x)\Big|_{x=0}. \tag{73}$$

When $F(0)$ and $G(0)$ are not zero one can normalize the functions in this bilinear product so that $F(0) = G(0) = 1$. We would like to obtain compact and generic formulas. If we write the functions as

$$F(z) = \exp\left(\sum_{n=1}^{\infty} L_n \frac{z^n}{n!}\right), \qquad G(z) = \exp\left(\sum_{n=1}^{\infty} V_n \frac{z^n}{n!}\right). \tag{74}$$

that is, as free exponentials, then by using Bell polynomials in the sets of variables $\mathbb{L}, \mathbb{V}$ (see [11, 16] for details), we obtain

$$\mathcal{H}(F,G) = \sum_{n\geq 0} \frac{z^n}{n!} \sum_{P_1,P_2\in UP_n} \mathbb{L}^{Type(P_1)}\mathbb{V}^{Type(P_2)} \tag{75}$$

where UP_n is the set of unordered partitions of $[1\cdots n]$. An unordered partition P of a set X is a collection of (nonempty) subsets of X, mutually disjoint and covering X (i.e. the union of all the subsets is X, see [18] for details).
The type of $P \in UP_n$ (denoted above by $Type(P)$) is the multi-index $(\alpha_i)_{i\in\mathbb{N}^+}$ such that α_k is the number of k-blocks, that is the number of members of P with cardinality k.

At this point the formula entangles and the diagrams of the theory arise.
Note particularly that

- the monomial $\mathbb{L}^{Type(P_1)}\mathbb{V}^{Type(P_2)}$ needs much less information than that which is contained in the individual partitions P_1, P_2 (for example, one can relabel the elements without changing the monomial),
- two partitions have an incidence matrix *from which it is still possible to recover the types of the partitions.*

The construction now proceeds as follows.

(i) Take two unordered partitions of $[1\cdots n]$, say P_1, P_2

(ii) Write down their incidence matrix $(\mathrm{card}(Y\cap Z))_{(Y,Z)\in P_1\times P_2}$

(iii) Construct the diagram representing the multiplicities of the incidence matrix : for each block of P_1 draw a black spot (resp. for each block of P_2 draw a white spot)

(iv) Draw lines between the black spot $Y \in P_1$ and the white spot $Z \in P_2$; there are $\mathrm{card}(Y\cap Z)$ such.

(v) Remove the information of the blocks $Y, Z, \cdots$.

In so doing, one obtains a bipartite graph with p $(= \mathrm{card}(P_1))$ black spots, q $(= \mathrm{card}(P_2))$ white spots, no isolated vertex and integer multiplicities. We denote the set of such diagrams by **diag**.

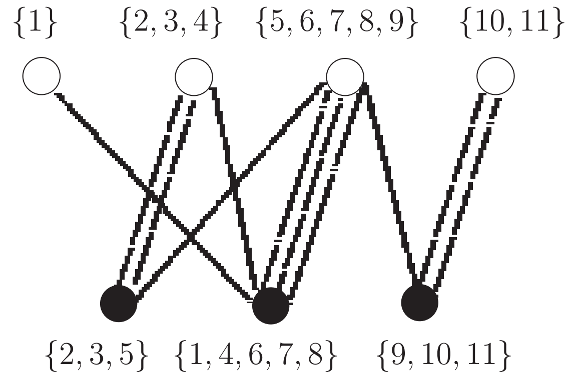


Fig 1. — *Diagram from $P_1,\ P_2$ (set partitions of $[1\cdots 11]$).*
$P_1 = \{\{2,3,5\},\{1,4,6,7,8\},\{9,10,11\}\}$ *and* $P_2 = \{\{1\},\{2,3,4\},\{5,6,7,8,9\},\{10,11\}\}$ *(respectively black spots for P_1 and white spots for P_2).*
The incidence matrix corresponding to the diagram (as drawn) or these partitions is $\begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$. *But, due to the fact that the defining partitions are unordered, one can permute the spots (black and white, between themselves) and, so, the lines and columns of this matrix can be permuted. Thus, the diagram could be represented by the matrix* $\begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 3 & 1 \end{pmatrix}$ *as well.*

The product formula now reads

$$\mathcal{H}(F,G) = \sum_{n\geq 0} \frac{z^n}{n!} \sum_{\substack{d\in diag \\ |d|=n}} mult(d)\mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)} \tag{76}$$

where $\alpha(d)$ (resp. $\beta(d)$) is the "white spots type" (resp. the "black spots type") i.e. the multi-index $(\alpha_i)_{i\in\mathbb{N}^+}$ (resp. $(\beta_i)_{i\in\mathbb{N}^+}$) such that α_i (resp. β_i) is the number of white spots (resp. black spots) of degree i (i lines connected to the spot) and $mult(d)$ is the number of pairs of unordered partitions of $[1\cdots|d|]$ (here $|d| = |\alpha(d)| = |\beta(d)|$ is the number of lines of d) with associated diagram d.

