Hopf Algebras of Diagrams and Deformations

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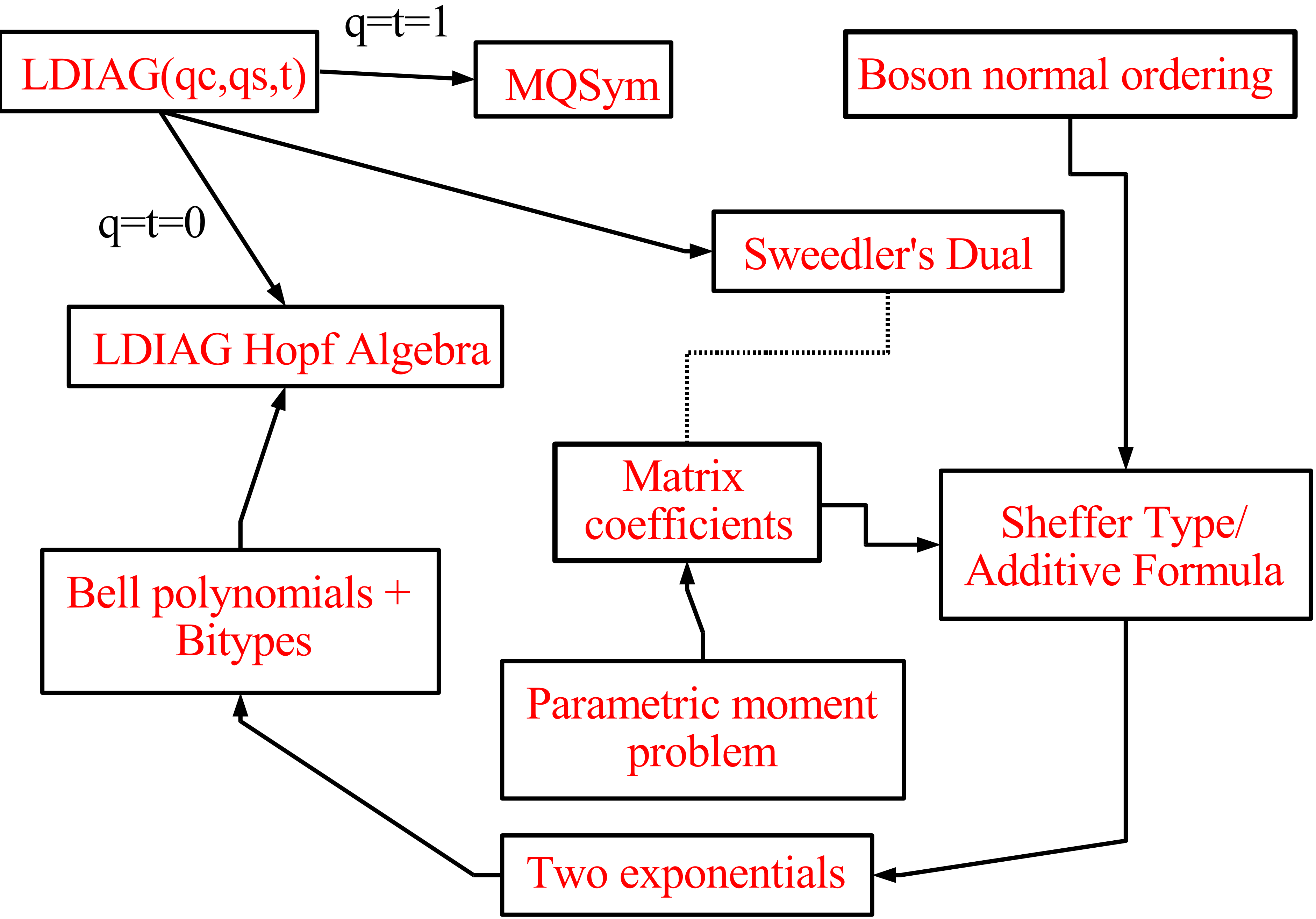

LDIAG(qc,qs,t)
q=t=1
MQSym
Boson normal ordering
q=t=0
Sweedler's Dual
LDIAG Hopf Algebra
Matrix coefficients
Sheffer Type/ Additive Formula
Bell polynomials + Bitypes
Parametric moment problem
Two exponentials

Warning :

What follows is a rather descriptive account of what has been done on the following subjects :

- Normal order of boson strings, substitutions, one-parameter groups (one exponential)

- Product formula and diagrams

- Hopf algebras of these diagrams and deformations

- Sweedlers duals and automata theory

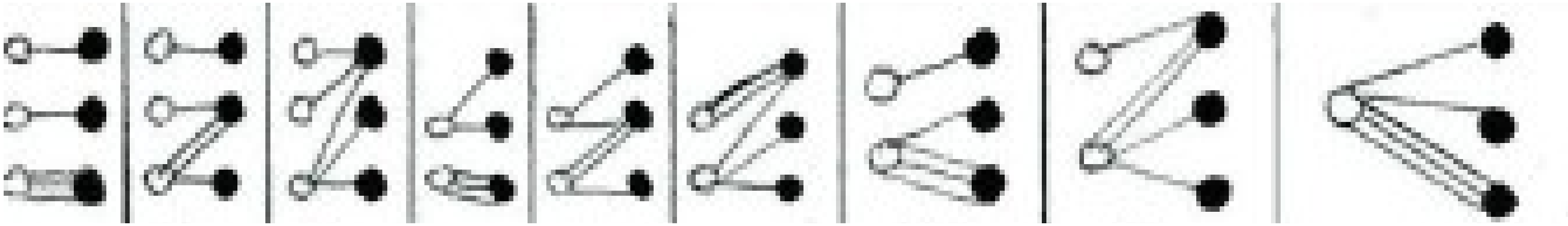
for details (necessarily missing here) **do not hesitate** to contact me (lunch, coffee-pause, email etc)

About the LDIAG Hopf algebra

In a relatively recent paper Bender, Brody and Meister (*) introduce a special Field Theory described by a product formula (a kind of Hadamard product for two exponential generating functions - EGF) in the purpose of proving that any sequence of numbers could be described by a suitable set of rules applied to some type of Feynman graphs (see third Part of this talk).

These graphs label monomials and are obtained in the case of special interest when the two EGF have a constant term equal to unity.

Bender, C.M, Brody, D.C. and Meister,
Quantum field theory of partitions, J. Math. Phys. Vol 40 (1999)



Some 5-line diagrams

If we write these functions as exponentials, we are led to witness a surprising interplay between the following aspects: **algebra** (of normal forms or of the exponential formula), **geometry** (of one-parameter groups of transformations and their conjugates) and **analysis** (parametric Stieltjes moment problem and convolution of kernels).

A single exponential

The normal ordering problem goes as follows.

- Weyl (two-dimensional) algebra defined as

$$< a^+, a \; ; [a \, , a^+]=1 > \mid aa^+ \text{--->} a^+a + 1$$

- Known to have no (faithful) representation by bounded operators in a Banach space.

There are many « combinatorial » (faithful) representations by operators. The most famous one is the Bargmann-Fock representation

$$a \text{--->} d/dx \; ; \; a+ \text{--->} x$$

where a has degree -1 and a^+ has degree 1.

A typical element in the Weyl algebra is of the form (normal form).

$$\Omega = \sum_{k,l \geq 0} c(k,l)(a^{+})^{k} a^{l}$$

When Ω is a single monomial, a word i.e. a product of generators a^{+}, a, there is solution to the normal ordering problem (and thus, by linearity to the general problem) using rook numbers.

Today, we will be interested with the use of matrix coefficients in two instances :

normal ordering --> infinite matrices (--> moments)

finite representations --> Sweedler's dual and automata

A word (boson string) and more generally an homogeneous operator (for the grading where a has degree -1 and a^+ has degree 1) of degree e reads

$$\Omega = \sum_{\substack{k,l \geq 0 \\ k-l=e}} c(k,l)(a^+)^k a^l$$

Due to the symmetry of the Weyl algebra, we can suppose, with no loss of generality that e≥0. For homogeneous operators one can define generalized Stirling numbers (GSN) by

$$\Omega^n = (a^+)^{ne} \sum_{k \geq 0} S_\Omega(n,k)(a^+)^k a^k \quad \text{(Eq1)}$$

The case of a pure string is of special interest for physics and can be solved combinatorally. The recipe, for a string W is the following:

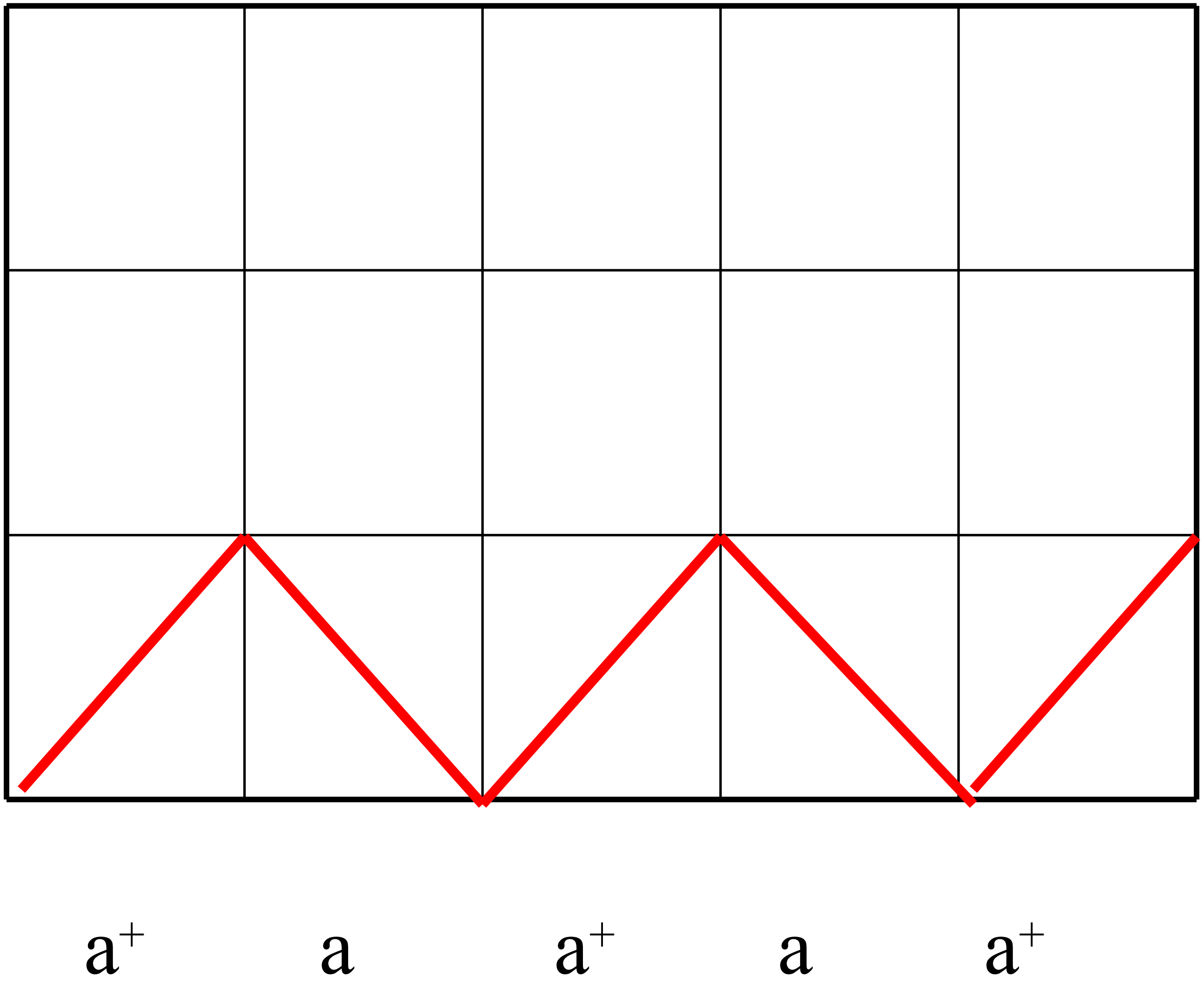
- associate a path with north east steps for every a^+ and a south east step for every a.
- construct the Ferrers diagram B over this path

