A Three-Parameter Hopf Deformation of the Algebra of Feynman-like Diagrams

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Abstract. We construct a three-parameter deformation of the Hopf algebra **LDIAG**. This is the algebra that appears in an expansion in terms of Feynman-like diagrams of the *product formula* in a simplified version of Quantum Field Theory. This new algebra is a true Hopf deformation which reduces to **LDIAG** for some parameter values and to the algebra of Matrix Quasi-Symmetric Fuctions (**MQSym**) for others, and thus relates **LDIAG** to other Hopf algebras of contemporary physics. Moreover, there is an onto linear mapping preserving products from our algebra to the algebra of Euler-Zagier sums.

Contents

1 Introduction **3**

2 How and why these Feynman-like Diagrams arise **3**

3 Non-commutative lifting (classical case) **6**

3.1 Free monoids 6

3.2 Labeling the nodes 7

3.3 Coding **ldiag** with "lists of monomials" 8

4 The Hopf algebra LDIAG (non-deformed case) **8**
4.1 The monoid $(\mathfrak{MDN}^+(X))^*$ and the submonoid of codes of diagrams . . . 9
4.2 The Hopf algebras $\mathcal{C}_{ldiag}$ and **LDIAG** 10
4.3 Subalgebras of **LDIAG** 13
4.3.1 Graphic primitive elements 13
4.3.2 Level subalgebras 14
4.3.3 **BELL** and **LBELL** 15

5 The algebra LDIAG(q_c, q_s, q_t) (deformed case) **15**
5.1 Counting crossings (q_c) and superpositions (q_s) 15
5.2 Modified laws 16

6 Coproducts **19**

7 More on LDIAG(q_c, q_s, q_t) : structure, images and the link with Euler-Zagier sums **20**

8 Concluding remarks **21**

1. Introduction

We briefly describe the passage from the product formula, as described by by Bender et al. [3], and the related Feynman-like diagrams, to the description of Hopf algebra structures [15] on the diagrams themselves compatible with their evaluations.
First, C. M. Bender, D. C. Brody, and B. K. Meister [3] introduced a special field theory which proved to be particularly rich in combinatorial links and by-products.
Second, the Feynman-like diagrams produced by this theory label monomials; these monomials combine in a manner compatible with the monomial multiplication and co-addition‡. This is the Hopf algebra **DIAG**.
Third, the natural noncommutative pull-back of this algebra, **LDIAG**, has a basis (the labeled diagrams) which is in one-to-one correspondence with that of the Matrix Quasi-Symmetric Functions (the *packed matrices* of **MQSym**), but their algebra and co-algebra structures are completely different. In particular, in this basis, the multiplication of **MQSym** implies a sort of *shifted shuffle* with overlappings reminiscent of Hoffmann's shuffle used in the theory of of polyzeta functions[11]. The superpositions and overlappings involved there are not present in the (non-deformed) **LDIAG** and, moreover, the coproduct of **LDIAG** is co-commutative while that of **MQSym** is not.

The aim of this paper is to introduce a "parametric algebra" which mediates between the two Hopf algebras **LDIAG** and **MQSym**. The striking result is that when we introduce parameters which count the crossings and overlappings of the shifted shuffle, one notes that the resulting law is associative (graded with unit). We also show how to interpolate with a coproduct which makes, at each stage, our algebra a Hopf algebra. The result is thus a three-parameter Hopf algebra deformation which reduces to **LDIAG** at $(0,0,0)$ and to **MQSym** at $(1,1,1)$. Moreover it appears that, for one set of parameters, the multiplication rule of **LDIAG** recovers that of Euler-Zagier sums.

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2. How and why these Feynman-like Diagrams arise

The beginning of the story was fully explained in [36, 37, 38, 28, 5, 6], and the Hopf algebra structure was made precise in [15, 40]. In this note we shall emphasize the latter part of the analysis, where the algebraic structure constructed on the diagrams themselves arise.

Our starting point is the formula (*product formula*) of Bender and al. [3], which can be considered as an expression of the *Hadamard product* for an *exponential generating series*. That is, using

‡ i.e. the commultiplication obtained by replacing each variable by the sum of two (independent) copies of it.

$$F(z) = \sum_{n\geq 0} a_n \frac{z^n}{n!},\ G(z) = \sum_{n\geq 0} b_n \frac{z^n}{n!},\ \mathcal{H}(F,G) := \sum_{n\geq 0} a_n b_n \frac{z^n}{n!} \tag{1}$$

one can check that

$$\mathcal{H}(F,G) = F\left(z\frac{d}{dx}\right) G(x)\Big|_{x=0}. \tag{2}$$

When $F(0)$ and $G(0)$ are not zero one can renormalize the functions in this bilinear product so that $F(0) = G(0) = 1$. We wish to obtain compact and generic formulas. If we write the functions as

$$F(z) = \exp\left(\sum_{n=1}^{\infty} L_n \frac{z^n}{n!}\right), \qquad G(z) = \exp\left(\sum_{n=1}^{\infty} V_n \frac{z^n}{n!}\right). \tag{3}$$

that is, as free exponentials, then by using Bell polynomials in the sets of variables $\mathbb{L}, \mathbb{V}$ (see [15] for details), we obtain

$$\mathcal{H}(F,G) = \sum_{n\geq 0} \frac{z^n}{n!} \sum_{P_1,P_2\in UP_n} \mathbb{L}^{Type(P_1)}\mathbb{V}^{Type(P_2)} \tag{4}$$

where UP_n is the set of unordered partitions of $[1\cdots n]$. An unordered partition P of a set X is a subset of $P \subset \mathfrak{P}(X) - \{\emptyset\}$§ (that is an unordered collection of blocks, i. e. non-empty subsets of X) such that

- the union $\bigcup_{Y\in P} Y = X$ (P is a covering)
- P consists of disjoint subsets, i. e.
  $Y_1, Y_2 \in P$ *and* $Y_1 \cap Y_2 \neq \emptyset \Longrightarrow Y_1 = Y_2$.

The type of $P \in UP_n$ (denoted above by $Type(P)$) is the multiindex $(\alpha_i)_{i\in\mathbb{N}^+}$ such that α_k is the number of k-blocks, that is the number of members of P with cardinality k.

At this point the formula entangles and the diagrams of the theory arise.
Note particularly that

- the monomial $\mathbb{L}^{Type(P_1)}\mathbb{V}^{Type(P_2)}$ needs much less information than that which is contained in the individual partitions $P_1,\ P_2$ (for example, one can relabel the elements without changing the monomial),
- two partitions have an incidence matrix *from which it is still possible to recover the types of the partitions.*

The construction now proceeds as follows.

§ The set of subsets of X is denoted by $\mathfrak{P}(X)$ (this notation [9] is that of the former German school).

(i) Take two unordered partitions of $[1 \cdots n]$, say P_1, P_2

(ii) Write down their incidence matrix $(\mathrm{card}(Y \cap Z))_{(Y,Z) \in P_1 \times P_2}$

(iii) Construct the diagram representing the multiplicities of the incidence matrix : for each block of P_1 draw a black spot (resp. for each block of P_2 draw a white spot)

(iv) Draw lines between the black spot $Y \in P_1$ and the white spot $Z \in P_2$; there are $\mathrm{card}(Y \cap Z)$ such.

(v) Remove the information of the blocks $Y, Z, \cdots$.

In so doing, one obtains a bipartite graph with p $(= \mathrm{card}(P_1))$ black spots, q $(= \mathrm{card}(P_2))$ white spots, no isolated vertex and integer multiplicities. We denote the set of such diagrams by **diag**.