8.1. Diagrams

One can design a (graphically) natural multiplicative structure on **diag** such that the arrow

$$d \mapsto \mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)}. \tag{77}$$

be a morphism. This is provided by the concatenation of the diagrams (the result, i.e. the diagram obtained in placing d_2 at the right of d_1, will be denoted by $[d_1|d_2]_D$). One must check that this product is compatible with the equivalence of the permutation of white and black spots among themselves, which is rather straightforward (see [11, 18]). We have

Proposition 8.1 *[18] Let* **diag** *be the set of diagrams (including the empty one).*
i) The law $(d_1, d_2) \mapsto [d_1|d_2]_D$ *endows* **diag** *with the structure of a commutative monoid with the empty diagram as neutral element(this diagram will, therefore, be denoted by* $1_{\mathbf{diag}}$*).*
ii) The arrow $d \mapsto \mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)}$ *is a morphism of monoids, the codomain of this arrow being the monoid of (commutative) monomials in the alphabet* $\mathbb{L} \cup \mathbb{V}$ *i.e.*

$$\mathfrak{MON}(\mathbb{L}\cup\mathbb{V}) = \{\mathbb{L}^{\alpha}\mathbb{V}^{\beta}\}_{\alpha,\beta\in(\mathbb{N}^+)^{(\mathbb{N})}} = \bigcup_{n,m\geq 1} \{L_1^{\alpha_1}L_2^{\alpha_2}\cdots L_n^{\alpha_n}V_1^{\beta_1}V_2^{\beta_2}\cdots V_m^{\beta_m}\}_{\alpha_i,\beta_j\in\mathbb{N}}. \quad (78)$$

iii) The monoid $(\mathbf{diag}, [-|-]_D, 1_{\mathbf{diag}})$ *is a free commutative monoid. The set on which it is built is the set of the connected (non-empty) diagrams.*

Remark 8.2 *The reader who is not familiar with the algebraic structure of* $\mathfrak{MON}(X)$ *can find rigorous definitions in paragraph (12.2).*

We denote $\phi_{mon,diag}$ the arrow **diag** $\mapsto \mathfrak{MON}(\mathbb{L} \cup \mathbb{V})$.

8.2. Labelled diagrams

We have seen the diagrams (of **diag**) are in one-to-one correspondence with classes of matrices as in Fig. 1. In order to fix one representant of this class, we have to number the black (resp. white) spots from 1 to, say p (resp. q). Doing so, one obtains a packed matrix [15] that is a matrix of integers with no raw nor column full of zeroes. In this way, we define the *labelled diagrams.*

Definition 8.3 *A labelled diagram of size* $p \times q$ *is a bi-coloured (vertices are* p *black and* q *white spots) graph*

- *with no isolated vertex*
- *every black spot is joined to a white spot by an arbitrary quantity (but a positive integer) of lines*
- *the black (resp. white) spots are numbered from 1 to* p *(resp. from 1 to* q*).*

As in paragraph (8.1), one can concatenate the labelled diagrams, the result, i.e. the diagram obtained in placing D_2 at the right of D_1, will be denoted by $[D_1|D_2]_L$). This time, one has not to check any compatibility with classes. We have a structure of free monoid (but not commutative this time)

Proposition 8.4 *[18] Let* **ldiag** *be the set of labeled diagrams (including the empty one).*
i) The law $(d_1, d_2) \mapsto [d_1|d_2]_L$ *endows* **ldiag** *with the structure of a noncommutative monoid with the empty diagram* $(p = q = 0)$ *as neutral element (which will, therefore,*

be denoted by $1_{\mathbf{ldiag}}$*).*
ii) The arrow from **ldiag** *to* **diag**, *which implies "forgetting the labels of the vertices" is a morphism of monoids.*
iii) The monoid $(\mathbf{ldiag}, [-|-]_L, 1_{\mathbf{ldiag}})$ *is a free (noncommutative) monoid which is constructed on the set of irreducible diagrams which are diagrams* $d \neq 1_{\mathbf{ldiag}}$ *which cannot be written* $d = [d_1|d_2]_L$; $d_i \neq 1_{\mathbf{ldiag}}$.