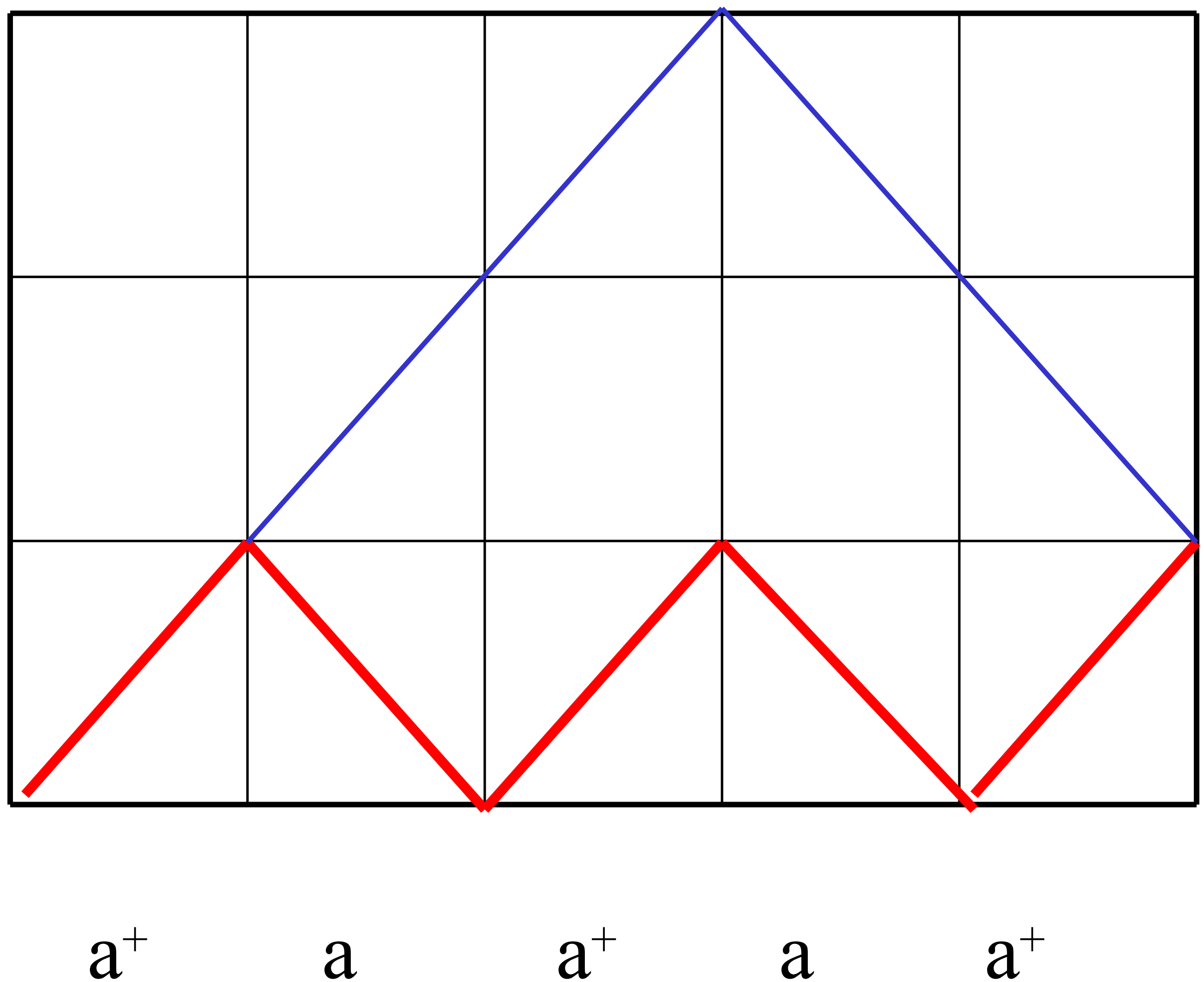
The normal form of W is

$$W = \sum_{k \geq 0} R(B,k)(a^+)^{r-k} a^{s-k}$$

where R(B,k) is the k-th rook number of the board B.

Example with $\Omega = a^+ a\ a^+ a\ a^+$




a+
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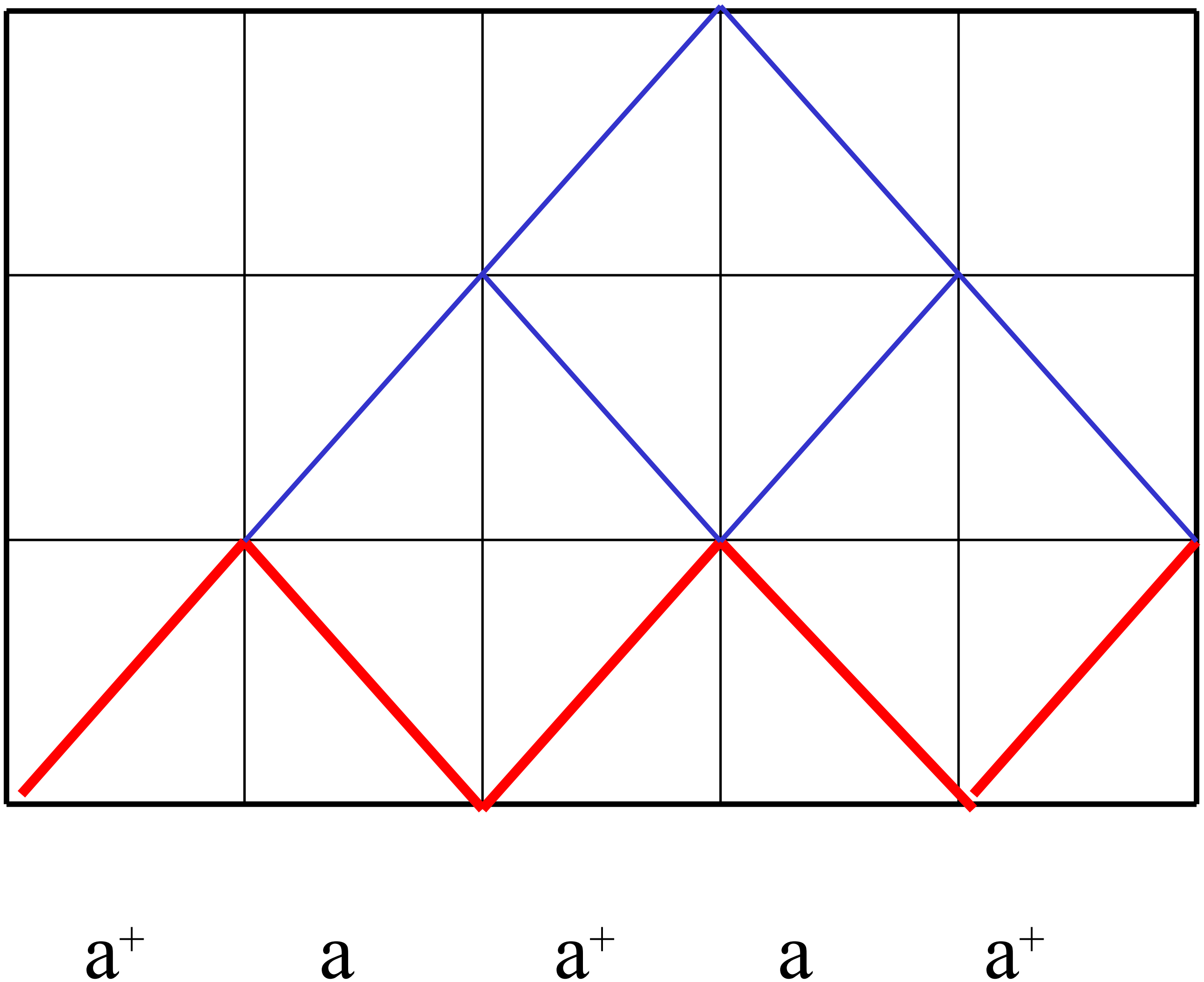
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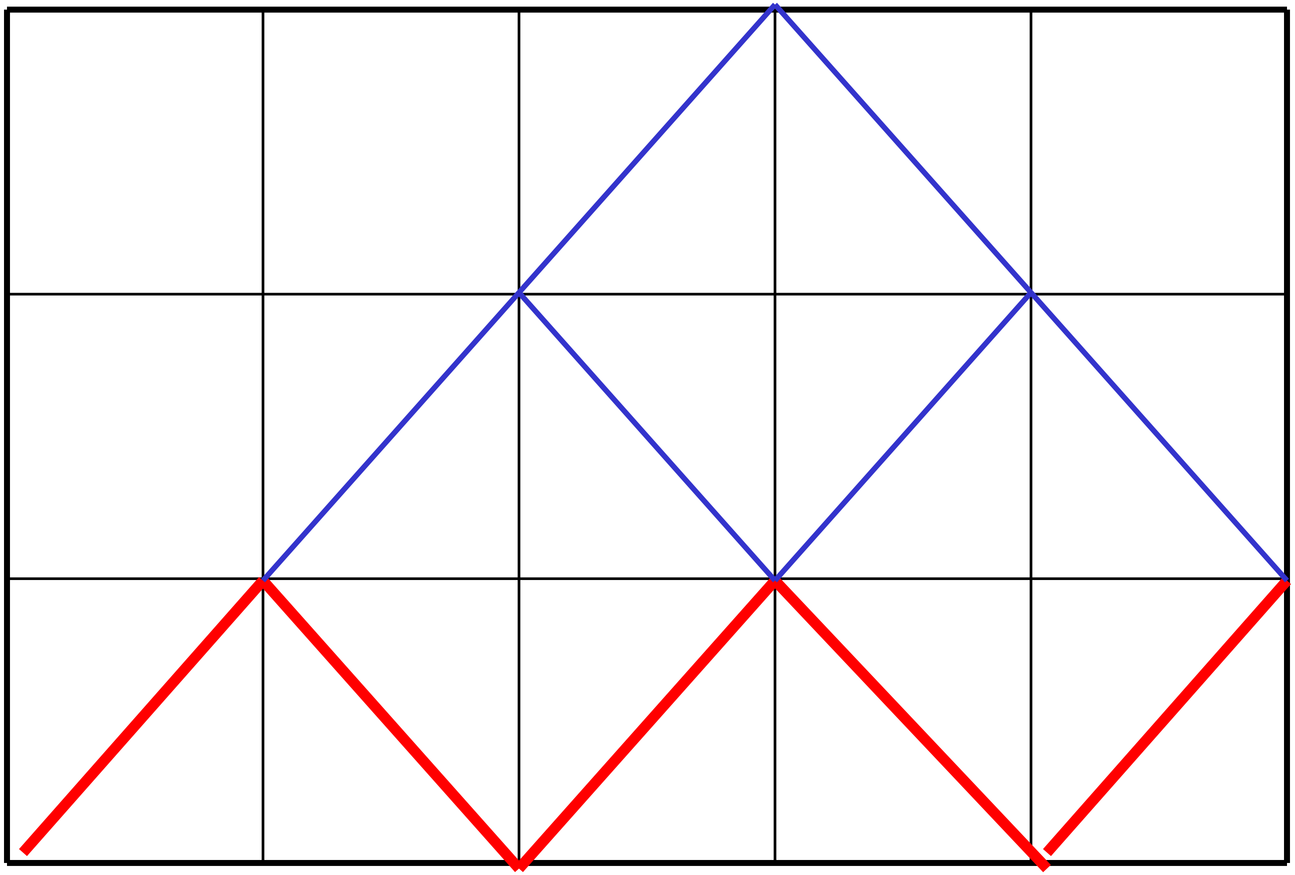

$$a^+aa^+aa^+ = 1\ a^+a^+a^+aa + 3\ a^+a^+a + 1\ a^+$$

In particular, the boson string $w=[(a^+)^r a^s]$ was considered by Penson, Solomon, Blasiak and al. In this case the GSN will be denoted by $S_{r,s}(n,k)$ and, due to the particular form of W, one has

$$\left[(a^+)^r a^s\right]^n = (a^+)^{n(r-s)} \sum_{k=s}^{ns} S_{r,s}(n,k)(a^+)^k a^k$$

For $w = a^+a$, one gets the usual matrix of Stirling numbers of the second kind.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 7 & 6 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 15 & 25 & 10 & 1 & 0 & \cdots \\ 0 & 1 & 31 & 90 & 65 & 15 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (3)$$

For $w = a^+ a a^+$, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 4 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 6 & 18 & 9 & 1 & 0 & 0 & 0 & \cdots \\ 24 & 96 & 72 & 16 & 1 & 0 & 0 & \cdots \\ 120 & 600 & 600 & 200 & 25 & 1 & 0 & \cdots \\ 720 & 4320 & 5400 & 2400 & 450 & 36 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{4}$$

For $w = a^+ a a a^+ a^+$, one gets

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 12 & 60 & 54 & 14 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 144 & 1296 & 2232 & 1296 & 306 & 30 & 1 & 0 & 0 & \cdots \\ 2880 & 40320 & 109440 & 105120 & 45000 & 9504 & 1016 & 52 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{5}$$

Setting

$$B_{r,s}(n,y) = \sum_{k=0}^{\infty} S_{r,s}(n,k) y^k$$

for the generating polynomials of the lines of the generalized Stirling matrix, one has the formulas

$$B_{r,s}(n,y) = \sum_{k=s}^{ns} S_{r,s}(n,k) y^k$$
$$= e^{-y} \sum_{k=s}^{\infty} \frac{1}{k!} \prod_{j=1}^{n} [(k+(j-1)(r-s))(k+(j-1)(r-s)-1)$$
$$\cdots (k+(j-1)(r-s)-s+1)] y^k.$$

... and, when s=1, the EGF of these polynomials is an exponential which gives an additive formula in the variable y (see the paper One-parameter Groups or below)

$$e^{y(e^x-1)} = \sum_{n=0}^{\infty}\left(\sum_{k=1}^{n} S_{1,1}(n,k)y^k\right)\frac{x^n}{n!}$$

and

$$\exp\left[y\left(\frac{1}{\sqrt[r-1]{1-(r-1)x^{r-1}}}-1\right)\right] = \sum_{n=0}^{\infty}\left(\sum_{k=1}^{n} S_{r,1}(n,k)y^k\right)\frac{x^n}{n!} \qquad r=2,3,\ldots.$$

For which, we have Dobi'nski-type relations

$$\frac{1}{e^y}\sum_{k=1}^{\infty}\frac{k^n}{k!}y^k = \sum_{k=1}^{n} S_{1,1}(n,k)y^k \qquad n=0,1,\ldots$$

$$\frac{(r-1)^n}{e^y}\sum_{k=1}^{\infty}\frac{\Gamma\left(n+\frac{k}{r-1}\right)}{k!\,\Gamma\left(\frac{k}{r-1}\right)}y^k = \sum_{k=1}^{n} S_{r,1}(n,k)y^k \qquad n=1,2,\ldots.$$

The matrices of coefficients for expressions with only a single « a » turn to be matrices of substitutions with prefunction factor. This is, in fact, due to a conjugacy phenomenon.