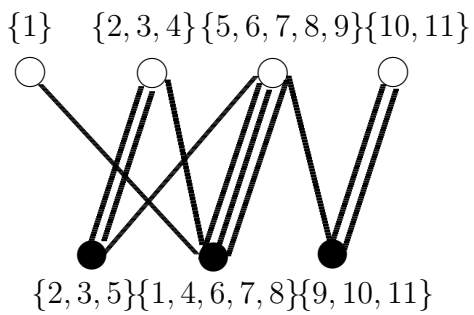

Fig 1. — *Diagram from* P_1, P_2 *(set partitions of* $[1 \cdots 11]$*).*
$P_1 = \{\{2,3,5\},\{1,4,6,7,8\},\{9,10,11\}\}$ *and* $P_2 = \{\{1\},\{2,3,4\},\{5,6,7,8,9\},\{10,11\}\}$ *(respectively black spots for* P_1 *and white spots for* P_2*).*
The incidence matrix corresponding to the diagram (as drawn) or these partitions is $\begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$*. But, due to the fact that the defining partitions are unordered, one can permute the spots (black and white, between themselves) and, so, the lines and columns of this matrix can be permuted. the diagram could be represented by the matrix* $\begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 3 & 1 \end{pmatrix}$ *as well.*

The product formula now reads

$$\mathcal{H}(F,G) = \sum_{n \geq 0} \frac{z^n}{n!} \sum_{\substack{d \in diag \\ |d|=n}} mult(d) \mathbb{L}^{\alpha(d)} \mathbb{V}^{\beta(d)} \tag{5}$$

where $\alpha(d)$ (resp. $\beta(d)$) is the "white spots type" (resp. the "black spots type") i.e. the multiindex $(\alpha_i)_{i \in \mathbb{N}^+}$ (resp. $(\beta_i)_{i \in \mathbb{N}^+}$) such that α_i (resp. β_i) is the number of white spots (resp. black spots) of degree i (i lines connected to the spot) and $mult(d)$ is the number of pairs of unordered partitions of $[1 \cdots |d|]$ (here $|d| = |\alpha(d)| = |\beta(d)|$ is the number of lines of d) with associated diagram d.

Now one may naturally ask

Q1) "Is there a (graphically) natural multiplicative structure on **diag** *such that the arrow*

$$d \mapsto \mathbb{L}^{\alpha(d)} \mathbb{V}^{\beta(d)} \tag{6}$$

be a morphism ?"

The answer is "yes". The desired product just consists in concatenating the diagrams (the result, i.e. the diagram obtained in placing d_2 at the right of d_1, will be denoted by $[d_1|d_2]_D$). One must check that this product is compatible with the equivalence of the permutation of white and black spots among themselves, which is rather straightforward (see [15]). We have

Proposition 2.1 *Let* **diag** *be the set of diagrams (including the empty one).*
i) The law $(d_1, d_2) \mapsto [d_1|d_2]_D$ *endows* **diag** *with the structure of a commutative monoid with the empty diagram as neutral element(this diagram will, therefore, be denoted by* $1_{\mathbf{diag}}$*).*
ii) The arrow $d \mapsto \mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)}$ *is a morphism of monoids, the codomain of this arrow being the monoid of (commutative) monomials in the alphabet* $\mathbb{L} \cup \mathbb{V}$ *i.e.*

$$\mathfrak{MON}(\mathbb{L} \cup \mathbb{V}) = \{\mathbb{L}^{\alpha}\mathbb{V}^{\beta}\}_{\alpha,\beta\in(\mathbb{N}^+)^{(\mathbb{N})}} = \bigcup_{n,m\geq 1} \{L_1^{\alpha_1}L_2^{\alpha_2}\cdots L_n^{\alpha_n}V_1^{\beta_1}V_2^{\beta_2}\cdots V_m^{\beta_m}\}_{\alpha_i,\beta_j\in\mathbb{N}}.$$

iii) The monoid $(\mathbf{diag}, [-|-]_D, 1_{\mathbf{diag}})$ *is a free commutative monoid. Its letters are the connected (non-empty) diagrams.*

Remark 2.2 *The reader who is not familiar with the algebraic structure of* $\mathfrak{MON}(\mathbb{X})$ *can find rigorous definitions in paragraph (3.1) where this structure is needed for the proofs relating to deformations.*

3. Non-commutative lifting (classical case)

The "classical" construction of the Hopf algebra **LDIAG** was given in [15]. We give the proofs below, using a *coding* through "lists of monomials" needed for the deformed (quantum) case. The entries of a list can be considered as "coordinate functions" for the diagrams (see introduction of section (4)).

3.1. Free monoids

We recall here the construction of the free and free-commutative monoids generated by a given set of variables (i.e. an alphabet).
Let $\mathbb{X}$, be a set. We denote by $\mathbb{X}^*$ the set of lists of elements of $\mathbb{X}$, including the empty one. In many works, and in the sequel, the list $[x_1, x_2, \cdots, x_n]$ will be considered as a word $x_1x_2\cdots x_n$ so that the concatanation of two lists $[x_1, x_2, \cdots, x_n]$, $[y_1, y_2, \cdots, y_m]$ is just the word $x_1x_2\cdots x_ny_1y_2\cdots y_m$. For this (associative) law, the empty list [ ] is the neutral element and will therefore be denoted by $1_{\mathbb{X}^*}$

Similarly, we denote by $\mathbb{N}^{(\mathbb{X})}$ [7] the set of multisubsets of $\mathbb{X}$ (i.e. the set of - multiplicity - mappings with finite support $\mathbb{X} \mapsto \mathbb{N}$). Every element α of $\mathbb{N}^{(X)}$ can be written multiplicatively, following the classical multiindex notation

$$\mathbb{X}^{\alpha} = \prod_{x\in\mathbb{X}} x^{\alpha(x)} \tag{7}$$

and the set $\mathfrak{MON}(X) = \{\mathbb{X}^\alpha\}_{\alpha\in\mathbb{N}^{(X)}}$ is exactly the set of (commutative) monomials with variables in $\mathbb{X}$. It is a monoid, indeed a (multiplicative) copy of $\mathbb{N}^{(X)}$ as $\mathbb{X}^\alpha\mathbb{X}^\beta = \mathbb{X}^{\alpha+\beta}$. The subset of its non-unit elements is a semigroup which will be denoted by $\mathfrak{MON}^+(X)$ $(= \mathfrak{MON}(X) - \{\mathbb{X}^0\})$.

3.2. Labeling the nodes

There are (at least) two good reasons to look for non-commutative structures which may serve as a noncommutative pullback for **diag**.

- Rows and Columns of matrices are usually (linearly) ordered and we have seen that a diagram is not represented by a matrix but by a class of matrices
- The complexity of **diag** and its algebra is not sufficient to relate it to other (non-commutative or non-cocommutative) algebras relevant to contemporary physics

The solution (of the non-deformed problem [15]) is simple and consists in labeling the nodes from left to right and from "1" to the desired number as follows.

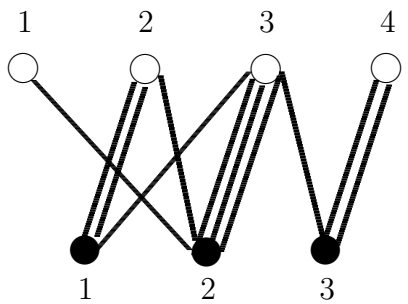

Fig 2. — *Labelled diagram of format* 3×4 *corresponding to the one of Fig 1.*

The set of these graphs (i.e. bipartite graphs on some product $[1..p] \times [1..q]$ with no isolated vertex) will be denoted by **ldiag**. The composition law is, as previously, concatenation in the obvious sense. Explicitly, if d_i, $i = 1, 2$ are two diagrams of dimension $[1..p_i] \times [1..q_i]$, one relabels the black (resp. white) spots of d_2 from $p_1 + 1$ to $p_1 + p_2$ (resp. from $q_1 + 1$ to $q_1 + q_2$) the result will be noted $[d_1|d_2]_L$. One has

Proposition 3.1 *Let* **ldiag** *be the set of labeled diagrams (including the empty one).*
i) The law $(d_1, d_2) \mapsto [d_1|d_2]_L$ *endows* **ldiag** *with the structure of a noncommutative monoid with the empty diagram* $(p = q = 0)$ *as neutral element(which will, therefore, be denoted by* $1_{\mathbf{ldiag}}$*).*
ii) The arrow from **ldiag** *to* **diag***, which implies "forgetting the labels of the vertices" is a morphism of monoids.*
iii) The monoid $(\mathbf{ldiag}, [-|-]_L, 1_{\mathbf{ldiag}})$ *is a free (noncommutative) monoid. Its letters are the irreducible diagrams (denoted from now on by* $irr(\mathbf{ldiag})$*).*

Remark 3.2 *i) In a general monoid* $(M, \star, 1_M)$*, the irreducible elements are the elements* $x \neq 1_M$ *such that* $x = y \star z \Longrightarrow 1_M \in \{y, z\}$.
ii) It can happen that an irreducible of **ldiag** *has an image in* **diag** *which splits, as shown by the simple example of the* cross *defined by the incidence matrix* $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

3.3. Coding **ldiag** *with "lists of monomials"*

One can code every labelled diagram by a "list of (commutative) monomials" in the following way.