Remark 8.5 *i) In a general monoid* $(M, \star, 1_M)$*, the irreducible elements are the elements* $x \neq 1_M$ *such that* $x = y \star z \Longrightarrow 1_M \in \{y, z\}$.
ii) It can happen that an irreducible of **ldiag** *has an image in* **diag** *which splits, as shown by the simple example of the* cross *defined by the incidence matrix* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

8.3. Hopf algebras **DIAG** *and* **LDIAG**

Let us first construct the Hopf algebra on the labelled diagrams (details can be found in [18]). In order to define the comultiplication, we need the notion of "restriction of a labelled diagram". Let indeed $d \in \mathbf{ldiag}$ of size $p \times q$. For any subset $I \subset [1..p]$, we define a labelled diagram $d[I]$ (of size $k \times l$, $k = \mathrm{card} I$) by taking the $k = |I|$ blackspots numbered in I and the edges (resp. white spots) that are connected to them. We take this subgraph and relabel increasingly the black (resp. white) spots.
The construction of the Hopf algebra **LDIAG** goes as follows :

(i) the algebra structure is that of algebra of the monoid **ldiag** so that the elements of **LDIAG** read
$$x = \sum_{d \in \mathbf{ldiag}} \alpha_d d \tag{79}$$
(the sum is finitely supported)

(ii) the comultiplication is given, on a labelled diagram $d \in \mathbf{ldiag}$ of size $p \times q$, by
$$\Delta_L(d) = \sum_{I+J=[1..p]} d[I] \times d[J] \tag{80}$$

(iii) the counity is "taking the coefficient of the void diagram", that is, for x as in Eq. (79),
$$\epsilon_L(x) = \alpha_{1_{\mathbf{ldiag}}}. \tag{81}$$

One can check that $(\mathbf{LDIAG}, [-|-]_L, 1_{\mathbf{ldiag}}, \Delta_L, \epsilon_L)$ is a bialgebra (for proofs see [18]). Now one can check that we are in conditions of exercise (11.6) question 5 and the antipode S_L can be computed by the procedure of question 4 of the same exercise (i.e. by formula (56)).
We have so far constructed the Hopf algebra $(\mathbf{LDIAG}, [-|-]_L, 1_{\mathbf{ldiag}}, \Delta, \epsilon, S_L)$.
The constructions above are compatible with the arrow

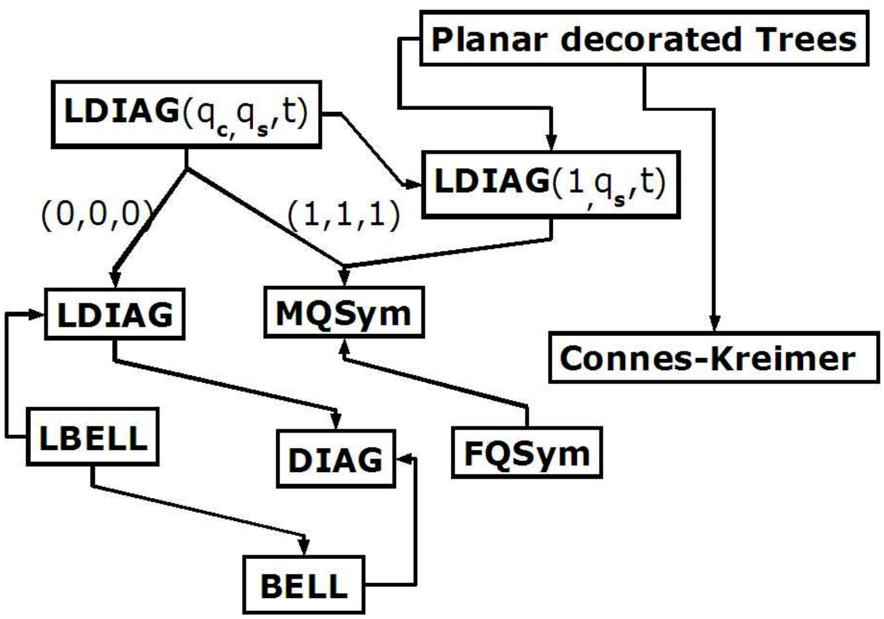
$$\phi_{\mathbf{DIAG},\mathbf{LDIAG}} \;:\; \mathbf{LDIAG} \mapsto \mathbf{DIAG}$$