Conjugacy trick: The one-parameter groups associated with the operators of type $\Omega=q(x)d/dx+v(x)$ are conjugate to vector fields on the line.

Let $u_2=\exp(\int(v/q))$ and $u_1=q/u_2$ then

$u_1u_2=q$; $u_1u'_2=v$ and the operator $q(a^+)a+v(a^+)$

reads, via the Bargmann-Fock correspondence

$$(u_2u_1)d/dx+ u_1u'_2=u_1(u'_2 + u_2d/dx)= u_1d/dx\, u_2 =$$

$$1/u_2\,(u_1\, u_2\, d/dx\,)\, u_2$$

Which is conjugate to a vector field and integrates as a substitution with prefunction factor.

Example: The expression $\Omega = a^{+2}a\ a^{+} + a^{+}a\ a^{+2}$ above corresponds to the operator (the line below ω is in form q(x)d/dx+v(x))

$$\omega = x^2 \frac{d}{dx} x + x \frac{d}{dx} x^2 =$$

$$2x^3 \frac{d}{dx} + 3x^2 = x^{-3/2}(2x^3 \frac{d}{dx})x^{3/2} = x^{-3/2}(\phi)x^{3/2}$$

Now, ϕ is a vector field and its one-parameter group acts by a one parameter group of substitutions. We can compute the action by another conjugacy trick which amounts to straightening ϕ to a constant field.

Thus set

$$\exp(\lambda\,\phi)[f(x)]=f(u^{-1}(u(x)+\lambda)) \text{ for some } u \ldots$$

By differentiation w.r.t. λ at ($\lambda=0$) one gets

$$u'=1/(2x^3)\ ;\ u=-1/(4x^2)\ ;\ u^{-1}(y)=(-4y)^{-1/2}$$

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One parameter group by f(v(u(x)+λ)); v is reciprocal of u

```
> T1(lambda,x):=(-4*(-1/(4*x^2)+lambda))^(-1/2);
```

$$\mathrm{T1}(\lambda, x) := \frac{1}{\sqrt{\frac{1}{x^2} - 4\lambda}}$$

We suppose x>0

```
> T1:=(lambda,x) -> x/((1-4*lambda*x^2)^(1/2));
```

$$T1 := (\lambda, x) \rightarrow \frac{x}{\sqrt{1 - 4\lambda x^2}}$$

Checking the tangent vector

```
> subs(lambda=0,diff(T1(lambda,x),lambda));
```

$$2x^3$$

... and the one-parameter group property

```
> simplify(T1(lambda1,T1(lambda2,x))^2-T1(lambda1+lambda2,x)^2);
```

$$0$$

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In view of the conjugacy established previously we have that exp(λ ω)[f(x)] acts as

$$U_\lambda(f) = x^{-\frac{3}{2}} f(s_\lambda(x)).(s_\lambda(x))^{\frac{3}{2}}$$

$$= \sqrt[4]{\frac{1}{(1-4\lambda x^2)^3}}\, f\left(\sqrt{\frac{x^2}{1-4\lambda x^2}}\right)$$

which explains the prefactor. Again we can check by computation that the composition $(U_\mu U_\lambda)$ amounts to simple addition of parameters !!

Now suppose that exp(λ ω) is in normal form.

In view of Eq1 (slide 24) we must have

$$\exp(\lambda\omega) = \sum_{n\geq 0} \frac{\lambda^n \omega^n}{n!} = \sum_{n\geq 0} \frac{\lambda^n}{n!} x^{ne} \sum_{k=0}^{ne} S_\omega(n,k) x^k \left(\frac{d}{dx}\right)^k$$

Hence, introducing the eigenfunctions of the derivative (a method which is equivalent to the computation with coherent states) one can recover the mixed generating series of $S_\omega(n,k)$ from the knowledge of the one-parameter group of transformations.

$$\exp(\lambda\omega)\left[e^{yx}\right] = \left(\sum_{n\geq 0}\frac{\lambda^n}{n!}x^{ne}\sum_{k=0}^{ne}S_\omega(n,k)x^k y^k\right)e^{yx}$$

Thus, one can state

Proposition (*): With the definitions introduced, the following conditions are equivalent (where f → U_λ[f] is the one-parameter group exp(λω)).

$$1. \sum_{n,k\geq 0} S_\omega(n,k)\frac{x^n}{n!}y^k = g(x)e^{y\phi(x)}$$

$$2. U_\lambda[f](x) = g(\lambda x^e)f(x(1+\phi(\lambda x^e)))$$

Remark : Condition 1 is known as saying that S_ω(n,k) is of « Sheffer » type.

Example : With $\Omega = a^{+2}a\, a^{+} + a^{+}a\, a^{+2}$ (Slide 11), we had e=2 and

$$U_\lambda[f](x) = \sqrt[4]{\frac{1}{(1-4\lambda x^2)^3}}\, f\left(\sqrt[2]{\frac{x^2}{1-4\lambda x^2}}\right)$$

Then, applying the preceding correspondence one gets

$$\sum_{n,k\geq 0} S_\omega(n,k)\frac{x^n}{n!}y^k = \sqrt[4]{\frac{1}{(1-4x)^3}}\, e^{y\left(\sqrt{\frac{1}{1-4x}}-1\right)} =$$

$$\sqrt[4]{\frac{1}{(1-4x)^3}}\, e^{y\left(\sum_{n\geq 1} c_n x^n\right)}$$

Where $c_n = \binom{2n}{n}$ are the central binomial coefficients.

```
> E1:=(1/((1-4*x)^3))^(1/4)*exp(y*(1/(1-4*x)^(1/2)-1));
```

$$E1 := \left(\frac{1}{(1-4x)^3}\right)^{(1/4)} \mathbf{e}^{\left(y\left(\frac{1}{\sqrt{1-4x}}-1\right)\right)}$$

```
> T1:=taylor(E1,x=0,6);
```

$$T1 := 1 + (2y+3)x + \left(12y + 2y^2 + \frac{21}{2}\right)x^2 + \left(59y + 18y^2 + \frac{4}{3}y^3 + \frac{77}{2}\right)x^3 + \left(270y + 115y^2 + 16y^3 + \frac{2}{3}y^4 + \frac{1155}{8}\right)x^4 + \left(\frac{4389}{8} + \frac{4767}{4}y + 637y^2 + 126y^3 + 10y^4 + \frac{4}{15}y^5\right)x^5 + \mathrm{O}(x^6)$$

```
> seq([sort(coeff(T1,x,n)*n!)],n=1..5);
```

$$[2y+3], [4y^2+24y+21], [8y^3+108y^2+354y+231], [16y^4+384y^3+2760y^2+6480y+3465], [32y^5+1200y^4+15120y^3+76440y^2+143010y+65835]$$

```
> M1:=matrix(5,5,(n,k)->coeff(coeff(T1,x,n)*n!,y,k));
```

$$M1 := \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 24 & 4 & 0 & 0 & 0 \\ 354 & 108 & 8 & 0 & 0 \\ 6480 & 2760 & 384 & 16 & 0 \\ 143010 & 76440 & 15120 & 1200 & 32 \end{bmatrix}$$

Autre exemple : transformation idempotente.

I(n,k)=nombre d'endofonctions de [1..n] idempotentes avec k points fixes.

$$\sum_{n,k} I(n,k)\frac{x^n}{n!}y^k = \sum_{n\geq 0}\sum_{k\leq n}\binom{n}{k}k^{(n-k)}\frac{x^n}{n!}y^k =$$

$$\sum_{k\leq n}\sum_{n\geq k}\binom{n}{k}k^{(n-k)}\frac{x^{(n-k)}x^k}{n!}y^k = e^{yxe^x}$$

ceci est un cas particulier de la
« formule exponentielle »

Maple 9 - [calc1.mws - [Server 1]]

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```
> g1:=exp(y*x*exp(x));d1:=taylor(g1,x=0,7);
```

$$g1 := e^{(y\,x\,e^{x})}$$

$$d1 := 1 + y\,x + \left(y + \frac{1}{2}y^2\right)x^2 + \left(\frac{1}{2}y + y^2 + \frac{1}{6}y^3\right)x^3 + \left(\frac{1}{6}y + y^2 + \frac{1}{2}y^3 + \frac{1}{24}y^4\right)x^4 + \left(\frac{1}{24}y + \frac{2}{3}y^2 + \frac{3}{4}y^3 + \frac{1}{6}y^4 + \frac{1}{120}y^5\right)x^5 + \left(\frac{1}{120}y + \frac{1}{3}y^2 + \frac{3}{4}y^3 + \frac{1}{3}y^4 + \frac{1}{24}y^5 + \frac{1}{720}y^6\right)x^6 + O(x^7)$$

```
> matrix(7,7,(i,j)->(i-1)!*coeff(coeff(d1,x,i-1),y,j-1));
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & 1 & 0 & 0 & 0 \\ 0 & 4 & 24 & 12 & 1 & 0 & 0 \\ 0 & 5 & 80 & 90 & 20 & 1 & 0 \\ 0 & 6 & 240 & 540 & 240 & 30 & 1 \end{bmatrix}$$

For these one-parameter groups and conjugates of vector fields

G. H. E. Duchamp, K.A. Penson, A.I. Solomon, A. Horzela and P. Blasiak,

One-parameter groups and combinatorial physics,

Third International Workshop on Contemporary Problems in Mathematical Physics (COPROMAPH3), Porto-Novo (Benin), November 2003. arXiv : quant-ph/0401126.

For the Sheffer-type sequences and coherent states

P Blasiak, A Horzela , K A Penson, G H E Duchamp and A I Solomon,

Boson Normal Ordering via Substitutions and Sheffer-type Polynomials,

(Published in Physics Letters A)

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Substitutions and the « connected graph theorem» « exponential formula»

A great, powerful and celebrated result:
(For certain classes of graphs)
If $C(x)$ is the EGF of CONNECTED graphs, then
exp(C(x)) is the EGF of ALL (non void) graphs.
(Touchard, Uhlenbeck, Mayer,...)

This implies that the matrix

M(n,k)=number of graphs with n vertices and having k connected components

is the matrix of a substitution (like $S_{\Omega}(n,k)$ previously but without prefactor).

Maple 9 - [Untitled (1) - [Server 1]]

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```
> f1:=exp(y*(x+x^2/2)*exp(x));d1:=taylor(f1,x=0,7);
```

$$f1 := e^{(y\,(x+1/2\,x^2)\,e^x)}$$

$$d1 := 1 + y\,x + \left(\frac{3}{2}y + \frac{1}{2}y^2\right)x^2 + \left(y + \frac{3}{2}y^2 + \frac{1}{6}y^3\right)x^3 + \left(\frac{5}{12}y + \frac{17}{8}y^2 + \frac{3}{4}y^3 + \frac{1}{24}y^4\right)x^4 + \left(\frac{1}{8}y + \frac{23}{12}y^2 + \frac{13}{8}y^3 + \frac{1}{4}y^4 + \frac{1}{120}y^5\right)x^5 + \left(\frac{7}{240}y + \frac{5}{4}y^2 + \frac{109}{48}y^3 + \frac{35}{48}y^4 + \frac{1}{16}y^5 + \frac{1}{720}y^6\right)x^6 + O(x^7)$$

```
> matrix(7,7,(i,j)->(i-1)!*coeff(coeff(d1,x,i-1),y,j-1));
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 6 & 9 & 1 & 0 & 0 & 0 \\ 0 & 10 & 51 & 18 & 1 & 0 & 0 \\ 0 & 15 & 230 & 195 & 30 & 1 & 0 \\ 0 & 21 & 900 & 1635 & 525 & 45 & 1 \end{bmatrix}$$

One can prove, using a Zariski-like argument, that, if *M* is such a matrix (with identity diagonal) then, all its powers (positive, negative and fractional) are substitution matrices and form a one-parameter group of substitutions, thus coming from a vector field on the line which could (in theory) be computed.