- Let $\mathbb{X} = \{x_i\}_{i \geq 1}$ be an infinite set of indeterminates and $d \in \mathbf{ldiag}_{p \times q}$ a diagram ($\mathbf{ldiag}_{p \times q}$ is the set of diagrams with p black spots and q white spots).
- Associate with d the multiplicity function $[1..p] \times [1..q] \to \mathbb{N}$ such that $d(i, j)$ is the number of lines from the black spot i to the white spot j.
- The code associated with d is $\varphi_{lm}(d) = [m_1, m_2, \cdots, m_p]$ such that $m_i = \prod_{j=1}^{q} x_j^{m(i,j)}$

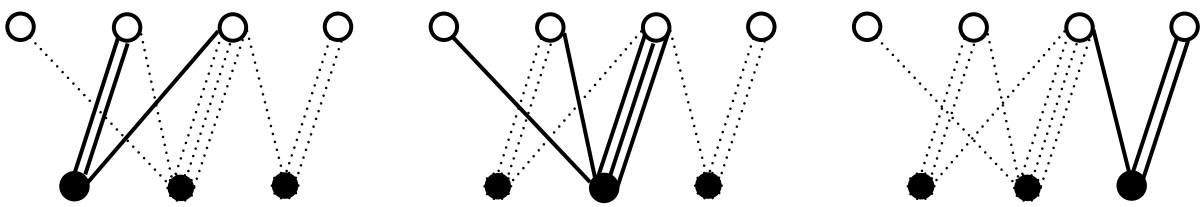

Fig 3. — *Coding the diagram of fig 2 by a word of monomials. The code here is* $[x_2^2 x_3\ ,\ x_1 x_2 x_3^3\ ,\ x_3 x_4^2]$

As a data structure, the lists of monomials are elements of $(\mathfrak{MON}^+(X))^*$, the free monoid whose letters are $\mathfrak{MON}^+(X) = \mathfrak{MON}(X) - \{\mathbb{X}^0\}$, the semigroup of non-unit monomials over $\mathbb{X}$.

It is not difficult to see that, through this coding, concatenation is reflected in the following formula

$$\varphi_{lm}([d_1|d_2]_L) = \varphi_{lm}(d_1) * T_{max(IndAlph(\varphi_{lm}(l_1)))}(\varphi_{lm}(d_2)) \tag{8}$$

where T_p is the translation operator which changes the variables according to $T_p(x_i) = x_{i+p}$ (which corresponds to the relabelling of the white spots) and p_1 is the number of black spots of d_1.
For example, one has

$$T_2([x_2^2 x_3, x_1 x_2 x_3^3, x_3 x_4^2]) = [x_4^2 x_5, x_3 x_4 x_5^3, x_5 x_6^2]\ ;\ T_6([x_1, x_2^2]) = [x_7, x_8^2] \tag{9}$$

4. The Hopf algebra LDIAG (non-deformed case)

In [15], we defined a Hopf algebra structure on the space of diagrams **LDIAG**. The aim of this section is to give complete proofs and details for this construction through

the use of the special space of coordinates constructed above (the complete vector of coordinates of a diagram being its code).

4.1. The monoid $(\mathfrak{MON}^+(X))^*$ and the submonoid of codes of diagrams

Formula (8) can be written using lists as

$$l_1 \bar{*} l_2 = l_1 * T_{max(IndAlph(l_1))}(l_2) \tag{10}$$

which defines a monoid structure on $(\mathfrak{MON}^+(X))^*$ (the set of lists of non-unit monomials) with the empty list as neutral (i.e. [ ] which will, therefore, be denoted by "$1_{(\mathfrak{MON}^+(X))^*}$" or simply "1" when the context is clear).
We will return to this construction (called shifting [20]) later.
The alphabet of a list is the set of variables occurring in the list. Formally

$$Alph([m_1, m_2, \cdots m_n]) = \bigcup_{1 \leq i \leq k} Alph(m_i) \tag{11}$$

where, classically, for a monomial $m = \mathbb{X}^\alpha$, $Alph(m) = \{x_i\}_{\alpha(i) \neq 0}$.
Now, we can define the "compacting operator" on $k\langle \mathfrak{MON}^+(X)\rangle$ by its action on the lists. This operator actually removes the holes in the alphabet of a list by pushing to the left the indices which are at the right of a hole. For example (we denote by *cpt* the operator)

$$cpt([x_2^2 x_{10}, x_3 x_4 x_8^3, x_3 x_4^2]) = [x_1^2 x_5, x_2 x_3 x_4^3, x_2 x_3^2] \tag{12}$$

the alphabet of the list on the LHS is $Alph(l) = Alph([x_2^2 x_{10}, x_3 x_4 x_8^3, x_3 x_4^2]) = \{x_2, x_3, x_4, x_8, x_{10}\}$, its indices are $IndAlph(l) = \{2, 3, 4, 8, 10\}$ and the re-indexing function is the unique strictly increasing mapping from $\{2, 3, 4, 8, 10\}$ to $[[5]]$. Here the compacting operator is just the substitution

$$x_1 \leftarrow x_2;\ x_2 \leftarrow x_3;\ x_3 \leftarrow x_4;\ x_4 \leftarrow x_8;\ x_5 \leftarrow x_{10}$$

The formal definitions are the following

- $IndAlph(l) = \{i \mid x_i \in Alph(l)\}$
- l being given, let ϕ_l be the unique increasing mapping from $IndAlph(l)$ to $[[card(IndAlph(l))]]$ (in fact, $card(IndAlph(l)) = card(Alph(l))$)
- let s_l be the substitution $x_i \leftarrow x_{\phi_l(i)}$ in the monomials.
- Then, if $l = [m_1, m_2, \cdots m_n]$, $cpt(l) = [s_l(m_1), s_l(m_2), \cdots s_l(m_n)]$.

Définition 4.1 *The compacting operator cpt* $: k\langle \mathfrak{MON}^+(X)\rangle \mapsto k\langle \mathfrak{MON}^+(X)\rangle$ *is the extension by linearity of the mapping cpt defined above.*

It can be checked easily that, for $l \in (\mathfrak{MON}^+(X))^*$, the following are equivalent

(i) $cpt(l) = l$

(ii) $IndAlph(l) = [[card(IndAlph(l))]]$

(iii) there is no hole in $Alph(l)$; that is, there exists no $i \geq 1$ s.t. $x_i \notin Alph(l)$ and $x_{i+1} \in Alph(l))$

(iv) l is the code of some (then unique) diagram d.

It follows from the preceding properties that cpt is a projector with range the subspace $\mathcal{C}_{ldiag}$ of $k\langle \mathfrak{MON}^+(\mathbb{X})\rangle$ generated by the codes of the diagrams. Formula (8) proves that $\mathcal{C}_{ldiag}$ is closed under the shifted concatenation defined by (10). More precisely

Proposition 4.2 *The algebra $\mathcal{C}_{ldiag}$ is a free algebra on the set of the codes of irreducible diagrams.*

These codes are also the non-empty lists l which are compact (i.e. $cpt(l) = l$) and cannot be factorized into a product of two non-empty lists i.e. $l = l_1 * l_2;\ l_i \neq [\ ]$ (one can check easily that, if $l_1 * l_2$ is compact, so are l_1 and l_2).

4.2. The Hopf algebras $\mathcal{C}_{ldiag}$ and **LDIAG**

The algebra **LDIAG** is endowed with the structure of a bi-algebra by the comultiplication

$$\Delta_{BS}(d) = \sum_{I+J=[1..p]} d[I] \otimes d[J] \tag{13}$$

where p is the number of black spots and $d[I]$ is the "restriction" of d to the black spots selected by the $I \subset [1..p]$.