deduced from the class-map $\phi_{\mathbf{diag},\mathbf{ldiag}}$: **ldiag** $\mapsto$ **diag** (a diagram is a class of labelled diagrams under permutations of black and white spots between themselves). So that, one can deduce "by taking quotients" a structure of Hopf algebra on the algebra of **diag**. This algebra being denoted **DIAG**, one has a natural structure of Hopf algebra $(\mathbf{DIAG}, [-|-]_D, 1_{\mathbf{diag}}, \Delta_D, \epsilon_D, S_D)$ and one can prove that this is the unique structure of Hopf algebra such that $\phi_{\mathbf{DIAG},\mathbf{LDIAG}}$ is a morphism for the algebra and coalgebra structures.

9. Link between LDIAG and other Hopf algebras

9.1. The deformed case

One can construct a three-parameter Hopf algebra deformation of **LDIAG** (see [18]), denoted $\mathbf{LDIAG}(q_c, q_s, q_t)$ such that $\mathbf{LDIAG}(0, 0, 0) = \mathbf{LDIAG}$ and $\mathbf{LDIAG}(1, 1, 1) = \mathbf{MQSym}$ the algebra of Matrix Quasi Symmetric Functions [15]). On the other hand, it was proved by L. Foissy [20] that one of the planar decorated trees Hopf algebra is isomorphic **MQSym** and even to $\mathbf{LDIAG}(1, q_s, t)$ for every q_s and $t \in \{0, 1\}$. The complete picture is given below.



10. Duals of Hopf algebras

The question of dualizing a Hopf algebra (i.e. endowing the dual - or a subspace of it - with a structure of Hopf algebra) is solved, in complete generality, by the machineray of Sweedler's duals. The procedure consists in taking the "representative" linear forms (instead of all the linear forms) and dualize w.r.t. the following table

$$\begin{array}{ccc}
\text{comultiplication} & \rightarrow & \text{multiplication} \\
\text{counit} & \rightarrow & \text{unit} \\
\text{multiplication} & \rightarrow & \text{comultiplication} \\
\text{unit} & \rightarrow & \text{counit} \\
\text{antipode} & \rightarrow & \text{trsnspose of the antipode.}
\end{array}$$

In the case when the Hopf algebra is free as an algebra (which is often the case with noncommutative hopf algebras of combinatorial physics), one can use rational expressions of Automata Theory to get a genuine calculus within this dual (see [19]).

11. EXERCISES

Exercise 11.1 ***Representative functions on*** $\mathbb{R}$. *A function* $\mathbb{R} \xrightarrow{f} \mathbb{K}$ *(*$\mathbb{K} = \mathbb{R}, \mathbb{C}$*) is said to be representative if there exist* $(f_i^{(1)})_{i=1}^n$ *and* $(f_i^{(2)})_{i=1}^n$ *such that, for all* $x, y \in \mathbb{R}$ *one has*

$$f(x+y) = \sum_{i=1}^{n} f_i^{(1)}(x) f_i^{(2)}(y). \tag{82}$$

1.1) Show that $\cos, \cos^2, \sin, \exp$ *and* $a \to a^n$ *are representative. Provide minimal sums of type Eq.(82) 1.2) Show that the following are equivalent*
i) f *is representative.*
ii) There exists group representation $(\mathbb{R}, +) \xrightarrow{\rho} \mathcal{M}(n, \mathbb{K})$*, a row vector* $\lambda \in \mathbb{K}^{1\times n}$*, a column vector* $\gamma \in \mathbb{K}^{n\times 1}$ *such that* $f(x) = \lambda\rho(x)\gamma$.
iii) $(f_t)_{t\in\mathbb{R}}$ *is of finite rank in* $\mathbb{K}^{\mathbb{R}}$ *(here* f_t*, the shift of* f *by* t*, is the function* $x \longrightarrow f(x+t)$*).*
1.3) Show that the minimal n *such that formula Eq.(82) holds is also the rank of* $(f_t)_{t\in\mathbb{R}}$.
1.4) a) If f *is continuous then* ρ *can be chosen so and* $\rho(x) = e^{xT}$ *for a certain matrix* $T \in \mathcal{M}(n, \mathbb{C})$.
b) In this case show that representative functions are linear combinations of products of polynomials and exponentials.
1.5) $f \in \mathbb{C}^{\mathbb{R}}$ *is representative iff* $\mathcal{R}e(f) = (f + \bar{f})/2$ *and* $\mathcal{I}m(f) = (f - \bar{f})/2i$ *are representative in* $\mathbb{R}^{\mathbb{R}}$.
1.6) Show that the set of representative functions of $\mathbb{K}^{\mathbb{R}}$ *is a vector space. This space will be denoted* $Rep_{\mathbb{K}}(\mathbb{R})$.
1.7) Show that the functions $\varphi_{n,\lambda} = x^n e^{\lambda x}$ *are a basis of* $Rep_{\mathbb{C}}(\mathbb{R}) \cap \mathcal{C}_o(\mathbb{R}; \mathbb{C})$.
2.) Let M *be a monoid (semigroup with unit) and* $\mathbb{K} = \mathbb{R}, \mathbb{C}$*. For a function* $M \xrightarrow{f} \mathbb{K}$ *one defines the shifts:*