For example, to begin with the Stirling substitution $z \to e^z - 1$. We know that there is a unique one-parameter group of substitutions $s_\lambda(z)$ such that, for λ integer, one has the value ($s_2(z)$ ←→ partition of partitions)

$$s_2(z) = e^{(e^z - 1)} - 1;\ s_3(z) = e^{(e^{(e^z - 1)} - 1)} - 1;\ s_{-1}(z) = log(1 + z)$$

But we have no nice description of this group nor of the vector field generating it.

Product formula

The Hadamard product of two sequences

$$(a_n)_{n\geq 0} \quad (b_n)_{n\geq 0}$$

is given by the pointwise product

$$(a_n b_n)_{n\geq 0}$$

We can at once transfer this law on EGFs by

$$F=\sum_{n\geq 0} a_n \frac{y^n}{n!};\ G=\sum_{m\geq 0} b_m \frac{y^m}{m!};\ \mathcal{H}(F,G):=\sum_{n\geq 0} a_n b_n \frac{y^n}{n!}$$

but, here, as $\frac{(y\frac{d}{dx})^n}{n!}\frac{x^m}{m!}\Big|_{x=0}=\delta_{mn}\frac{y^n}{n!}$

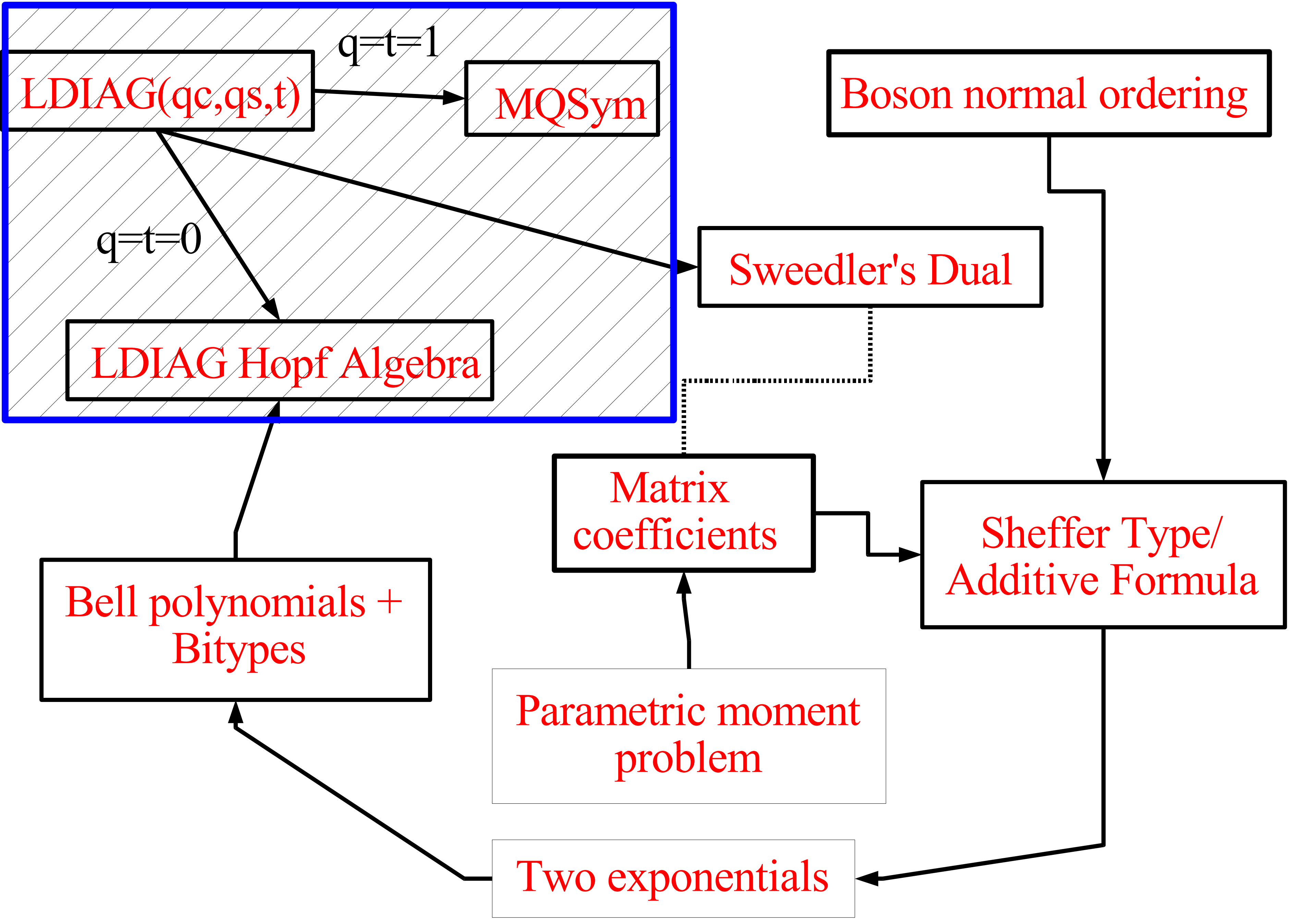
we get $\mathcal{H}(F,G)=F(y\frac{d}{dx})G(x)|_{x=0}$

- Writing F and G as free exponentials we shall see that these diagrams are in fact labelling monomials. We are then in position of imposing two types of rule:

    - On the diagrams (Selection rules) : on the outgoing, ingoing degrees, total or partial weights.
    - On the set of diagrams (Composition and Decomposition rules) : product and coproduct of diagram(s)

- This leads to structures of Hopf algebras for spaces freely generated by the two sorts of diagrams (labelled and unlabelled).

Labelled diagrams generate the space of Matrix QuasiSymmetric Functions, we thus obtain a new Hopf algebra structure on this space.



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Construction of the Hopf algebra LDIAG

How these diagrams arise and which data structures are around them

Let F, G be two EGFs.

$$F=\sum_{n\geq 0} a_n \frac{y^n}{n!};\; G=\sum_{m\geq 0} b_m \frac{y^m}{m!};\; \mathcal{H}(F,G):=\sum_{n\geq 0} a_n b_n \frac{y^n}{n!}$$

$$\mathcal{H}(F,G)=F(y\frac{d}{dx})G(x)|_{x=0}$$

Called « product formula » in the QFTP of Bender, Brody and Meister.

In case *F(0)=G(0)=1*, one can set

$$F(y) = exp(\sum_{n\geq 1} L_n \frac{y^n}{n!}) \quad G(x) = exp(\sum_{n\geq 1} V_m \frac{x^m}{m!})$$

and then,

$$\mathcal{H}(F,G) = F(y\frac{d}{dx})G(x)|_{x=0} =$$

$$\sum_{n\geq 0} \frac{y^n}{n!} \sum_{|\alpha|=|\beta|=n} numpart(\alpha)numpart(\beta)\mathbb{L}^{\alpha}\mathbb{V}^{\beta}$$

with $\alpha, \beta \in \mathbb{N}^{(\mathbb{N}^*)}$ multiindices

$$numpart(\alpha) = \frac{|\alpha|!}{(1!)^{a_1}(2!)^{a_2}\cdots(r!)^{a_r}(a_1)!(a_2)!\cdots(a_r)!}$$

We will adopt the notation

$$\alpha = 1^{a_1} 2^{a_2} \cdots r^{a_r}$$

for the *type* of a (set) partition which means that there are a_1 singletons a_2 pairs a_3 3-blocks a_4 4-blocks and so on.

The number of set partitions of type α as above is well known (see Comtet for example)

$$numpart(\alpha) = \frac{|\alpha|!}{(1!)^{a_1}(2!)^{a_2}\cdots(r!)^{a_r}(a_1)!(a_2)!\cdots(a_r)!}$$

Then, with

$$F(y) = exp(\sum_{n\geq 1} L_n \frac{y^n}{n!}) \quad G(x) = exp(\sum_{n\geq 1} V_m \frac{x^m}{m!})$$

$$F(y) = exp(\sum_{n\geq 1} L_n \frac{y^n}{n!}) \quad G(x) = exp(\sum_{n\geq 1} V_m \frac{x^m}{m!})$$

one has

$$\mathcal{H}(F,G) = F(y\frac{d}{dx})G(x)|_{x=0} =$$

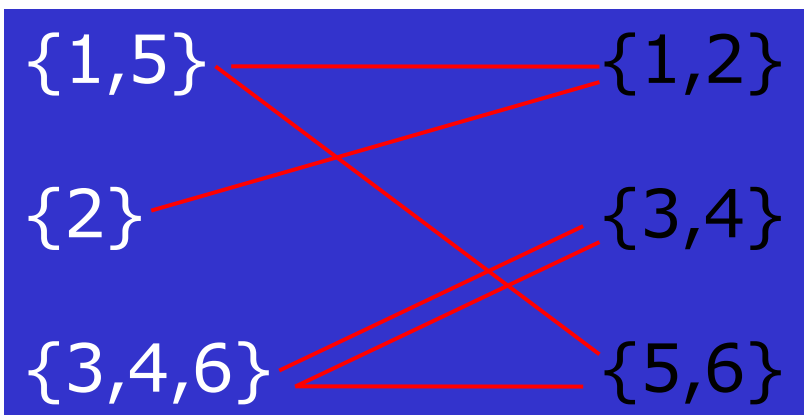
$$\sum_{n\geq 0} \frac{y^n}{n!} \sum_{|\alpha|=|\beta|=n} numpart(\alpha)numpart(\beta)\mathbb{L}^{\alpha}\mathbb{V}^{\beta}$$

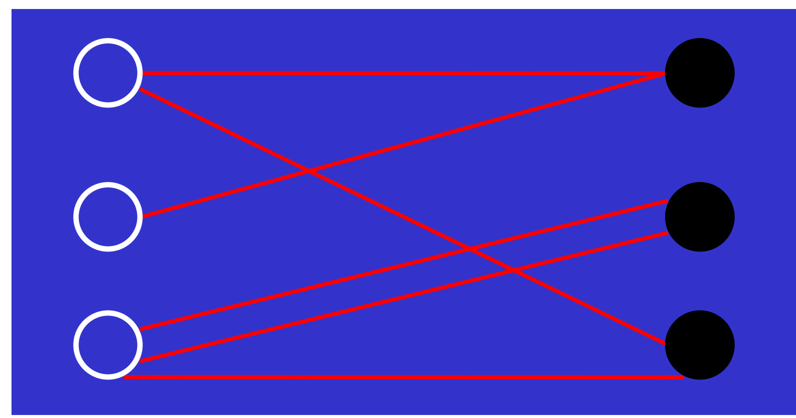
Now, one can count in another way the term *numpart(α)numpart(β)*. Remarking that this is the number of pairs of set partitions (P1,P2) with type(P1)=α, type(P2)=β. But every pair of partitions (P1,P2) has an intersection matrix ...

	{1,5}	{2}	{3,4,6}
{1,2}	1	1	0
{3,4}	0	0	2
{5,6}	1	0	1

Classes of packed matrices see NCSF VI (GD, Hivert, and Thibon)

Feynman-type diagram (Bender & al.)