On the other hand, we have a standard Hopf algebra structure on the free algebra, expressed in terms of concatenation and subwords [26, 34]. Let $\mathbb{A}$ be an alphabet (a set of letters) and $w \in \mathbb{A}^*$ a word, if we write w a a sequence of letters $w = a_1 a_2 \cdots a_n;\ a_i \in \mathbb{A}$, the length $|w|$ of w is n and if $I = \{i_1, i_2, \cdots i_k\} \subset [1..n]$, the subword $w[I]$ is $a_{i_1} a_{i_2} \cdots a_{i_k}$ (this notation is slightly different from that of [34] where it is $w|_I$). Then, the free algebra $k\langle \mathbb{A}\rangle$ is a Hopf algebra with comultiplication [34, 26].

$$\Delta_{LieHopf}(w) = \sum_{I+J=[1..n]} w[I] \otimes w[J]. \tag{14}$$

One has the following relation between restrictions of diagrams and subwords

$$\varphi_{lm}(d[I]) = cpt(\varphi_{lm}(d)[I]) \tag{15}$$

this suggests that the coproduct

$$\Delta_{list}(l) = \sum_{I+J=[1..n]} cpt(l[I]) \otimes cpt(l[J]) \tag{16}$$

could be a Hopf algebra comultiplication for the shifted algebra $(k\langle \mathfrak{MON}^+(\mathbb{X})\rangle, \bar{*}, [\ ])$. Unfortunately, this fails due to the lack of counit (i and ii of the following Theorem),

but the "ground subalgebra" $\mathcal{C}_{ldiag}$ is a genuine Hopf algebra (which is exactly what we do need here).

Theorem 4.3 *Let $\mathcal{A} = (k\langle \mathfrak{MON}^+(\mathbb{X})\rangle, \bar{*}, [\,])$ be the algebra of lists of (non-unit) monomials endowed with the shifted concatenation of formula (10). Then*
i) $\mathcal{A}$ is a free algebra.
ii) The coproduct Δ_{list} (recalled below) is co-associative and a morphism of algebras $\mathcal{A} \mapsto \mathcal{A} \otimes \mathcal{A}$ (i.e. $\mathcal{A}$ is a bi-algebra without counit).

$$\Delta_{list}(l) = \sum_{I+J=[1..n]} cpt(l[I]) \otimes cpt(l[J]) \tag{17}$$

iii) The algebra $\mathcal{C}_{ldiag}$ is a sub-algebra and coalgebra of $\mathcal{A}$ which is a Hopf algebra for the following co-unit and antipode.

- Counit

$$\varepsilon(l) = \delta_{l,[\,]} \qquad \text{(Kronecker delta)} \tag{18}$$

- Antipode

$$S(l) = \sum_{r\geq 0} \sum_{\substack{I_1+I_2+\ldots I_r=[1..p] \\ I_j\neq\emptyset}} (-1)^r cpt(l[I_1])cpt(l[I_2])\cdots cpt(l[I_r]) \tag{19}$$

Proof — i) Throughout the proof, we will denote by $*$ the concatenation between lists and $\bar{*}$ the shifted concatenation defined by the formula (10). We first remark that, if $l = l_1 \bar{*} l_2$, then $max(IndAlph(l_1)) < min(IndAlph(l_2))$. This leads us to define, for a (non-shifted) factorization $l = l_1 * l_2 = l[1..t] * l[t+1..p]$ ($p = |l|$), a gauge of the degree of overlapping of the intervals (of integers) $[1..max(IndAlph(l1))]$ and $[min(IndAlph(l2))..\infty[$, thus the function

$$\omega_l(t) = card\Big([1..max(IndAlph(l[1..t])] \cap [min(IndAlph(l[t+1..p])..\infty[\Big) = \Big(max(IndAlph(l[1..t])) - min(IndAlph(l[t+1..p])) + 1\Big)^+. \tag{20}$$

(We recall that, for a real number x, x^+ is its positive part $x^+ = max(x,0) = \frac{1}{2}(|x|+x)$ [8]). It can be easily checked that the points t where $\omega_l(t) = 0$ determine the (unique) factorisation of l in irreducibles. It follows that the monoid $((\mathfrak{MON}^+(\mathbb{X}))^*, \bar{*}, [\,])$ is free and so is its algebra $(k\langle \mathfrak{MON}^+(\mathbb{X})\rangle, \bar{*}, [\,])$.

ii) If we denote $\Delta : \mathcal{A} \mapsto \mathcal{A} \otimes \mathcal{A}$ the standard coproduct given, for a list l of length p, by formula (14), one can remark that

(i) $cpt(l_1)\bar{*}cpt(l_2) = cpt(l_1\bar{*}l_2)$
(ii) $\Delta_{list} = (cpt \otimes cpt) \circ \Delta$
(iii) $\Delta_{list} \circ cpt = \Delta_{list}$

(iv) $(\forall n \in \mathbb{N})(cpt(T_n(l)) = cpt(l))$

(v) $(\forall n \in \mathbb{N})(\Delta \circ T_n = (T_n \otimes T_n) \circ \Delta)$.

Coassociativity of Δ_{list}. —

One has

$$
\begin{aligned}
&(\Delta_{list} \otimes Id) \circ \Delta_{list} = (\Delta_{list} \otimes Id) \circ (cpt \otimes cpt) \circ \Delta = \text{‘} \\
&((\Delta_{list} \circ cpt) \otimes cpt) \circ \Delta = (\Delta_{list} \otimes cpt) \circ \Delta = \\
&(((cpt \otimes cpt) \circ \Delta) \otimes cpt) \circ \Delta = \\
&(cpt \otimes cpt \otimes cpt) \circ (\Delta \otimes Id) \circ \Delta = (cpt \otimes cpt \otimes cpt) \circ (Id \otimes \Delta) \circ \Delta \\
&(cpt \otimes ((cpt \otimes cpt) \circ \Delta)) \circ \Delta = (cpt \otimes \Delta_{list}) \circ \Delta = \\
&(cpt \otimes (\Delta_{list} \circ cpt)) \circ \Delta = \\
&(Id \otimes \Delta_{list}) \circ (cpt \otimes cpt) \circ \Delta = (Id \otimes \Delta_{list}) \circ \Delta_{list}
\end{aligned} \tag{21}
$$

Δ_{list} is a morphism. —

For two lists $u, v \in$, let us compute $\Delta_{list}(u \bar{*} v)$. With $p = max(IndAlph(u))$, one has

$$
\begin{aligned}
&\Delta_{list}(u \bar{*} v) = (cpt \otimes cpt) \circ \Delta(l_1 * T_p(v)) = \\
&(cpt \otimes cpt)(\Delta(u) *^{\otimes 2} \Delta(T_p(v))) = \\
&(cpt \otimes cpt)(\Delta(u) *^{\otimes 2} (T_p \otimes T_p)\Delta(v) = \\
&(cpt \otimes cpt)(\sum_{(1)(2)} u_{(1)} \otimes u_{(2)}) *^{\otimes 2} (T_p \otimes T_p)(\sum_{(3)(4)} v_{(3)} \otimes v_{(4)}) = \\
&(cpt \otimes cpt)(\sum_{(1)(2)(3)(4)} u_{(1)} * T_{p_1}(T_{p-p_1}(v_{(3)})) \otimes u_{(2)} * T_{p_2}(T_{p-p_2}(v_{(4)})))
\end{aligned} \tag{22}
$$