$$f_z : x \longrightarrow f(zx), \tag{83}$$
$$_y f : x \longrightarrow f(xy), \tag{84}$$
$$_y f_z : x \longrightarrow f(zxy). \tag{85}$$
$$\tag{86}$$

2.1)a) Check the following formulas

$$(f_{y_1})_{y_2} = f_{y_1 y_2}, \tag{87}$$
$$_{y_2}(_{y_1}f) =_{y_2 y_1} f. \tag{88}$$

b) Define from Eqs.(87) and (88) the two canonical M*-module structures of* $\mathbb{K}^M$.
Do they commute ?
2.2)a) Show that the following are equivalent
i) $(f_z)_{z\in M}$ *is of finite rank in* $\mathbb{K}^M$.
ii) $(_y f)_{y\in M}$ *is of finite rank in* $\mathbb{K}^M$.
iii) $(_y f_z)_{y,z\in M}$ *is of finite rank in* $\mathbb{K}^M$.

iv) There exist two families $(f_i^{(1)})_{i=1}^n$ *and* $(f_i^{(2)})_{i=1}^n$ *such that*

$$f(xy) = \sum_{i=1}^{n} f_i^{(1)}(x) f_i^{(2)}(y). \tag{89}$$

v) There exists a representation of M $\rho : M \longrightarrow \mathbb{C}^{n\times n}$*, a row vector* $\lambda \in \mathbb{C}^{1\times n}$*, a column vector* $\gamma \in \mathbb{C}^{n\times 1}$ *such that* $f(x) = \lambda\rho(x)\gamma$ *for all* $x \in M$*.*
b) Using (v) above show that the (pointwise) product of two representative functions is representative.

Exercise 11.2 *Let* $\mathcal{A}$ *be an AAU and* $\Delta : \mathcal{A} \mapsto \mathcal{A}\otimes\mathcal{A}$ *a comultiplication. We build (tensor) products of two representations by the formula (23), more precisely by*

$$can \circ (\rho_1 \otimes \rho_2) \circ \Delta \tag{90}$$

where $can : End(V_1)\otimes End(V_2) \mapsto End(V_1\otimes V_2)$ *is the canonical mapping.*
a) Prove that, if Δ *is a morphism of algebras, then the linear mapping*

$$\rho_1 \boxed{\times} \rho_2 : \mathcal{A} \mapsto End(V_1 \otimes V_2)$$

defined by the composition (90) is a morphism of AAU (and hence a representation).
b) Prove that, if $\rho_1 \boxed{\times} \rho_2$ *is a representation for any pair* ρ_1, ρ_2 *of representations of* $\mathcal{A}$*, then* Δ *is morphism of AAU (use* $\rho_1 = \rho_2$*, the - regular - representation of* $\mathcal{A}$ *on itself by multiplications on the left).*

Exercise 11.3 *We consider the canonical isomorphisms*

$$\begin{aligned} &can_{1|23} : V_1 \otimes (V_2 \otimes V_3) \mapsto V_1 \otimes V_2 \otimes V_3 \\ &can_{12|3} : (V_1 \otimes V_2) \otimes V_3 \mapsto V_1 \otimes V_2 \otimes V_3 \end{aligned} \tag{91}$$

Show that, in order to have (36) for every triple $\rho_i,\ i = 1, 2, 3$ of representations it is necessary and sufficient that

$$can_{12|3} \circ (\Delta \otimes Id_{\mathcal{A}}) \circ \Delta = can_{1|23} \circ (Id_{\mathcal{A}} \otimes \Delta) \circ \Delta \tag{92}$$

(for the necessary condition, consider again the left regular representations).