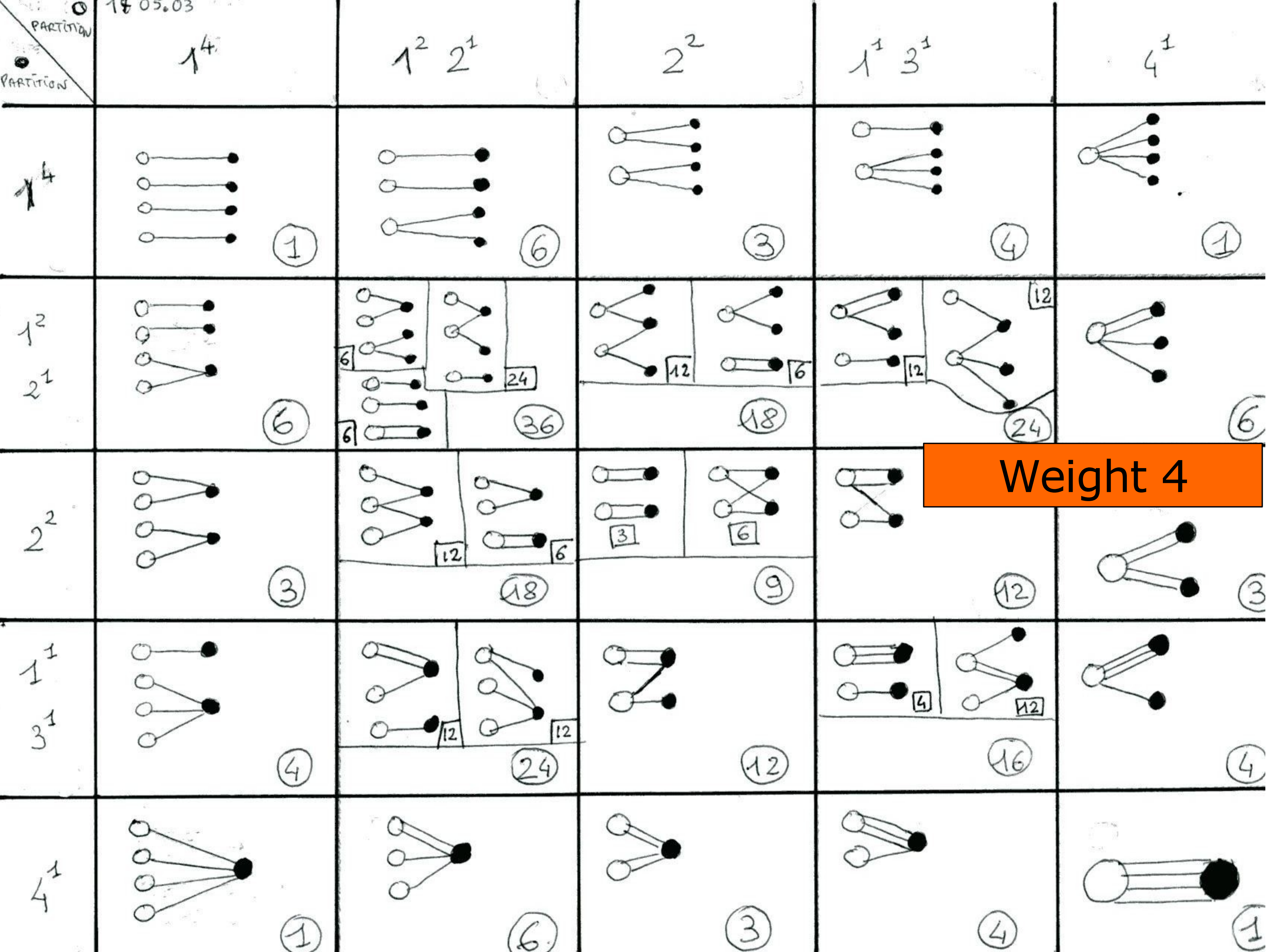


Now the product formula for EGFs reads

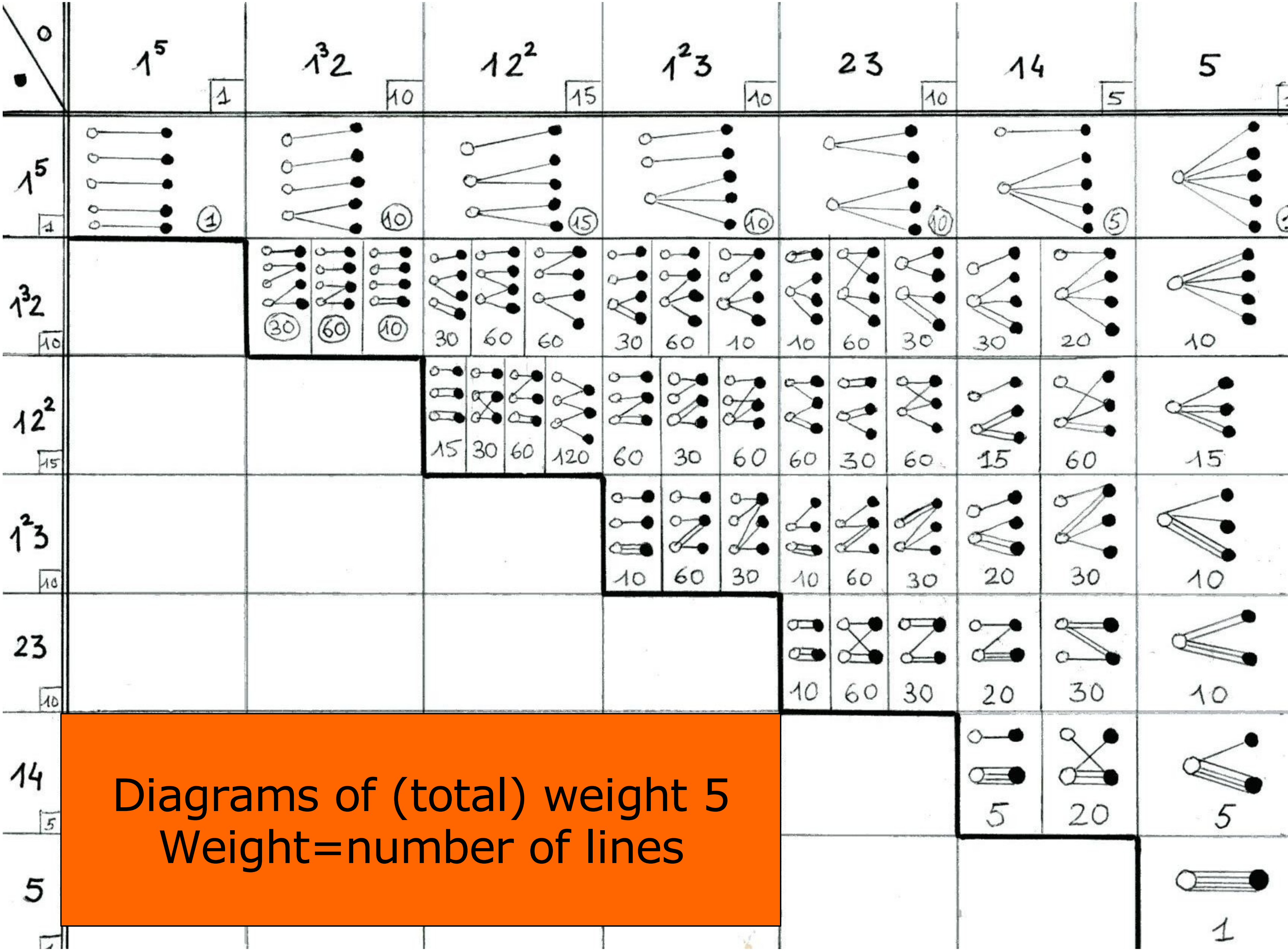
$$\mathcal{H}(F,G) = \sum_{d\ FB-diagram} \frac{y^{|d|}}{|d|!} \mathbb{L}^{\alpha(d)} \mathbb{V}^{\beta(d)}$$

$$\mathcal{H}(F,G) = \sum_{d \in \mathbf{diag}} \frac{y^{|d|}}{|d|!} mult(d) \mathbb{L}^{\alpha(d)} \mathbb{V}^{\beta(d)}$$

The main interest of these new forms is that we can impose rules on the counted graphs and we can call these (and their relatives) graphs : Feynman Diagrams of this theory (here, the simplified model of Quantum Field Theory of Partitions).


18.05.03
PARTITION
PARTITION
1^4
1^2 2^1
2^2
1^1 3^1
4^1
1
6
3
4
1
6
6 6 24 36
12 6 18
12 12 24
6
3
12 6 18
3 6 9
12
3
4
12 12 24
12
4 12 16
4
1
6
3
4
1

Weight 4


Diagrams of (total) weight 5
Weight=number of lines

One has now 3 types of diagrams :

- the diagrams with labelled edges (from 1 to |d|). Their set is denoted (see above) FB-diagrams.

- the unlabelled diagrams (where permutation of black and white spots). Their set is denoted (see above) **diag.**

- the diagrams, as drawn, with black (resp. white) spots ordered i.e. labelled. Their set is denoted **ldiag.**

Hopf algebra structure

$$(H,\mu,\Delta,1_H,\varepsilon,\alpha)$$

Satisfying the following axioms

- $(H,\mu,1_H)$ is an associative k-algebra with unit (here k will be a – commutative - field)
- (H,Δ,ε) is a coassociative k-coalgebra with counit
- $\Delta : H \to H\otimes H$ is a morphism of algebras
- $\alpha : H \to H$ is an anti-automorphism (the antipode) which is the inverse of Id for convolution.

Convolution is defined on End(H) by

$$\varphi \bullet \psi = \mu\,(\varphi \otimes \psi)\,\Delta$$

with this law End(H) is endowed with a structure of associative algebra with unit $1_H\varepsilon$.

First step: Defining the spaces

$Diag = \oplus_{d \in \text{diagrams}} \mathbf{C}\, d$ $\qquad$ $LDiag = \oplus_{d \in \text{labelled diagrams}} \mathbf{C}\, d$

(functions with finite supports on the set of diagrams). At this stage, we have a natural arrow *LDiag* → *Diag*.

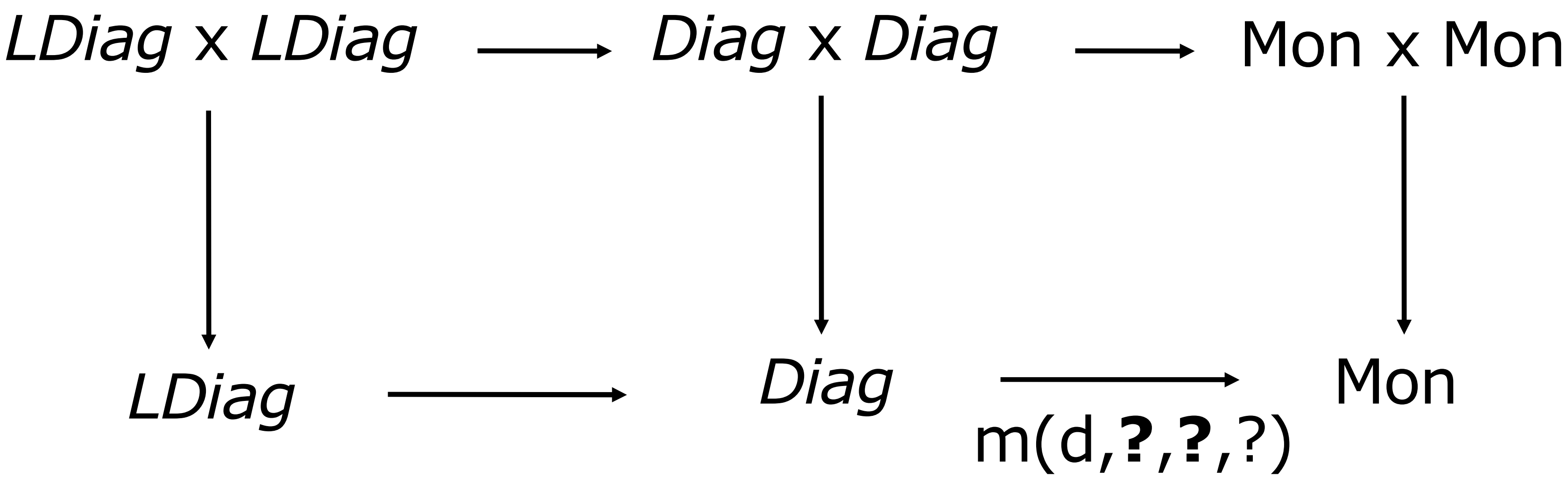
Second step: The product on *Ldiag* is just the concatenation of diagrams

$$d_1 \star d_2 = d_1 \; d_2$$

And, setting $m(d,\mathbf{L},\mathbf{V},z)=\mathbf{L}^{\alpha(d)}\mathbf{V}^{\beta(d)} z^{|d|}$
one gets

$$m(d_1 * d_2,\mathbf{L},\mathbf{V},z) = m(d_1,\mathbf{L},\mathbf{V},z)m(d_2,\mathbf{L},\mathbf{V},z)$$

This product is associative with unit (the empty diagram). It is compatible with the arrow *LDiag* → *Diag* and so defines the product on *Diag* which, in turn, is compatible with the product of monomials.

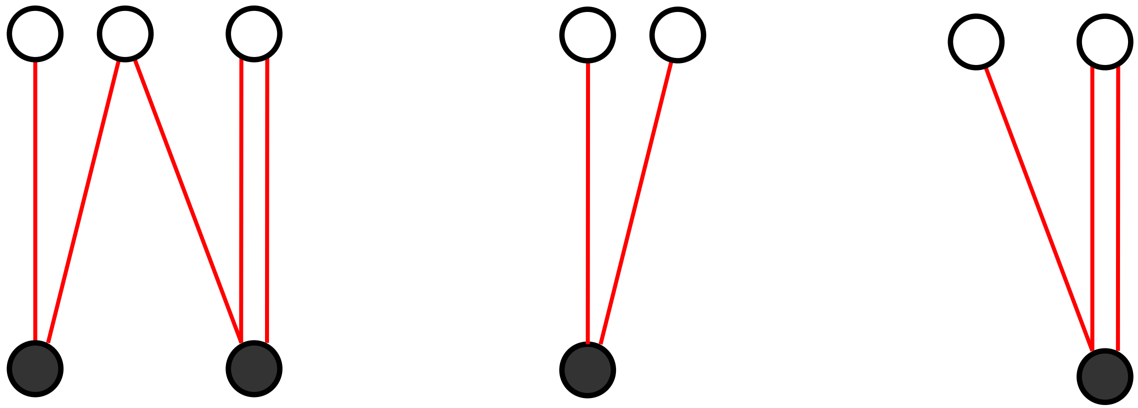


The coproduct needs to be compatible with m(d,**?**,**?**,?). One has two symmetric possibilities. The « black spots coproduct » reads

$$\Delta_{BS}(d)=\sum d_I \otimes d_J$$

the sum being taken over all the decompositions, (I,J) of the Black Spots of d into two subsets.