with, for each term in the sum

$$p_1 = max(IndAlph(u_{(1)})) \leq p \ ; \ p_2 = max(IndAlph(u_{(2)})) \leq p$$

so, the quantity in (22) is

$$
\begin{aligned}
&(cpt \otimes cpt)(\sum_{(1)(2)(3)(4)} u_{(1)} \bar{*} (T_{p-p_1}(v_{(3)})) \otimes u_{(2)} \bar{*} (T_{p-p_2}(v_{(4)}))) = \\
&\sum_{(1)(2)(3)(4)} cpt(u_{(1)} \bar{*} (T_{p-p_1}(v_{(3)}))) \otimes cpt(u_{(2)} \bar{*} (T_{p-p_2}(v_{(4)}))) = \\
&\sum_{(1)(2)(3)(4)} \Big(cpt(u_{(1)}) \bar{*} cpt(T_{p-p_1}(v_{(3)}))\Big) \otimes \Big(cpt(u_{(2)}) \bar{*} cpt(T_{p-p_2}(v_{(4)}))\Big) = \\
&\sum_{(1)(2)(3)(4)} \Big(cpt(u_{(1)}) \bar{*} cpt(v_{(3)})\Big) \otimes \Big(cpt(u_{(2)}) \bar{*} cpt(v_{(4)})\Big) = \\
&\Big(\sum_{(1)(2)} cpt(u_{(1)}) \otimes cpt(u_{(2)})\Big) \bar{*}^{\otimes 2} \Big(\sum_{(3)(4)} cpt(v_{(3)}) \otimes cpt(v_{(4)})\Big) =
\end{aligned}
$$

$$\Delta_{list}(u)\bar{*}^{\otimes 2}\Delta_{list}(v) \tag{23}$$

iii) As $\mathcal{C}_{ldiag}$ is generated by the image of cpt it is clear that this space is a sub-coalgebra of $\mathcal{A}$. Moreover, cpt is a (multiplicative) morphism $\mathcal{A} \mapsto \mathcal{A}$ and thus its image $\mathcal{C}_{ldiag}$ is a subalgebra of $\mathcal{A}$. We now supply the missing ingredients to complete the proof of the Hopf algebra structure.

ε IS A COUNIT. —

Let $l = cpt(l)$ be a compact list. We remark that, for any list u, one has $cpt(u) = [\ ] \Longleftrightarrow u = [\ ]$. Then, with $\mu_l\ :\ k \otimes \mathcal{A} \mapsto \mathcal{A}$ the scaling operator

$$\mu_l(\varepsilon \otimes Id)\Delta_{list}(l) = \sum_{I+J=[1..n]} \varepsilon(cpt(l[I]))cpt(l[J]) =$$
$$\sum_{\substack{I+J=[1..n]\\ I=\emptyset}} \varepsilon(cpt(l[I]))cpt(l[J]) + \sum_{\substack{I+J=[1..n]\\ I\neq\emptyset}} \varepsilon(cpt(l[I]))cpt(l[J]) = cpt(l) + 0 = l \tag{24}$$

the proof of the fact that ε is a left counit is similar.

S IS THE ANTIPODE. —

One has $\mathcal{C}_{ldiag} = k.1 \oplus ker(\varepsilon)$, let us denote Id^+ the projection $\mathcal{C}_{ldiag} \mapsto ker(\varepsilon)$ according to this decomposition.
Then, for every list l,

$$\sum_{r\geq 0} \sum_{\substack{I_1+I_2+\ldots I_r=[1..p]\\ I_j\neq\emptyset}} (-1)^r cpt(l[I_1])cpt(l[I_2])\cdots cpt(l[I_r])$$

is well defined as the first sum is locally finite. Thus, the operator

$$\sum_{r\geq 0} \sum_{\substack{I_1+I_2+\ldots I_r=[1..p]\\ I_j\neq\emptyset}} (-1)^r \underbrace{(Id^+ * Id^+ * \cdots * Id^+)}_{r\ times}$$

is well defined and is the convolutional inverse of Id.

4.3. Subalgebras of **LDIAG**

4.3.1. Graphic primitive elements The problem of Graphic Primitive Elements (GPE) is the following.
Let $\mathcal{H}$ be a Hopf algebra with (linear) basis G, a set of graphs. The GPE are the primitive elements $\Gamma \in G$ which are primitive i.e.

$$\Gamma \text{ is a GPE} \Longleftrightarrow \Gamma \in G \text{ and } \Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma. \tag{25}$$

It is not difficult to check that, in any case, the subalgebra $\mathcal{H}^{\text{GPE}}$ generated by these elements is also a sub-coalgebra.
We make an extra hypothesis (which is often fulfilled)

$$1_{\mathcal{H}} \in G \text{ and } (\Gamma \in G - \{1_{\mathcal{H}}\} \Longrightarrow \varepsilon(\Gamma) = 0). \tag{26}$$

Then (if (26) is fulfilled) $\mathcal{H}^{\text{GPE}}$ is a sub-Hopf algebra as the antipode of the product $\Gamma_1\Gamma_2\cdots\Gamma_p$ of (GPE) is

$$S(\Gamma_1\Gamma_2\cdots\Gamma_p) = (-1)^p\ \Gamma_p\Gamma_{p-1}\cdots\Gamma_1. \tag{27}$$

The following proposition helps to determine $\mathbf{LDIAG}^{\text{GPE}}$.

Proposition 4.4 *In* **LDIAG** *(with basis $G = \mathbf{ldiag}$), the following are equivalent*
i) d is a GPE
ii) d has only one black spot.

Then, the Hopf algebra $\mathbf{LDIAG}^{\text{GPE}}$ is generated by the product of "one-black-spot" diagrams.

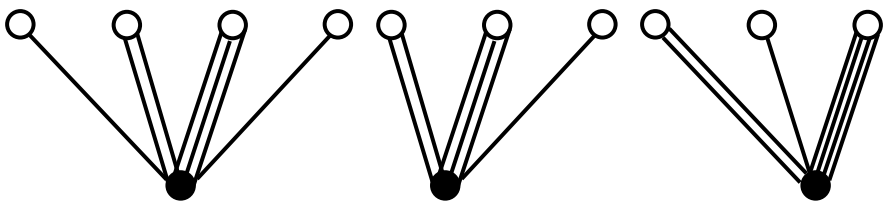

Fig 4. — *Graphic Primitive Elements of* **LDIAG** *have only one black spot and therefore are coded by the sequence of the ingoing degrees of their white spots (a composition). The first one here has code* $[1,2,3,1]$. *The picture shows an element of the monoid generated by Graphic Primitive Elements (a linear basis of* $\mathbf{LDIAG}^{GPE}$*) which is then coded by a list of compositions, here* $\Big[[1,2,3,1],[2,3,1],[2,1,4]\Big]$.

4.3.2. Level subalgebras

One can also impose limitations on the incoming degrees of the white spots in a way compatible with the coproduct. In this case, one defines an infinity of Hopf-subalgebras of **LDIAG** which we will call "level subalgebras".
More precisely, given an integer $l > 0$, one can ask for spaces generated by the diagrams d for which every white spot has an incoming degree $\leq l$. This amounts to say that the "white spot type" of every diagram d is of the form

$$\alpha(d) = (\alpha_1, \alpha_2, \cdots \alpha_k, 0, 0\cdots 0, \cdots)\ ;\ (\text{ all the } \alpha_i \leq l \text{ for } i \leq k \text{ and } \alpha_i = 0 \text{ for } i > k)$$

We denote by $\mathbf{LDIAG}^{\leq l}$ the subspace generated by these diagrams. One has a chain of Hopf algebras

$$\mathbf{LDIAG}^{\leq 1} \subset \mathbf{LDIAG}^{\leq 2} \subset \cdots \mathbf{LDIAG}^{\leq l} \subset \mathbf{LDIAG}^{\leq l+1} \subset \cdots \subset \mathbf{LDIAG} \tag{28}$$

In the next paragraph, we will specially be interested in

$$\mathbf{LBELL} = \mathbf{LDIAG}^{\leq 1} \cap \mathbf{LDIAG}^{\text{GPE}}.$$

4.3.3. **BELL** *and* **LBELL**

The algebras **BELL** and **LBELL** were defined in [39].

The algebra **LBELL** is the intersection $\mathbf{LDIAG}^{\leq 1} \cap \mathbf{LDIAG}^{\mathrm{GPE}}$ and since they are subspaces generated by subsets of **ldiag**, **LBELL** is generated by diagrams that

- are concatenations of one-black-spot-diagrams
- such that the incoming degree of every white spot is one.

Let d_k be the diagram with code $[x_1, x_2, \cdots x_k]$. **LBELL** is generated by concatenations of these diagrams. Indeed, the diagrams d_k are a subalphabet of the free monoid **ldiag** so that they generate a free submonoid which we will denote here **lbell**.

