If $f \in K^{rat}\langle\langle A \rangle\rangle$, it exists a morphism of monoids $\mu : A^* \to K^{n \times n}$ (square matrices of size $n \times n$), a row λ in $k^{1 \times n}$ and a column ξ in $k^{n \times 1}$ such that, for all word w in A^*, $f(w)=\lambda\mu(w)\xi$. Then

$$f = \sum_{w \in A^*} f(w)w = \sum_{w \in A^*} \lambda\mu(w)\xi\, w = \lambda\Big(\sum_{w \in A^*} \mu(w)w\Big)\xi =$$

$$\lambda\Big(\sum_{w \in A^*} \mu(w)w\Big)\xi = \lambda\Big(\sum_{n \geq 0} \sum_{|w|=n} \mu(w)w\Big)\xi$$

But, as words and scalars commute (it is so by construction of the convolution algebra $K^{n \times n}\langle\langle A \rangle\rangle$), one has

$$\sum_{n \geq 0} \sum_{|w|=n} \mu(w)w = \sum_{n \geq 0} \Big(\sum_{a \in A} \mu(a)a\Big)^n = \Big(\sum_{a \in A} \mu(a)a\Big)^*$$

hence

$$f = \lambda\Big(\sum_{a \in A} \mu(a)a\Big)^* \xi$$

where the “star” stands for the sum of the geometric series.