For example, with the following diagrams d, d_1 and d_2



one has $\Delta_{BS}(d)=d\otimes\emptyset + \emptyset\otimes d + d_1\otimes d_2 + d_2\otimes d_1$

If we concentrate on the multiplicative structure of *Ldiag*, we remark that the objects are in one-to-one correspondence with the so-called packed matrices of NCSFVI (Hopf algebra MQSym), but the product of MQSym is given (w.r.t. a certain basis **MS**) according to the following example

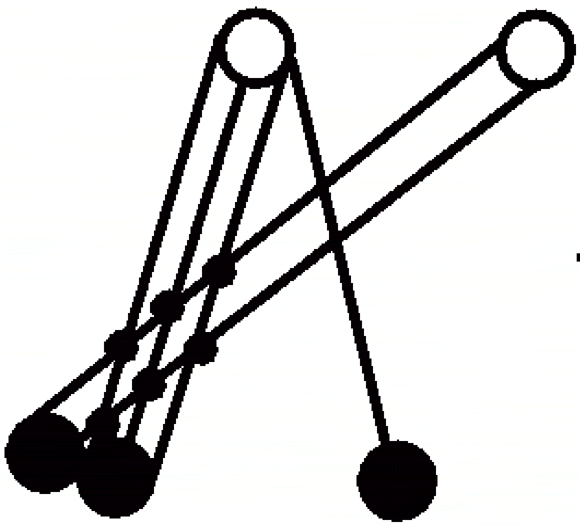
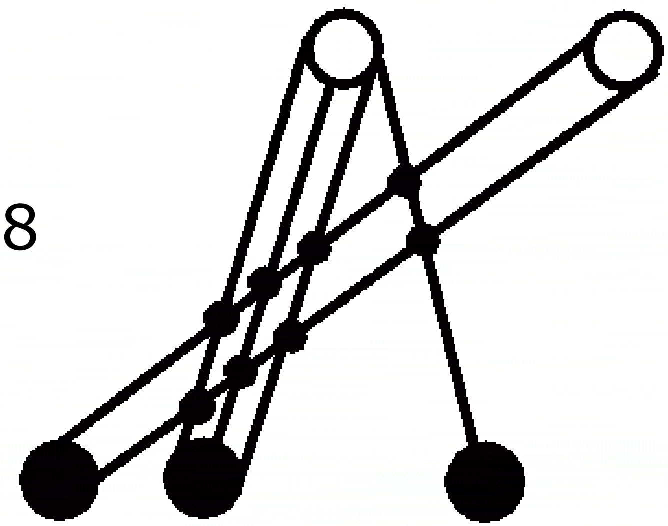
$$\mathbf{MS}\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{MS}_{[3\ 1]} =$$

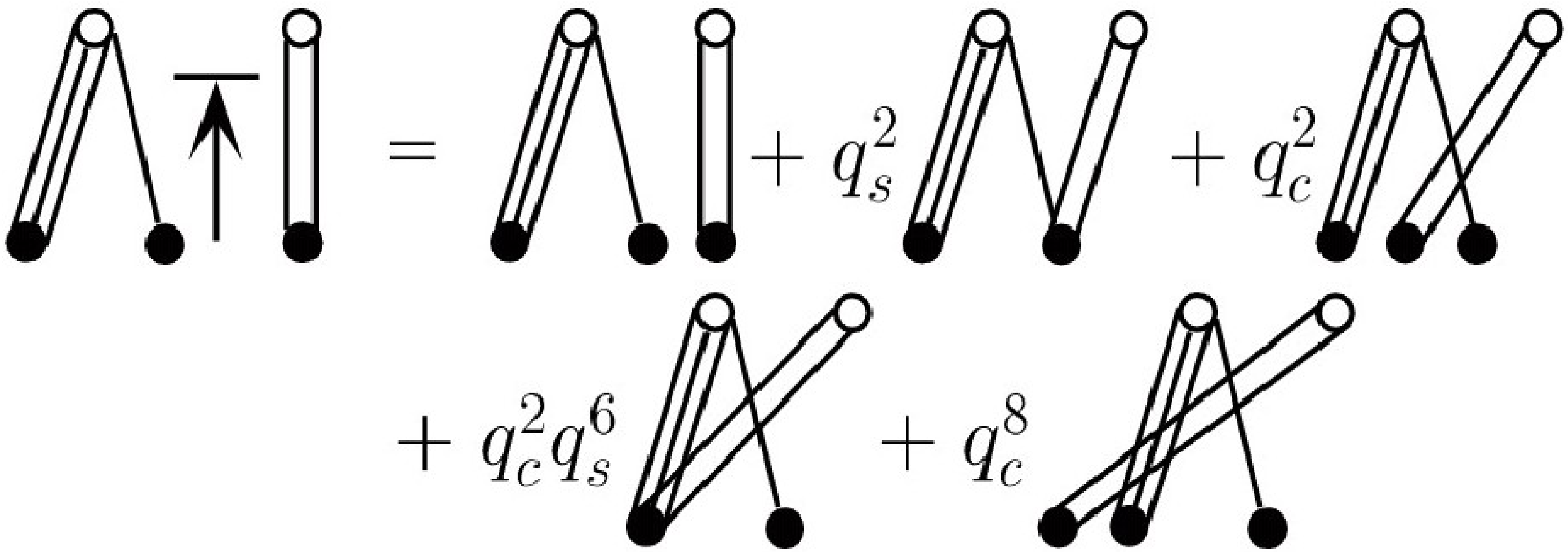
$$\mathbf{MS}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix} + \mathbf{MS}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 1 \end{bmatrix} + \mathbf{MS}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \mathbf{MS}\begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \mathbf{MS}\begin{bmatrix} 0 & 0 & 3 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

It is possible to (re)connect these Hopf algebras to MQSym and others of interest for physicists, by deforming the product with two parameters.
The double deformation goes as follows

- Concatenate the diagrams
- Develop according to the rules :
  - Every crossing "pays" a q_c
  - Every node-stacking "pays" a q_s

In the expansion, the weights are given by the intersection numbers.

$q_c^2 q_s^6$  $+ q_c^8$ 



By an unexpected “stroke of luck”, we could check that this law is associative (now three independent proofs + many verifications).

For example, by direct computation

$$(au \uparrow bv) \uparrow cw = \left(a(u \uparrow bv) + q^{|u||b|} t^{|a||b|} \begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v) + q^{|au||b|} b(au \uparrow v)\right) \uparrow cw$$

$$\left[a((u \uparrow bv) \uparrow cw) + q^{(|u|+|bv|)|c|} t^{|a||c|} \begin{bmatrix} c \\ a \end{bmatrix} ((u \uparrow bv) \uparrow w) + q^{(|au|+|bv|)|c|} c(a(u \uparrow bv) \uparrow w)\right]$$

$$\left[q^{|u||b|} t^{|a||b|} \begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v \uparrow cw) + q^{|u||b|+(|u|+|v|)|c|} t^{|a||b|} t^{(|a|+|b|)|c|} \begin{bmatrix} c \\ b \\ a \end{bmatrix} (u \uparrow v \uparrow w)\right.$$

$$\left. q^{|u||b|+(|au|+|bv|)|c|} t^{|a||b|} c((\begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v)) \uparrow w)\right]$$

$$\left[q^{|au||b|} b((au \uparrow v) \uparrow cw) + q^{|au||b|+(|au|+|v|)|c|} t^{|b||c|} \begin{bmatrix} c \\ b \end{bmatrix} (au \uparrow v \uparrow w) + q^{|au||b|+(|au|+|bv|)|c|} c(b(au \uparrow v) \uparrow w\right.$$

$$au \uparrow (bv \uparrow cw) = au \uparrow \left(b(v \uparrow cw) + q^{|v||c|}t^{|b||c|} \begin{bmatrix} c \\ b \end{bmatrix} (v \uparrow w) + q^{|bv||c|}c(bv \uparrow w)\right) =$$

$$\left[a(u \uparrow b(v \uparrow cw)) + q^{|u||b|}t^{|a||b|} \begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v \uparrow cw) + q^{|au||b|}b(au \uparrow v \uparrow cw)\right] +$$

$$\left[q^{|v||c|}t^{|b||c|}a(u \uparrow \begin{bmatrix} c \\ b \end{bmatrix} (v \uparrow w)) + q^{|v||c|+|u|(|c|+|b|)}t^{|b||c|+|a|(|b|+|c|)} \begin{bmatrix} c \\ b \\ a \end{bmatrix} (u \uparrow v \uparrow w) + \right.$$

$$\left. q^{|v||c|+|au|(|b|+|c|)}t^{|b||c|} \begin{bmatrix} c \\ b \end{bmatrix} (au \uparrow v \uparrow w)\right] +$$

$$\left[q^{|bv||c|}a(u \uparrow c(bv \uparrow w)) + q^{(|u|+|bv|)|c|}t^{|a||c|} \begin{bmatrix} c \\ a \end{bmatrix} (u \uparrow bv \uparrow w) + q^{(|au|+|bv|)|c|}c(au \uparrow bv \uparrow w)\right] \quad (3)$$

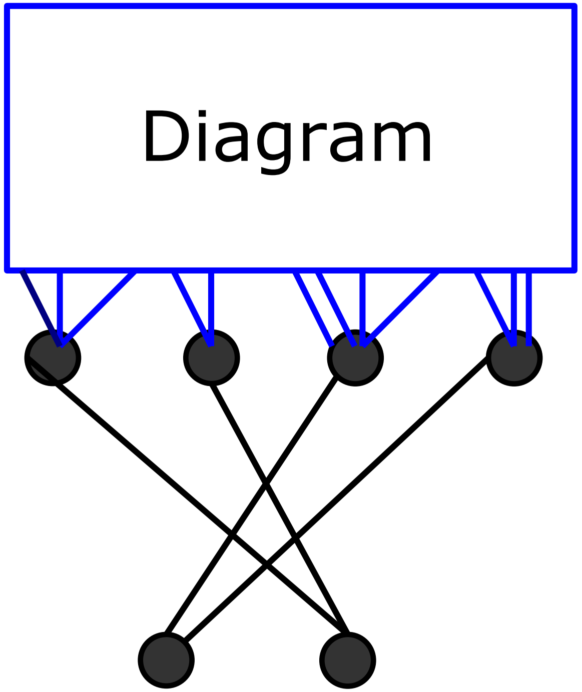
dans la deuxième expression, on regroupe les trois termes de tête des crochets et on trouve

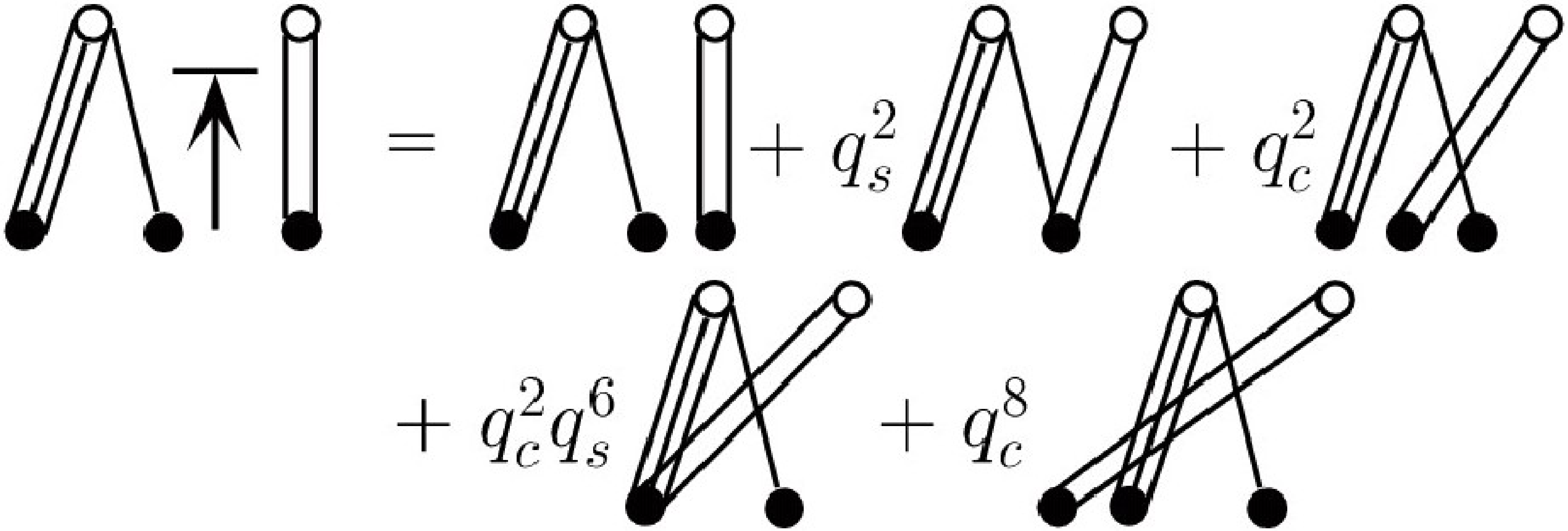
$$a(u \uparrow b(v \uparrow cw)) + q^{|v||c|}t^{|b||c|}a(u \uparrow \begin{bmatrix} c \\ b \end{bmatrix} (v \uparrow w)) + q^{|bv||c|}a(u \uparrow c(bv \uparrow w)) = a(u \uparrow bv \uparrow cw) \quad (4)$$

dans la première expression, on regroupe les trois termes de queue des crochets et on trouve

$$q^{(|au|+|bv|)|c|}c(a(u \uparrow bv) \uparrow w) + q^{|u||b|+(|au|+|bv|)|c|}t^{|a||b|}c((\begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v)) \uparrow w) +$$
$$q^{|au||b|+(|au|+|bv|)|c|}c(b(au \uparrow v) \uparrow w) = q^{(|au|+|bv|)|c|}c(au \uparrow bv \uparrow w) \quad (5)$$