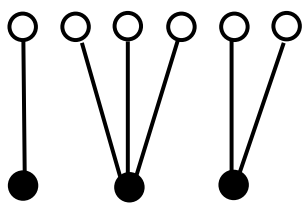

Fig 5. — *An element of* **lbell**, *concatenation* $d_1 d_3 d_2$.

The algebras **LDIAG** and **LBELL** are both enveloping algebras. They are generated by their primitive elements which are in general linear combinations of diagrams and not pure diagrams. For an analysis of "graphic primitive elements" see section (4.3.1).

5. The algebra $\mathbf{LDIAG}(q_c, q_s, q_t)$ (deformed case)

5.1. Counting crossings (q_c) and superpositions (q_s)

The preceding coding is particularly well adapted to the the deformation we want to construct here. The philosophy of the deformed product is expressed by the descriptive formula‖.

$$[d_1|d_2]_{L(q_c,q_s)} = \sum_{\substack{cs(?)\ all\ crossing\ and \\ superpositions\ of\ black\ spots}} q_c^{nc\times weight} q_s^{weight\times weight} cs([d_1|d_2]_L) \quad (29)$$

where

- $q_c, q_s \in \mathbb{C}$ or q_c, q_s formal. These and other cases may be unified by considering the set of coefficients as belonging to a ring K.
- the exponent of $q_c^{nc\times weight}$ is the number of crossings of "what crosses" times its weight

‖ Exact definition of the coefficient $q_c^{nc\times weight} q_s^{weight\times weight}$ is the result of crossing and shifting processes which will be detailed in paragraph (5.2).

- the exponent of $q_s^{weight \times weight}$ is the product of the weights of "what is overlapped"
- $cs()$ are the diagrams obtained from $[d_1|d_2]_L$ by the process of crossing and superposing the black spots of d_2 on to those of d_1, the order and distinguishability of the black spots of d_1 (i.e. d_2) being preserved.

What is striking is that this law is associative. This result will be established after the following paragraph.

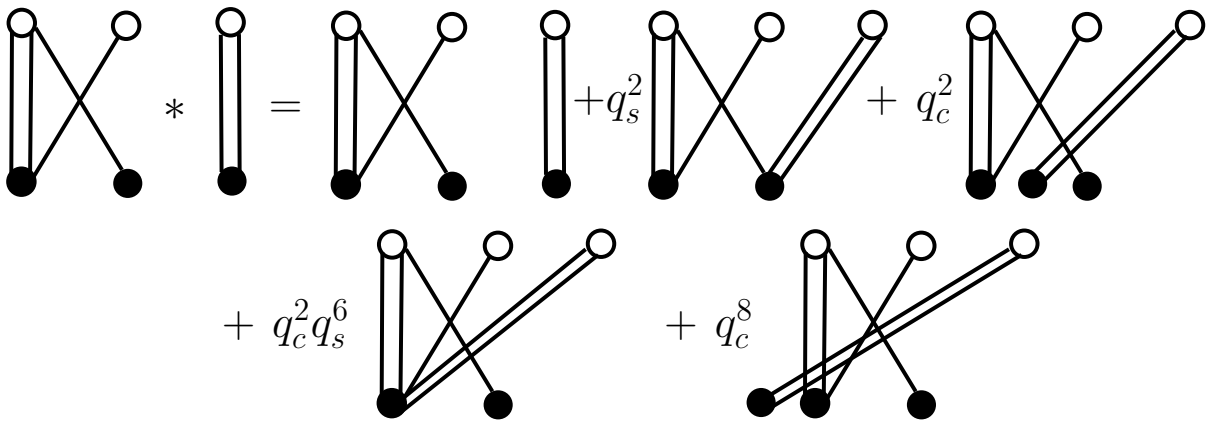

Fig 5. — *Counting crossings and superposings produces an associative law.*

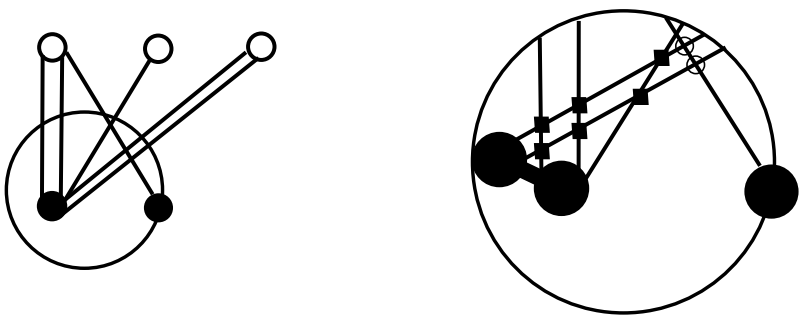

Fig 6. — *Detail of the fourth monomial (with coefficient $q_c^2 q_s^6$), crossings (circles) and superposings (black squares) are counted the same way but with a different variable.*

5.2. Modified laws

- *Twisting*

Proposition 5.1 *Let $A = (A_n)_{n \in \mathbb{N}}$ a graded semigroup and A^* the set of lists (denoted by $[a_1, a_2, \cdots a_k]$) with letters in A.*

For convenience we define the operator $*$ (left append) $A \times A^* \mapsto A^*$ by

$$a * [b_1, b_2, \cdots b_n] := [a, b_1, b_2, \cdots b_n] \tag{30}$$

Let $q_c, q_s \in k$ be two elements in a ring k. We define on $k < A >= k[A^*]$ a new law $\uparrow$ by

$$\begin{aligned} & w \uparrow 1_{A^*} = 1_{A^*} \uparrow w = w \\ & a * u \uparrow b * v = a * (u \uparrow b * v) + q_c^{|a*u||b|} b * (a * u \uparrow v) + q_c^{|u||b|} q_s^{|a||b|} ab * (u \uparrow v) \end{aligned} \tag{31}$$

where the weights ($|x| = n$ if $x \in A_n$) are extended additively to lists by

$$\Big|[a_1, a_2, \cdots, a_k]\Big| = \sum_{i=1}^{k} |a_i|$$

Then the new law $\uparrow$ is graded, associative with 1_{A^*} as unit.

Proof — It suffices to prove the identity $x \uparrow (y \uparrow z) = (x \uparrow y) \uparrow z$; x, y, z being lists (as the two members are trilinear). It is obviously true when one of the factors is the empty list. We show it when the three factors are non-empty (throughout the computation, the law $*$ will have priority over other operators).

$$\begin{aligned} & (a * u \uparrow b * v) \uparrow c * w = \\ & (a * (u \uparrow b * v) + q^{|u||b|} t^{|a||b|} (ab)(u \uparrow v) + q^{|a*u||b|} b(a * u \uparrow v)) \uparrow c * w = \\ & \Big[a * ((u \uparrow b * v) \uparrow c * w) + q^{(|u|+|b*v|)|c|} t^{|a||c|} (ac)((u \uparrow b * v) \uparrow w) \\ & + q^{(|a*u|+|b*v|)|c|} c * (a * (u \uparrow b * v) \uparrow w)\Big] + \\ & \Big[q^{|u||b|} t^{|a||b|} (ab)(u \uparrow v \uparrow c * w) + q^{|u||b|+(|u|+|v|)|c|} t^{|a||b|} t^{(|a|+|b|)|c|} (abc)(u \uparrow v \uparrow \ w) + \\ & q^{|u||b|+(|a*u|+|b*v|)|c|} t^{|a||b|} c(((ab)(u \uparrow v)) \uparrow w)\Big] + \\ & \Big[q^{|a*u||b|} b((a * u \uparrow v) \uparrow c * w) + q^{|a*u||b|+(|a*u|+|v|)|c|} t^{|b||c|} (bc)(au \uparrow v \uparrow w) + \\ & q^{|a*u||b|+(|a*u|+|b*v|)|c|} c(b(a * u \uparrow v) \uparrow w)\Big] \end{aligned} \tag{32}$$

