On the use of rational expressions in Combinatorial Physics: an example.

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Abstract

In a joint work with J. Katriel, one of us gave the solution of the problem of matrix coefficients in the Fock space of carriers between two levels. We give here a complete proof of these continued fractions-type formulas and possible extensions of the method. [0]

1 Introduction

2 Brief review of some known formulas

In [1] was considered, as Fock space, a general vector space V over a field k with basis $|e_n\rangle\ \ n = 0,\, 1,\, \cdots$, equipped with its natural grading

$$V = \oplus_{n\in\mathbf{Z}} V_n \text{ with } V_n := ke_n;\ V_{-n-1} := (0) \text{ for } n \geq 0 \tag{1}$$

and scalar product defined by $\langle e_n|e_m\rangle = \delta_{n,m}$. and $f,\ g$ two linear operators in V of degrees $-1,\ +1$, respectively. Generically, they read

$$\text{for } n \geq 0;\ f|e_0\rangle := 0;\ f|e_{k+1}\rangle = \alpha_{k+1}|e_k\rangle;\ g|e_k\rangle := \beta_{k+1}|e_{k+1}\rangle \tag{2}$$

We consider the words in $f,\ g$:

$$w(f,g) := f^{p_1}g^{q_1}f^{p_2}g^{q_2}\cdots f^{p_n}g^{q_n} \tag{3}$$

the degree (excess) of which is $\sum_{k=1}^{n}(q_k - p_k)$ (this is, for algebraists, the degree of the graded operator $f^{p_1}g^{q_1}f^{p_2}g^{q_2}\cdots f^{p_n}g^{q_n}$). This provides a representation μ of a two letters free monoid $\{b_-, b_+\}^*$ on V by $\mu(b_-) = f;\ \mu(b_+) = g$, which is graded for the weight on $\{b_-, b_+\}^*$. In order to keep the reading of a word from left to right, one performs the the action on the right. Thus V becomes a $\{b_-, b_+\}^*$ right module by

$$e_0.b_- = 0\ ;\ e_{n+1}.b_- = \alpha_{n+1}e_n\ ;\ e_n.b_+ = \beta_{n+1}e_{n+1} \tag{4}$$

one is interested by the matrix elements

$$\langle e_n.\{b_-, b_+\}^i|e_m\rangle = \omega^{(i)}_{n\to m}\ . \tag{5}$$

The following propsition gives the support i. e. the set of words w of length i such that $\langle e_n.w|e_m\rangle \neq 0$.

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Proposition 2.1 *Define* $W^{(i)}_{n\to m}$ *as follows*

$$W^{(i)}_{n\to m} = \{w \in \{b_-, b_+\}^i \mid (\pi_e(w) = m - n) \text{ and } (w = uv \Longrightarrow \pi_e(w) \geq -n)\} \qquad (6)$$

then
i) If the all the weights $\alpha_n;\ n \geq 1,\ \beta_n, n \geq 0$ *are not zero* $W^{(i)}_{n\to m}$ *is exactly the set of words of length* i *such that* $\langle e_n.w|e_m\rangle \neq 0$.
ii) In all cases the latter is a subset of $W^{(i)}_{n\to m}$ *i. e.*

$$\{w \in \{b_-, b_+\}^i \mid \langle e_n.w|e_m\rangle \neq 0\} \subset W^{(i)}_{n\to m} \qquad (7)$$

As a result,

$$\mu(W^{(i)}_{m-n})|n\rangle = \omega^{(i)}_{n\to m}|m\rangle\ ;\ T_{n\to n+k} := \sum_{i\geq 0} t^i \omega^{(i)}_{n\to n+k}\ . \qquad (8)$$

One can expand $T_{n\to n+k}$ as a product of continued fractions. Let

$$F^+_n = \cfrac{1}{1 - \cfrac{t^2\alpha_{n+1}\beta_{n+1}}{1 - \cfrac{t^2\alpha_{n+2}\beta_{n+2}}{1 - \cfrac{t^2\alpha_{n+3}\beta_{n+3}}{1 - \cdots}}}} = \frac{1}{1 - E^+_n}$$

$$F^-_n = \cfrac{1}{1 - \cfrac{t^2\alpha_{n}\beta_{n}}{1 - \cfrac{t^2\alpha_{n-1}\beta_{n-1}}{1 - \cfrac{t^2\alpha_{n-2}\beta_{n-2}}{1 - \cdots}}}} = \frac{1}{1 - E^-_n} \qquad (9)$$

and

$$F_n = \frac{1}{1 - E^+_n - E^-_n}\ . \qquad (10)$$

Then, if $k \geq 0$, we have

$$T_{n\to n+k} = t^k F_{n+k} \prod_{i=0}^{k-1} F^-_{n+i} = t^k F_n \prod_{i=1}^{k} F^+_{n+i} \qquad (11)$$

And, if $k \leq 0$

$$T_{n\to n+k} = t^{-k} F_{n+k} \prod_{i=0}^{-k-1} F^+_{n-i} = t^{-k} F_n \prod_{i=1}^{-k} F^-_{n-i} \qquad (12)$$

References

References

[1] Katriel J., Duchamp G., *Ordering relations for q-boson operators, continued fractions techniques, and the q-CBH enigma*. Journal of Physics A **28** 7209-7225 (1995).