This amounts to use a monoidal action with two parameters. Associativity provides an identity in an algebra which acts on a diagram as the algebra of the symmetric semigroup. Here, it is the symmetric semigroup which acts on the black spots





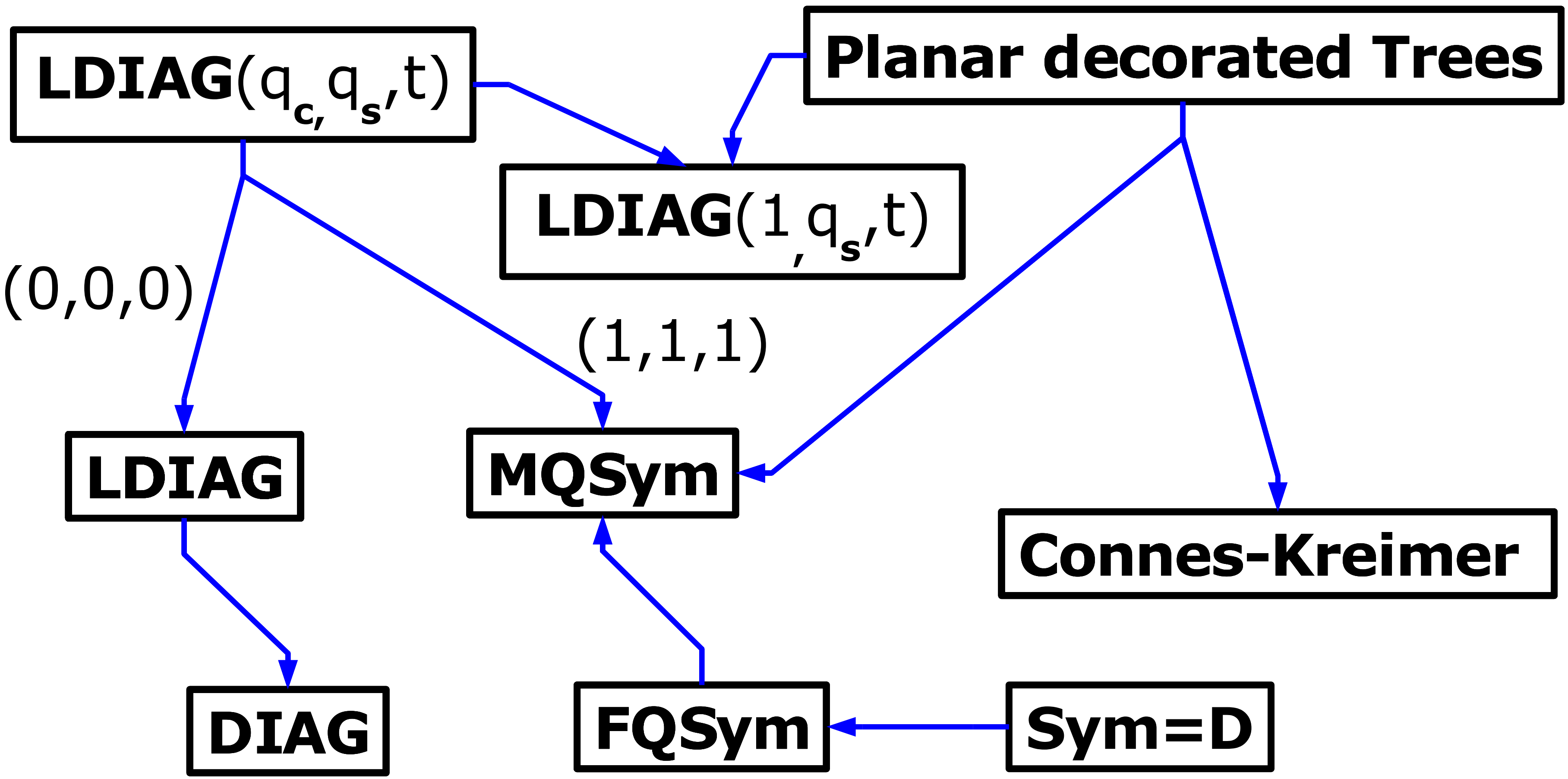
The labelled diagrams are in one to one correspondence with the packed matrices of MQSym and we can see easily that the product of the latter is obtained for

$$q_c = 1 = q_s$$

Hopf interpolation : One can see that the more intertwined the diagrams are the less connected components they have. This is the main argument to prove that LDIAG($q_{c,}$ q_s) is free on indecomposable diagrams. Therefore one can define a coproduct on these generators by

$$\Delta_t=(1-t)\Delta_{BS}+t\,\Delta_{MQSym}$$

this is LDIAG($q_{c,}$ q_s,t).



Notes :

i) The arrow *Planar Dec. Trees* → *LDIAG(1,q_s,t)* is due to L. Foissy

ii) **LDIAG**(q_c,q_s,t), through a noncommutative alphabetic realization shows to be a bidendriform algebra (FPSAC07 paper with ...).

The legacy of Schützenberger or how to harness Sweedler's duals using Automata Theory

Sweedler's dual of a Hopf algebra

i) Multiplication

$$\mathcal{A}\otimes\mathcal{A} \xrightarrow{\mu} \mathcal{A}$$

ii) By dualization one gets

$$(\mathcal{A})^* \xrightarrow{{}^t\mu} (\mathcal{A}\otimes\mathcal{A})^*$$

but not a "stable calculus" as

$$(\mathcal{A})^* \otimes (\mathcal{A})^* \subseteq (\mathcal{A}\otimes\mathcal{A})^*$$

(strict in general). We ask for elements $x\in\mathcal{A}$ such that

$${}^t\mu(x) \in (\mathcal{A})^* \otimes (\mathcal{A})^*$$

Here, we will be concerned with the case $\mathcal{A} = k\langle A \rangle$ (non-commutative polynomials with coefficients in a field k).

Indeed, we have the following theorem (the beginning can be found in [ABE : Hopf algebras]) and the end is the starting point of Schützenberger's school of automata and language theory.

Theorem A: TFAE (the notations being as above)

i) $^{t}\mu(c) \in (\mathcal{A})^{*} \otimes (\mathcal{A})^{*}$

ii) There are functions f_i ,g_i i=1,2..n such that

$$c(uv)=\sum_{i=1}^{n} f_i(u)\, g_i(v)$$

u,v words in A^* (the free monoid of alphabet A).

iii) There is a morphism of monoids $\mu: A^* \to k^{n \times n}$ (square matrices of size n x n), a line λ in $k^{1 \times n}$ and a column ξ in $k^{n \times 1}$ such that, for all word w in A^*

$$c(w)=\lambda\mu(w)\xi$$

iv) (Schützenberger) (If A is finite) c lies in the rational closure of A within the algebra k<<A>>.

We can safely apply the first three conditions of **Theorem A** to *Ldiag.* The monoid of labelled diagrams is free, but with an infinite alphabet, so we cannot keep Schützenberger's equivalence at its full strength and have to take more "basic" functions. The modification reads

iv) (A is infinite) c is in the rational closure of the weighted sums of letters

$$\sum_{a \in A} p(a)\, a$$

within the algebra k<<A>>.

iii) Schützenberger's theorem could be rephrased in saying that functions in Sweedler's dual are behaviours of finite (state and alphabet) automata.

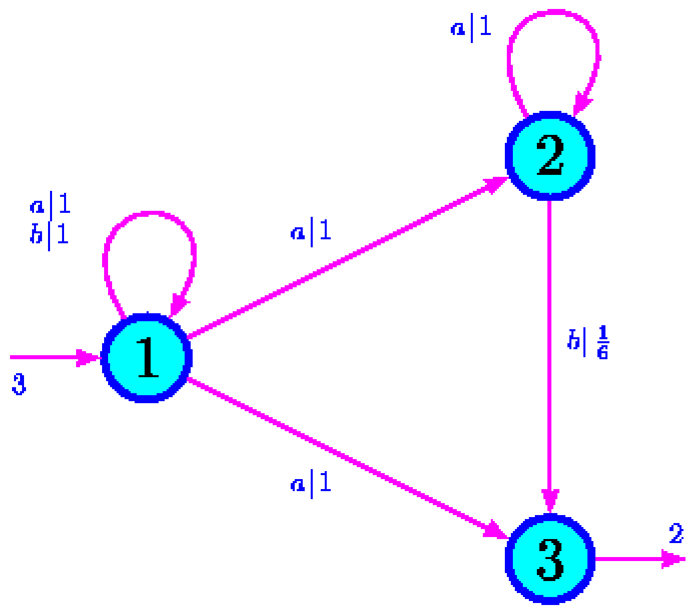
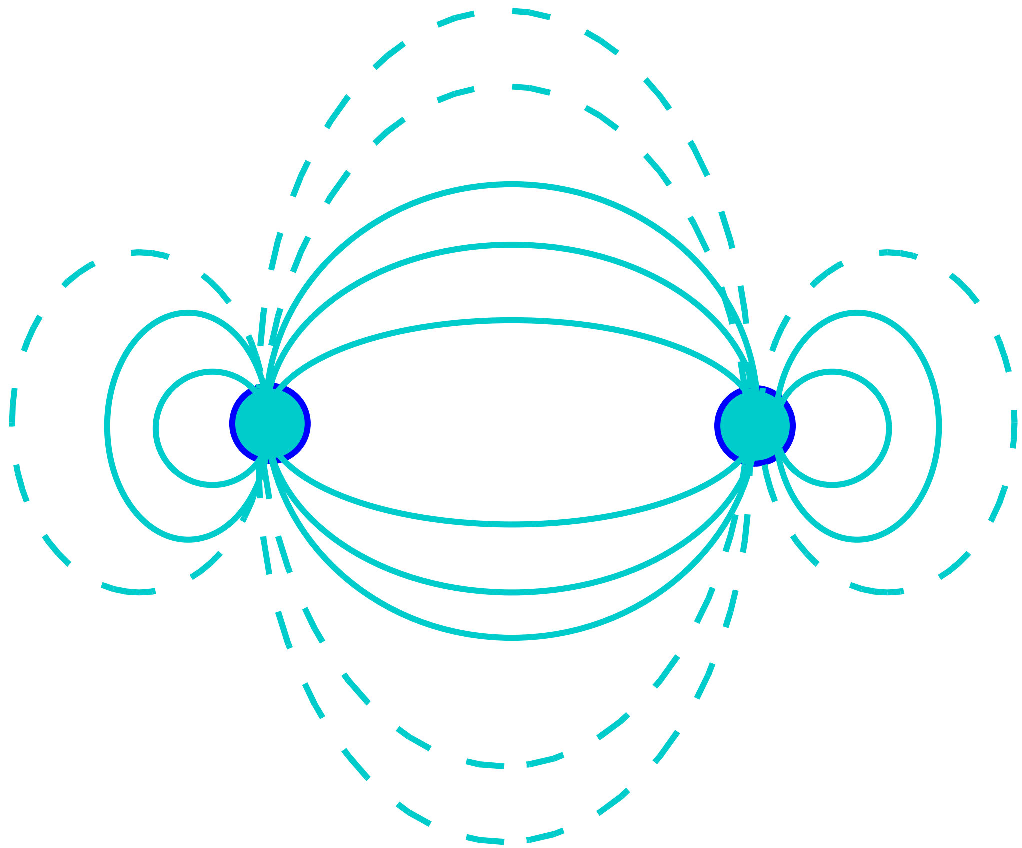


FIG. 1 – Un $\mathbb{Q}$-automate $\mathcal{A}$.

Le comportement de $\mathcal{A}$ est :

$$\text{comportement}(\mathcal{A}) = \sum_{a,b \in A} (a+b)^*(6 + a^*b).$$

In our case, we are obliged to allow infinitely many edges.