Exercise 11.4 *Let* $(\mathcal{A}, \Delta)$ *be a coalgebra (*Δ *is an arbitrary - but fixed - linear mapping) and* $(\mathcal{A}^*, *_\Delta)$ *be its dual algebra. Explicitely, for* $f, g \in \mathcal{A}^*$ *and* $x \in \mathcal{A}$ *(for convenience, the law is written in infix denotation)*

$$\langle f *_\Delta g | x\rangle = \langle f \otimes g | \Delta(x)\rangle \tag{93}$$

Prove the following equivalences

$$\Delta \text{ is co-associative } \Longleftrightarrow *_\Delta \text{ is associative} \tag{94}$$

$$(\forall \epsilon \in \mathcal{A}^*)\Big(\epsilon \text{ is a unity for } (\mathcal{A}^*, *_\Delta) \Longleftrightarrow \epsilon \text{ is a co-unity for } (\mathcal{A}, \Delta)\Big) \tag{95}$$

Exercise 11.5 *The mappings can_l, can_r are as in (40). Prove that, in order that, for any representation ρ of $\mathcal{A}$, one* has

$$can_l \circ (\epsilon \otimes \rho) \circ \Delta = can_r \circ (\rho \otimes \epsilon) \circ \Delta \tag{96}$$

it is necessary and sufficient that ϵ be a counity.

Exercise 11.6 *1) Let $(\mathcal{B}, *, 1, \Delta, \epsilon)$ be a bialgebra, we denote by $\mathcal{B}^+$ the kernel of ϵ.*
a) Prove that $\mathcal{B} = \mathcal{B}^+ \oplus k.1_{\mathcal{B}}$.
We denote I^+ the projection of $\mathcal{B}^+$ with respect to the preceding decomposition.
b) Prove that, for every $x \in \mathcal{B}^+$, one can write

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \sum_{(1)(2)} x_{(1)} \otimes x_{(2)} \text{ with } x_{(i)} \in \mathcal{B}^+ \tag{97}$$

2) One sets, for $x \in \mathcal{B}^+$,

$$\Delta^+(x) = \Delta(x) - (x \otimes 1 + 1 \otimes x) = \sum_{(1)(2)} x_{(1)} \otimes x_{(2)} \tag{98}$$

a) Check that $(\mathcal{B}^+, \Delta^+)$ is a coassociative coalgebra.
One sets

$$(\mathcal{B}^*)^+ = \{f \in \mathcal{B}^* | f(1) = 0\} \tag{99}$$

b) Prove that $(\mathcal{B}^*)^+$ is a subalgebra of $(\mathcal{B}, *_\Delta)$ and that its law is dual of Δ^+.
c) Prove that the algebra $(\mathcal{B}^*, *_\Delta$ is obtained from $(\mathcal{B}^*)^+, *_{\Delta^+}$ by adjunction of the unity ϵ.

3) The bialgebra is called locally finite if

$$(\forall x \in \mathcal{B})(\exists k \in \mathbb{N}^*)(\Delta^{+(k)}(x) = 0). \tag{100}$$

The projection I^+ being as above, show that, in case $\mathcal{B}$ is locally finite,

$$(\forall x \in \mathcal{B})(\exists N \in \mathbb{N}^*)(\forall k \geq N)((I^+)^k(x) = 0) \tag{101}$$

and that

$$S = \sum_{n \in \mathbb{N}} (-I^+)^n \tag{102}$$

is an antipode for $\mathcal{B}$.
4) Let G be a group and $\mathcal{H} = (\mathbb{C}[G], ., 1_G, \Delta, \epsilon, S)$ be the Hopf algebra of G.
a) Show that $\{(g-1)\}_{g \in G - \{1\}}$ is a basis of $\mathcal{H}^+$ and that $\Delta^+(g-1) = (g-1) \otimes (g-1)$.
b) Show that, if $G \neq \{1\}$, $\mathcal{H}^+$ is not locally finite, but $\mathcal{H}$ admits an antipode.
5) Prove that if the coproduct of H is graded (i.e. it exists a decomposition $H = \oplus_{n \in \mathbb{N}} H_n$ with $\Delta(H_n) \subset \sum_{a+b=n} H_a \otimes H_b$) and $H_0 = k.1_H$ then the comultiplication is locally finite.
6) Define the degree of a labelled diagram as its number of edges and **LDIAG**$_n$ *as the vector space generated by the diagrams of degree n and check that we are in conditions of exercise question 5.*