$$\begin{aligned} & a * u \uparrow (b * v \uparrow c * w) = \\ & a * u \uparrow (b * (v \uparrow c * w) + q^{|v||c|} t^{|b||c|} (bc)(v \uparrow w) + q^{|b*v||c|} c(b * v \uparrow w)) = \\ & \Big[a * (u \uparrow b * (v \uparrow c * w)) + q^{|u||b|} t^{|a||b|} (ab)(u \uparrow v \uparrow c * w) + q^{|a*u||b|} b(a * u \uparrow v \uparrow c * w)\Big] + \\ & \Big[q^{|v||c|} t^{|b||c|} a * (u \uparrow (bc)(v \uparrow w)) + q^{|v||c|+|u|(|c|+|b|)} t^{|b||c|+|a|(|b|+|c|)} (abc)(u \uparrow v \uparrow w) + \\ & q^{|v||c|+|a*u|(|b|+|c|)} t^{|b||c|} (bc)(a * u \uparrow v \uparrow w)\Big] + \\ & \Big[q^{|b*v||c|} a * (u \uparrow c(b * v \uparrow w)) + q^{(|u|+|b*v|)|c|} t^{|a||c|} (ac)(u \uparrow b * v \uparrow w) + \\ & q^{(|a*u|+|b*v|)|c|} c * (a * u \uparrow b * v \uparrow w)\Big] \end{aligned} \tag{33}$$

in the second expression, one gathers the three terms which we find first in the square brackets and we get

$$a * (u \uparrow b * (v \uparrow cw)) + q^{|v||c|} t^{|b||c|} a * (u \uparrow (bc) * (v \uparrow w)) + \\ q^{|b*v||c|} a * (u \uparrow c * (b * v \uparrow w)) = a * (u \uparrow b * v \uparrow c * w) \tag{34}$$

in the first expression, one gathers the three terms which we find last in the square brackets and we get

$$q^{(|a*u|+|b*v|)|c|} c * (a * (u \uparrow b * v) \uparrow w) + \\ q^{|u||b|+(|a*u|+|b*v|)|c|} t^{|a||b|} c * (((ab) * (u \uparrow v)) \uparrow w) + \\ q^{|a*u||b|+(|a*u|+|b*v|)|c|} c * (b * (a * u \uparrow v) \uparrow w) = \\ q^{(|au|+|bv|)|c|} c * (a * u \uparrow b * v \uparrow w) \tag{35}$$

and one finds the 7-term expression

$$a * (u \uparrow b * v \uparrow c * w) + q^{|a*u|} b * (a * u \uparrow v \uparrow c * w) + \\ q^{|a*u|+|b*v|} c * (a * u \uparrow b * v \uparrow w) + q^{|u||b|} t^{|a||b|} (ab) * (u \uparrow v \uparrow c * w) + \\ q^{(|u|+|b*v|)|c|} t^{|a||c|} (ac) * (u \uparrow b * v \uparrow w) + \\ q^{|v||c|(|b|+|c|)|au|} t^{|b||c|} (bc) * (a * u \uparrow v \uparrow w) \\ + q^{|v||c|+|u|(|c|+|b|)} t^{|b||c|+|a|(|b|+|c|)} (abc) * (u \uparrow v \uparrow w) \tag{36}$$

□

The framework with diagrams will need another proposition on shifted laws.

• *Shifting*

We begin by the "shifting lemma" [20].

Lemma 5.2 *Let $\mathcal{A}$ be an associative algebra (which law will be denoted by $\star$) and $\mathcal{A} = \oplus_{n \in \mathbb{N}} \mathcal{A}_n$ a decomposition of $\mathcal{A}$ in direct sum. Let $T \in \mathrm{End}(\mathcal{A})$ be an endomorphim of the algebra $\mathcal{A}$. We will denote $T^n = T \circ T \circ \cdots \circ T$ the n-th compositional power of T. We suppose that the shifted law*

$$a \bar{\star} b = a \star T^{\alpha}(b) \tag{37}$$

for $a \in \mathcal{A}_\alpha$ is graded for the decomposition $\mathcal{A} = \oplus_{n \in \mathbb{N}} \mathcal{A}_n$.
Then, if the law $\star$ is associative so is the law $\bar{\star}$.

Remark 5.3 *The hypothesis that the shifted law given by eq.(37) be graded is automatically satisfied if $\mathcal{A} = \oplus_{n \in \mathbb{N}} \mathcal{A}_n$ is a graded algebra and if all the morphisms T_n are of degree 0.*

This lemma will be applied to the decomposition given by $n = sup(Alph(w))$ (the highest index of variables appearing in w) and the morphism given by $T(x_i) = x_{i+1}$.

What do these statements mean for us ?

Here the graded semigroup is $\mathfrak{MON}^+(X)$ and we do not forget the coding arrow $\varphi_{lm} : \mathbf{ldiag} \to (\mathfrak{MON}^+(X))^*$. The image of φ_{lm} is exactly the set of lists of monomials $w = [m_1, m_2, \cdots, m_k]$ such that the set of variables involved $Alph(w)$ is of the form $x_1 \cdots x_l$ (the labelling of the white spots is without hole). By abuse of language we will say that a list of monomials "is in **ldiag**" in this case. It is not difficult to see, from formulas (31,37) that if $w_i,\ i = 1, 2$ are in **ldiag** so are all the factors of $w_1 \bar{\uparrow} w_2$, this defines a new law on $K[\mathbf{ldiag}]$ and this algebra will be called $\mathbf{LDIAG}(q_c, q_s)$. The properties of this algebra will be made precise in the following proposition.

Proposition 5.4 *Let $\mathcal{C}_{ldiag}$ be the subspace of $(K < \mathfrak{MON}^+(\mathbb{X}) >, \bar{\uparrow})$ generated by the codes of the diagrams (i.e. the lists $w \in \mathfrak{MON}^+(\mathbb{X})$ such that $Alph(w)$ is without hole). Then*
i) $(\mathcal{C}_{ldiag}, \bar{\uparrow})$ is a unital subalgebra of $(K < \mathfrak{MON}^+(\mathbb{X}) >, \bar{\uparrow})$
ii) $(\mathcal{C}_{ldiag}, \bar{\uparrow})$ is a free algebra. More precisely, for any diagram decomposed in irreducibles $d = d_1.d_2 \cdots d_k$ let

$$B(d) := \varphi_{lm}(d_1) \bar{\uparrow} \varphi_{lm}(d_2) \cdots \bar{\uparrow} \varphi_{lm}(d_k) \tag{38}$$

then
α) $(B(d))_{d \in ldiag}$ is a basis of $\mathcal{C}_{ldiag}$
β) $B(d_1.d_2) = B(d_1) \bar{\uparrow} B(d_2)$

As $k[\mathbf{ldiag}]$ is isomorphic to $\mathcal{C}_{ldiag}$ as a linear space, we denote $\mathbf{LDIAG}(q_c, q_s)$ the new algebra structure of $k[\mathbf{ldiag}]$ inherited from $\mathcal{C}_{ldiag}$. one has

$$\mathbf{LDIAG}(0, 0) \simeq \mathbf{LDIAG};\ \mathbf{LDIAG}(1, 1) \simeq \mathbf{MQSym} \tag{39}$$

6. Coproducts

We must now define a parametrized (say, by q_t) coproduct such that $(\mathbf{LDIAG}(q_c, q_s), \bar{\uparrow}, 1_{\mathbf{ldiag}}, \Delta_{q_t}, \varepsilon)$ is a graded bialgebra (as in the non-deformed Hopf algebra of [15], the counit ε is just the "constant term" linear form).
We will take advantage of the freeness of $\mathbf{LDIAG}(q_c, q_s)$ through the following lemma.

Lemma 6.1 *Let $\mathbb{Y}$ be an alphabet, k a ring and $k < \mathbb{Y} >= k[\mathbb{Y}^*]$ be the free algebra constructed on $\mathbb{Y}$. For every mapping $\Delta : A \to k < \mathbb{Y} > \otimes k < \mathbb{Y} >$, we denote $\bar{\Delta} : k < \mathbb{Y} > \mapsto k < \mathbb{Y} > \otimes k < \mathbb{Y} >$ its extension as a morphism of algebras ($k < \mathbb{Y} > \otimes k < \mathbb{Y} >$ being endowed with its non-twisted structure of tensor product of algebras). Then, in order to be coassociative, it is necessary and sufficient that*

$$(\bar{\Delta} \otimes I) \circ \Delta \text{ and } (I \otimes \bar{\Delta}) \circ \Delta \tag{40}$$

coincide on $\mathbb{Y}$.