Rational expressions To Be Done

Concluding remarks and future

i) *The diagrams of* ***diag*** *are well suited to* ***n!*** *Denominators (i.e. EGF). What are the good data structures for other ones ?*

ii) One can change the constants $V_k=1$ to a condition with level (i.e. $V_k=1$ for $k\leq N$ and $V_k= 0$ for $k>N$). We obtain then sub-Hopf algebras of the one constructed above. These can apply to the manipulation of partition functions of physical models including Free Boson Gas, Kerr model and Superfluidity.

Concluding remarks and future (cont'd)

iii) The deformation above is likely to be decomposed in two deformation processes ; twisting (already investigated in NCSFIII) and shifting (ongoing work with JGL and al.). Also, it could have a connection with other well known associators.

iv) The identity on the symmetric semigroup can be lifted to a more general monoid which takes into account the operations of concatenation and stacking which are so familiar to Computer Scientists (ongiong work in LIPN).

Partition Function

$$Z = tr\exp(-\beta H)$$

$$H = \varepsilon a^{+} a$$

Free Boson Gas (single mode)
Coherent state basis

$$\frac{1}{\pi}\int d^2 z |z><z| = I$$

$$Z = (1/\pi)\int d^2 z < z|\exp(-\beta\varepsilon a^{+} a)| z >$$

$$= (1/\pi)\int d^2 z < z|:\exp(a^{+} a(\exp(-\beta\varepsilon)-1):| z >$$

$$= (1/\pi)\int_0^{2\pi} d\theta \int_0^{\infty} rdr\exp(r^2(\exp(-\beta\varepsilon)-1))$$

$$= (1-\exp(-\beta\varepsilon))^{-1} \qquad r = |z|$$

$$Z = \int_0^{\infty} dy\exp(y(e^x - 1)) \qquad y = r^2, x = -\beta\varepsilon$$

$$= \int_0^{\infty} dy B(x, y)$$

Graphs for Bell numbers *B(n)*

$$G(x) = \exp(x/1! + x^2/2! + x^3/3! + \ldots)$$

End of the talk

Merci

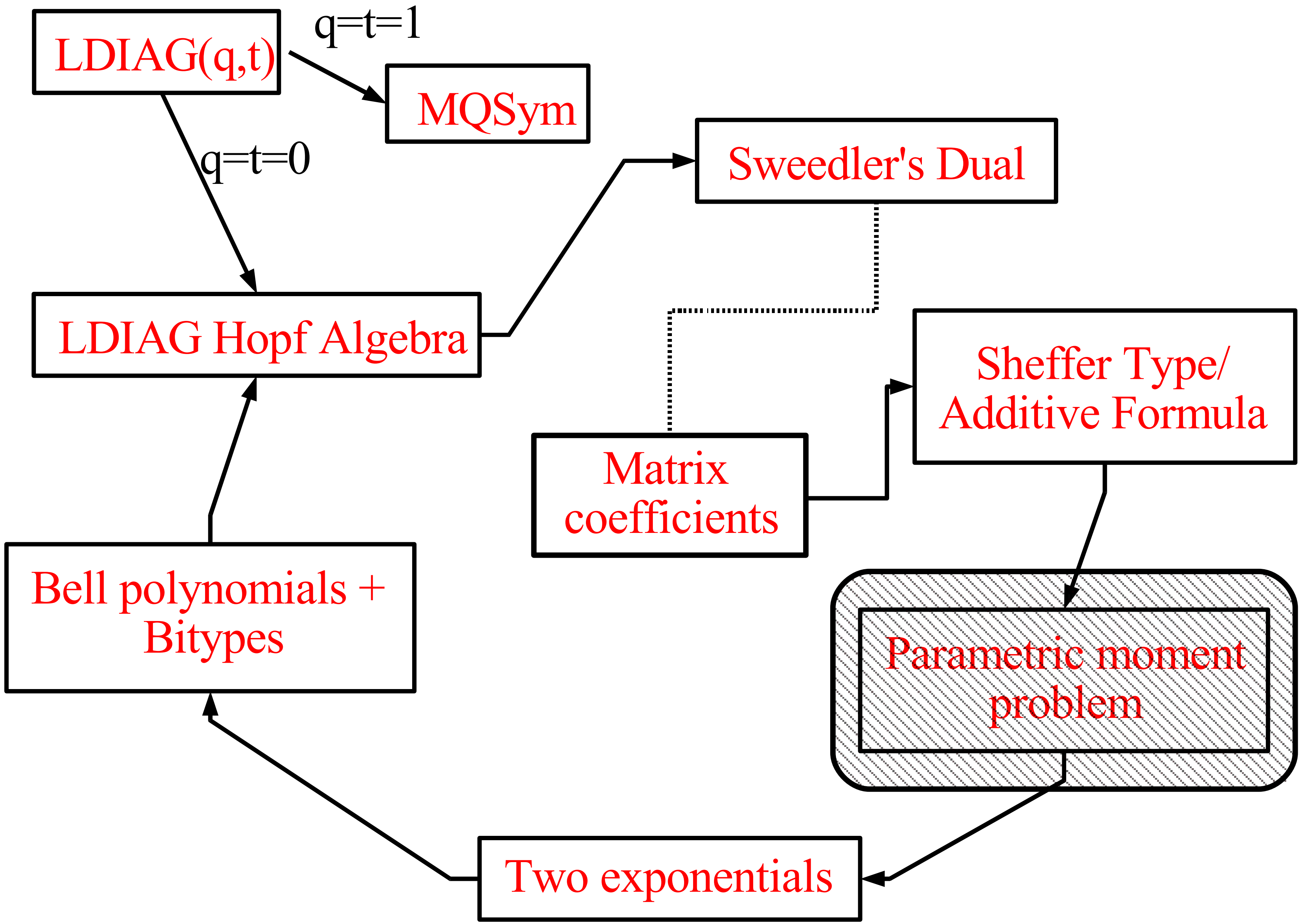
Thank you

Dziękuję

Classical Stieltjes moment problem

Consider a sequence of real numbers *B(n)*. The classical Stieltjes moment problem consists in finding a positive measure *W(x)dx* on the half-line]0,+∞[such that

$$B(n) = \int_0^{+\infty} x^n W(x) dx$$



LDIAG(q,t)
q=t=1
MQSym
q=t=0
Sweedler's Dual
LDIAG Hopf Algebra
Sheffer Type/
Additive Formula
Matrix
coefficients
Bell polynomials +
Bitypes
Parametric moment
problem
Two exponentials

Parametric Stieltjes moment problem

Consider a sequence of real functions *B(n,y)*. The parametric Stieltjes moment problem consists in finding a family of positive measures *W(x,y)dx* on the half-line]0,+∞[such that

$$B(n,y) = \int_0^{+\infty} x^n W(x,y) dx$$

Using the first Dobinski relation of slide (10), one can solve the parametric Stieltjes moment problem for the classical Stirling numbers as

$$S_{1,1}(n,y) = \int_0^{+\infty} x^n W_1(x,y)\,dx$$

with

$$W_1(x,y) = \mathrm{e}^{-y} \sum_{k=1}^{\infty} \frac{y^k \delta(x-k)}{k!}$$

which is a Poisson distribution on the half-line]0,+∞[.

Using an inverse Mellin transform, one can solve the second parameric moment problem, which gives, this time, a continuous measure

$$S_{2,1}(n,y) = \int_0^{+\infty} x^n W_2(x,y) dx$$

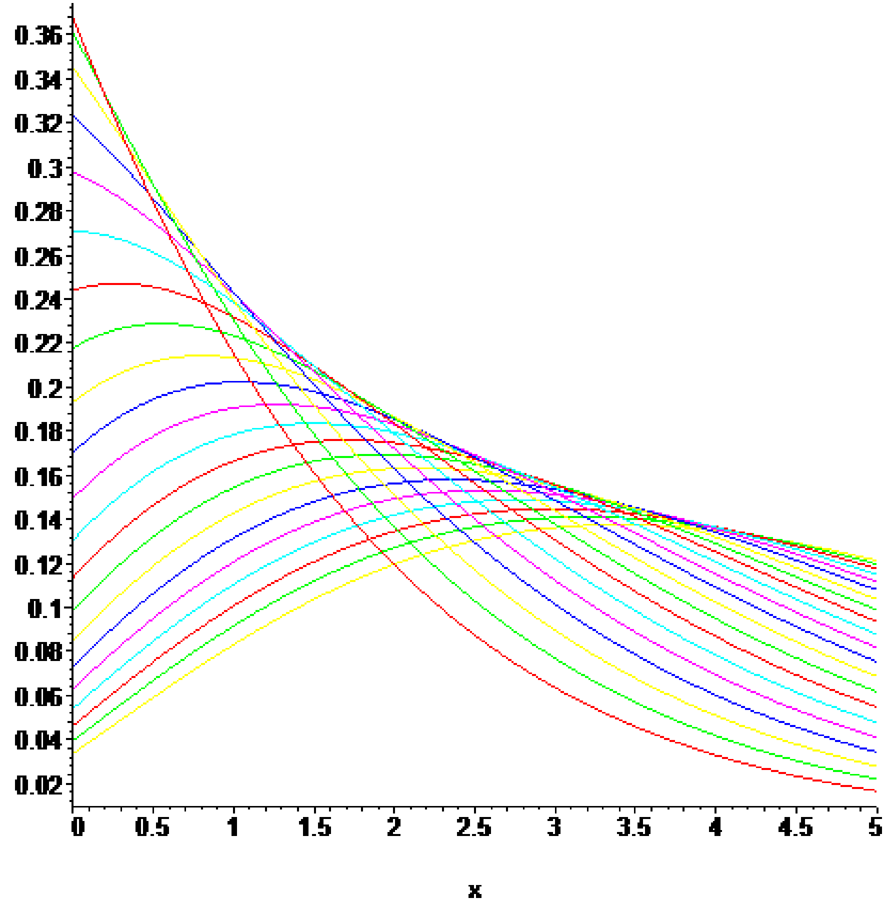
with

$$W_2(x,y) = y\, e^{-(x+y)} \frac{I_1(2\sqrt{xy})}{\sqrt{xy}}$$

```
> f1:=(x,y)->y*exp(-(x+y))*BesselI(1,2*sqrt(x*y))/sqrt(x*y);
```

$$f1 := (x, y) \rightarrow \frac{y\, e^{(-x-y)}\, \mathrm{BesselI}(1, 2\sqrt{x\,y})}{\sqrt{x\,y}}$$

```
> plot([seq(f1(x,1+0.2*k),k=0..20)],x=0..5,tickmarks=[8,10]);
```




```
> seq(evalf(f1(0.001,1+0.2*k),3),k=0..20);
```

0.367, 0.361, 0.345, 0.322, 0.297, 0.271, 0.245, 0.218, 0.193, 0.171, 0.150, 0.131, 0.115, 0.0985, 0.0851, 0.0731, 0.0633, 0.0543, 0.0468, 0.0397, 0.0337

Ongoing work

Realizations of the product for some types of infinite matrices

Convolution of kernels: We first suppose given two infinite matrices F(n,k), G(n,k) (n,k integers) admitting solutions for the parametric moment problem (PMP) which means that there are two (parametric) measures W_F, W_G such that

$$B_F(n,y) = \int_0^{+\infty} x^n W_F(x,y)dx$$

$$B_G(n,y) = \int_0^{+\infty} x^n W_G(x,y)dx$$

Then one can check easily that, if the two kernels W_F *and* W_G are convolable, then FG admits a solution for the PMP and

$$W_{FG}(x,y) = \int_0^{+\infty} W_F(x,z)W_G(z,y)dz$$

Questions: Q1) What are the types of matrices for which there is a PMP solution ?

Q2) Which are the ones for which the kernel is discrete ? Continuous ?

Q3) Are there general laws for convolution of these types of kernels.

Link with grafting: Certain classes of graphs (i.e. closed by relabelling and extraction of connected components) provide lower triangular matrices via

M(n,k)=number of graphs with labels {1, 2,..n} and k connected components

the product of the matrices associated with two classes corresponds to the grafting obtained by considering the connected components of a graph of the first kind as vertices of a graph of the second kind.

Question: What are the legal types of grafting when we change denominators ? Link with renormalisation ?

Substitutions: An infinite matrice F(n,k) with finite rows can be seen as defining a transformation between EGF. The transformation is of the form f --> u(x)f(v(x)) with u(x)=1+... and v(x)=λx+... if the sequence of polynomials $B_F(n,y)$ is of Extended Sheffer Type (EST).

There is a « calculus » using vector fields on the half-line and their conjugates. (see SLC Viennot - Lucelle - and Myczcowce talks)

Questions: Q1) Combinatorial fields ? What is the «Stirling field » for instance ?

Q2) Make precise the dictionnaries (formal or analytic)
vector fields ↔ combinatorial matrices

Q3) What are the matrices coming from classes of graphs