Exercise 11.7 *1) Show that, in order that a family $(\lambda_{i,j}^k)_{i,j,k\in I}$ be the family of structure constants of some algebra it is necessary and sufficient that*

$$(\forall (i,j)\in I^2)\Big((\lambda_{i,j}^k)_{k\in I} \text{ is finitely supported}\Big) \tag{103}$$

2) Similarly show that in order that a family $(\lambda_i^{j,k})_{i,j,k\in I}$ be the family of structure constants of some algebra it is necessary and sufficient that

$$(\forall i\in I)\Big((\lambda_i^{j,k})_{(j,k)\in I^2} \text{ is finitely supported}\Big) \tag{104}$$

3) Give examples of mappings $\lambda : I^3 \mapsto k$ such that the corresponding families satisfy

i) (103) and (104)

ii) (103) and not (104)

iii) (104) and not (103)

iv) none of (103) and (104)

4) Give again examples such as in 3) i-iii but with associative (resp. coassociative) multiplications (resp. comultiplications).

Exercise 11.8 *Let $(\mathcal{C},\Delta)$ be a coalgebra and $(\mathcal{A},\mu)$ ba an algebra on the same (commutative) field of scalars k. We define a multiplication (called convolution) on $Hom_k(\mathcal{C},\mathcal{A})$ by*

$$f*g = \mu\circ(f\otimes g)\circ\Delta \tag{105}$$

so that, if $\Delta(x)=\sum_{(1)(2)} x_{(1)}x_{(2)}$,

$$f*g(x) = \sum_{(1)(2)} f(x_{(1)})g(x_{(2)}).$$

1) If $\mathcal{A}$ is associative and $\mathcal{C}$ coassociative, show that the algebra $(Hom_k(\mathcal{C},\mathcal{A}),*)$ is associative.
2) We suppose moreover that $\mathcal{C}$ admits a counit $\epsilon : \mathcal{C}\mapsto k$ and $\mathcal{A}$ a unit $1_{\mathcal{A}}$ (identified with the linear mapping $k\mapsto\mathcal{A}$ given by $\lambda\to\lambda 1_{\mathcal{A}}$).
Show that $1_{\mathcal{A}}\circ\epsilon$ (traditionnally denoted $1_{\mathcal{A}}\epsilon$) is the unit of the algebra $(Hom_k(\mathcal{C},\mathcal{A}),*)$.
3) Let $(\mathcal{B},*,1,\Delta,\epsilon)$ be a bialgebra. The convolution under consideration will be that constructed between the coalgebra $(\mathcal{B},\Delta,\epsilon)$ and the algebra $(\mathcal{B},*,1)$.
a) Let $S\in End(\mathcal{B})$. Show that the following are equivalent
i) S is an antipode for $\mathcal{B}$
ii) S is the inverse of $Id_{\mathcal{B}}$ in $(End_k(\mathcal{B}),*)$.
b) Deduce from (b) that the antipode, if it exists, is unique.
c) Prove that the bialgebra $(k\langle A\rangle,*,\Delta_h,\epsilon_{aug})$ defined around equation (49) admits no antipode (if the alphabet A is not empty).
4) Let $(\mathcal{C},\Delta,\epsilon)$ be a coalgebra coassociative with counit. We define Δ_2 by $T_{2,3}\circ\Delta\otimes\Delta$ where $T_{2,3} : \mathcal{C}^{\otimes 4}\mapsto\mathcal{C}^{\otimes 4}$ is the flip between the 2nd and the 3rd component

$$T_{2,3}(x_1\otimes x_2\otimes x_3\otimes x_4) = x_1\otimes x_3\otimes x_2\otimes x_4 \tag{106}$$

a) Show that $(\mathcal{C}\otimes\mathcal{C},\Delta_2,\epsilon\otimes\epsilon)$ (with $\epsilon\otimes\epsilon(x\otimes y)=\epsilon(x)\epsilon(y)$) is coassociative coalgebra with counit.
Let $(\mathcal{H},\mu,1,\Delta,\epsilon,S)$ be a Hopf algebra. The convolution $*$ here will be that constructed between the coalgebra $(\mathcal{H}\otimes\mathcal{H},\Delta_2,\epsilon\otimes\epsilon)$ and the algebra $(\mathcal{H},\mu,1)$. We consider the two elements $\nu_i\in(Hom_k(\mathcal{H}\otimes\mathcal{H},\mathcal{H}),*)$ defined by $\nu_1=S\circ\mu$ and $\nu_2(x\otimes y)=S(y)S(x)$.
b) Show that the elements ν_i are the convolutional inverses of μ. Deduce from this that $S:\mathcal{H}\mapsto\mathcal{H}$ is an antimorphism of algebras.