The preceding lemma expresses the fact that, for a free algebra, the "phase space" of the possible coproducts is a linear subspace. This will be transparent in formula (43).

We now consider the structure constants of the coproduct of **MQSym** [18] expressed with respect to the family of free generators

$$\{MS_P\}_{P\in\mathcal{PM}^c}$$

where $\mathcal{PM}^c$ is the set of connex packed matrices (similarly, $\mathcal{PM}$ is the set of packed matrices).

$$\Delta_{\mathbf{MQSym}}(MS_P) = \sum_{Q,R\in\mathcal{PM}} \alpha_P^{Q,R}\ MS_Q \otimes MS_R \tag{41}$$

For the irreducible diagram d, we set

$$\Delta_1(d) = \sum_{d_1,d_2\in irr(\mathbf{ldiag})} \alpha_{\varphi_{lm}(d)}^{\varphi_{lm}(d_1),\varphi_{lm}(d_2)} d_1 \otimes d_2 \tag{42}$$

and $\Delta_0(d) = \Delta_{WS}(d)$. Then proposition (6.1) proves that, for $q_t \in \{0,1\}$

$$\Delta_t = \overline{(1-q_t)\Delta_0 + q_t\Delta_1} \tag{43}$$

is a coproduct of graded bialgebra for $(\mathbf{LDIAG}(q_c,q_s),\uparrow,1_{\mathbf{ldiag}})$.

We sum up the results

Proposition 6.2 *i) With the operations defined above, q_c, q_s complex or formal and q_t boolean ($q_t \in \{0,1\}$),*

$$\mathbf{LDIAG}(q_c,q_s,q_t) := \big(\mathbf{LDIAG}(q_c,q_s),\bar{\uparrow},1_{\mathbf{ldiag}},\Delta_{q_t},\varepsilon\big)$$

is a Hopf algebra.
ii) At parameters $(0,0,0)$, one has $\mathbf{LDIAG}(0,0,0) \simeq \mathbf{LDIAG}$
iii) At parameters $(1,1,1)$, one has $\mathbf{LDIAG}(1,1,1) \simeq \mathbf{MQSym}$

7. More on $\mathbf{LDIAG}(q_c,q_s,q_t)$: structure, images and the link with Euler-Zagier sums

It has been proved recently that $\mathbf{LDIAG}(q_c,q_s,q_t)$ is a *tridendriform* Hopf Algebra [25] and that $\mathbf{LDIAG}(1,q_s,q_t)$ is a homomorphic image of the algebra of planar decorated trees of Foissy [24]. *Bidendriformity* of the algebra $\mathbf{LDIAG}(q_c,q_s)$ can also be established through a bi-word realization providing yet another (statistical) interpretation of the (q_c,q_s) deformation [20].
We will now make clear the relations between the (q_c,q_s) deformation and Euler-Zagier sums.
According the notation of [31], one has

$$\zeta(s_1,\cdots,s_n;\sigma_1,\cdots,\sigma_n) = \sum_{0<i_1<\cdots<i_n} \frac{\sigma_1^{i_1}\cdots\sigma_n^{i_n}}{i_1^{s_1}\cdots i_n^{s_n}} \tag{44}$$

with $\sigma_i \in \{-1,1\}$ and $s_1 > 1$ if $\sigma_1 = 1$. Here we are more interested in the multiplication mechanism, so we extend the notation to formal variables and use, for indices, the bi-word notation. Hence

$$\zeta_{FP}\begin{pmatrix} m_1 & \cdots & m_n \\ s_1 & \cdots & s_n \end{pmatrix} = \sum_{0<i_1<\cdots<i_n} \frac{m_1^{i_1}\cdots m_n^{i_n}}{i_1^{s_1}\cdots i_n^{s_n}}. \tag{45}$$

We remark that the indices are taken as words (i.e. lists) with variables located in the semigroup $\mathfrak{MON}(Z) \times \mathbb{N}^+$ with $Z = \{z_i\}_{i\geq 1}$. The set of these functions is closed under multplication and will be called below $FP(Z)$, *formal polyzeta functions in the variables* Z. Hence, the multiplication of these sums fits in the hypotheses of Proposition (5.1) with $q_c = q_s = 1$ (quasi-shuffle in [12]). From this, we deduce an arrow

$$LDIAG(1,1) \to FP(Z). \tag{46}$$

More precisely, if d is a diagram with code $[m_1, m_2 \cdots, m_p]$ we make correspond

$$\zeta_{FP}\begin{pmatrix} m_1 & \cdots & m_n \\ \deg(m_1) & \cdots & \deg(m_n) \end{pmatrix} \tag{47}$$

where $\deg(m_i)$ is the total degree of m_i. We will denote $\zeta_{D2FP}(d)$ this value (47).
One has

$$\zeta_{D2FP}(d_1)\zeta_{D2FP}(d_2) = \zeta_{D2FP}(d_1 \uparrow_{11} d_2) \tag{48}$$

the law $\uparrow_{11}$ being unshifted and specialized to $(q_c, q_s) = (1,1)$.
When restricted to "convergent" diagrams (i.e. diagrams with $\deg(m_1) \geq 2$ which form a subalgebra of $\mathbf{LDIAG}_u(q_c, q_s)$) and specializing all the variables to 1, we recover the "usual" Euler-Zagier sums by just counting the outgoing degrees of the black spots and the arrow of (46) becomes

$$d \to \zeta(\deg(m_1), \cdots, \deg(m_n)) \tag{49}$$

(usual Euler-Zagier sums). Denoting the last (49) value $\zeta_{D2EZ}(d)$, one has

$$\zeta_{D2EZ}(d_1)\zeta_{D2EZ}(d_2) = \zeta_{D2EZ}(d_1 \uparrow_{11} d_2) \tag{50}$$

8. Concluding remarks

For a diagram d with r black spots, the code $[m_1, m_2, \cdots, m_r]$ can be temporarily seen as a "vector of coordinates" for the given diagram, but we prefer to stick to the structure of lists as, firstly, the dimension of the vector varies with the diagram and secondly, we have to concatenate the codes. The coordinate functions of the diagram d are therefore the family $(a_i)_{i>0}$ defined by $a_i(d) = m_i$ for $i \leq r$ and $a_i(d) = 0$ for $i > r$. From this perspective the "q_t" of our three parameter deformation is a quantization in the sense of Moyal's deformed products [1] on the algebra of coordinate functions (but without the first order condition; see the introduction of [13]), by the formula

$$a_{i_1} * a_{i_1} \cdots * a_{i_k}(d) = \mu(a_{i_1} \otimes a_{i_1} \otimes \cdots \otimes a_{i_k}(\Delta_{q_t}^{[k]}(d))) \tag{51}$$

where μ is the ordinary multiplication of polynomials.
The crossing parameter q_c is also a quantization parameter as, for $q_s = 0$, one has

$$code(d_1 * d_2) = code(d_1) \sqcup_{q_c} T(code(d_2)) \tag{52}$$

where T is a suitable translation of the variables and $\sqcup_{q_c}$ is the quantum shuffle [35] for the braiding on $V = \mathbb{C}[x_i;\ i \geq 1]$ defined by

$$B(x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k} \otimes y_{j_1}^{\beta_1} y_{j_2}^{\beta_2} \cdots y_{j_l}^{\beta_l}) = q_c^{(\sum \alpha_i)(\sum \beta_j)}\ y_{j_1}^{\beta_1} y_{j_2}^{\beta_2} \cdots y_{j_l}^{\beta_l} \otimes x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k} \tag{53}$$

Let us add that q_s and q_c are of different nature as q_s is the coefficient of a perturbation of the shuffle product (better seen on the coproduct). This kind of perturbation occurs in various domains as : computer science by means of the infiltration product introduced by Ochsenschläger [33] (see also [17] and [16]), algebra of the Euler-Zagier sums [27] and noncommutative symmetric functions [18]. The mathematics of this dual aspect is of geometrical nature and will be developped in [19].

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