12. APPENDIX

12.1. Function spaces

Throughout the text, we use basic construction of set theory and algebra (see [4, 5]). The set of mappings between two sets X and Y being denoted Y^X, if k is a field one notes

$$k^X=\{f:X\longrightarrow K\} \tag{107}$$

the vector space of all functions defined on X with values in k. For each function $f\in k^X$, we call "support of f", the set of points $x\in X$ such that $f(x)$ is not zero[Σ]

$$supp(f)=\{x\in X:f(x)\neq 0\} \tag{108}$$

the set of functions with finite support is a vector subspace of k^X which is denoted $k^{(X)}$.

12.2. Basic structures

Definition 12.1 *(Semigroup) A semigroup $(S,*)$ is a set S endowed with an associative law, this means that, for all $x,\ y,\ z\in S$ one has $x*(y*z)=(x*y)*z$.*

`http://en.wikipedia.org/wiki/Semigroup`

Definition 12.2 *(Monoid) A monoid $(M,*)$ is a semigroup which possesses a neutral, i.e. an element $e\in M$ such that, for all $x\in M$:*

$$e*x=x*e=x. \tag{109}$$

Such an element, if it exists is unique. The neutral is often denoted 1_M.

`http://en.wikipedia.org/wiki/Monoid`

The free monoid of alphabet X is the set of *strings* $x_1x_2\cdots x_n$ with letters $x_i\in X$. This set is denoted X^* and its law is the concatenation. It is easily seen that this monoid is free in the following sense. For `any` "set-theoretical" mapping $\phi:X\mapsto M$, where $(M,*)$ is a monoid, ϕ `can be uniquely extended to morphism` $\bar\phi$ `so that`

$$\begin{tikzcd} \mathbb{X} \arrow[r, "\phi"] \arrow[dr, "can"'] & M \\ & \mathbb{X}^* \arrow[u, "\bar\phi"'] \end{tikzcd}$$

Σ In integration theory, the support of a function is the closure of what we define as the (algebraic) support.

The free commutative monoid of alphabet X is the set of *monomials* X^α ($\alpha \in \mathbb{N}^{(X)}$). This set is denoted $\mathfrak{MON}(X)$ and its law is the multiplication of monomials

$$X^\alpha X^\beta = X^{\alpha+\beta} \tag{110}$$

It is easily seen that this monoid is free in the following sense. For all "set-theoretical" mapping $\phi : X \mapsto M$, where $(M, *)$ is a commutative monoid, ϕ cn be extended to monomials such that

$$\begin{tikzcd} \mathbb{X} \arrow[r, "\phi"] \arrow[dr, "can"'] & M \\ & \mathbb{X}^* \arrow[u, "\bar{\phi}"'] \end{tikzcd}$$

Definition 12.3 *(Group) A group $(G, *)$ is a monoid such that for each $x \in G$ there exists y such that*

$$x * y = y * x = e. \tag{111}$$

For fixed x such an element is unique and is usually denoted by x^{-1} and colled the inverse of x.

Definition 12.4 *(Algebra of a monoid) Let k be a field (scalars, if not familiar one can think $k = \mathbb{R}$ or $\mathbb{C}$). The algebra $k[M]$ of a monoid M (with coefficients in k) is the set of mappings $K^{(M)}$ endowed with the convolution product*

$$f * g(w) = \sum_{uv=w} f(u)g(v). \tag{112}$$

The algebra $(k[M], *)$ is an AAU.

Each $m \in M$ may be identified with its characteristic function (i.e. the Dirac function δ_m with value 1 at m and 0 elsewhere). These functions form a basis of $k[M]$ and then, every $f \in k[M]$ can be written as a finite sum $f = \sum_w f(w)w$. Through this identification the unity of M and $k[M]$ coincide.

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