A multipurpose Hopf deformation of the Algebra of Feynman-like Diagrams

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Abstract. We construct a three parameter deformation of the Hopf algebra **LDIAG**. This new algebra is a true Hopf deformation which reduces to **LDIAG** on one hand and to **MQSym** on the other, relating **LDIAG** to other Hopf algebras of interest in contemporary physics. Further, its product law reproduces that of the algebra of polyzeta functions.

1 Introduction

The complete journey between the first appearance of a product formula by Bender et al. [1] and their related Feynman-like diagrams to the discovery of a Hopf algebra structure [8] on the diagrams themselves, goes roughly as follows.
Firstly, Bender, Brody, and Meister [1] introduced a special field theory which proved to be particularly rich in combinatorial links and by-products [11] (not to mention the link with vector fields and one-parameter groups [7, 10]).
Secondly, the Feynman-like diagrams of this theory label monomials which combine naturally in a way compatible with monomial multiplication and co-addition. This is the Hopf algebra **DIAG** [8]. The (Hopf-)subalgebra of **DIAG** generated by the primitive graphs is the Hopf algebra **BELL** described in Solomon's talk at this conference [12].
Thirdly, the natural noncommutative pull-back of this algebra, **LDIAG**, has a basis (the *labelled* diagrams) which is in one-to-one correspondence with that of the *Matrix Quasi-Symmetric Functions* [13] (the *packed matrices* of **MQSym**), but their algebra and co-algebra structures are completely different. In particular, multiplication in **MQSym** involves a sort of shifted shuffle with overlappings reminiscent of Hoffmann's shuffle used in the theory of

polyzeta functions [2]. The superpositions and overlappings involved there are not present in (non-deformed) **LDIAG** and, moreover, the coproduct of **LDIAG** is co-commutative while that of **MQSym** is not.

The aim of this paper is to announce the existence of a Hopf algebra deformation which connects **LDIAG** to other Hopf algebras relevant to physics (Connes-Kreimer, Connes-Moscovici, Brouder-Frabetti, see [6]) and other fields (noncommutative symmetric functions, Euler-Zagier sums).
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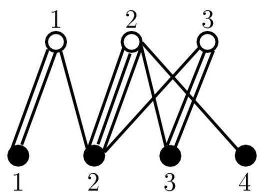
2 Labelled Diagrams and Diagrams

Product formula involves a summation over all diagrams of a certain type [12] a labelled version of which is described below.
Labelled diagrams can be identified with their weight functions which are mappings $\omega : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}$ such that the supporting subgraph

$$\Gamma_\omega = \{(i,j) \in \mathbb{N}^+ \times \mathbb{N}^+ \mid w(i,j) \neq 0\} \tag{1}$$

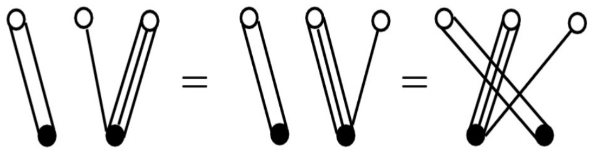
has specific projections i.e. $pr_1(\Gamma_\omega) = [1..p]$; $pr_2(\Gamma_\omega) = [1..q]$ for some $p,q \in \mathbb{N}$ (notice that when one of p,q is zero so too is the other and the diagram is empty).
These graphs are represented by labelled diagrams as follows



The labelled diagrams form the set **ldiag** and prescribe monomials through the formula $\mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)}$ where $\alpha(d)$ (resp. $\beta(d)$) is the "white spot type" (resp. the "black spot type") i.e. the multi-index $(\alpha_i)_{i\in\mathbb{N}^+}$ (resp. $(\beta_i)_{i\in\mathbb{N}^+}$) such that α_i (resp. β_i) is the number of white spots (resp. black spots) of degree i (i lines connected to the spot).
There is a (graphically) natural multiplicative structure on **ldiag** such that the arrow

$$m_{(\mathbb{L},\mathbb{V})} : d \mapsto \mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)} \tag{2}$$

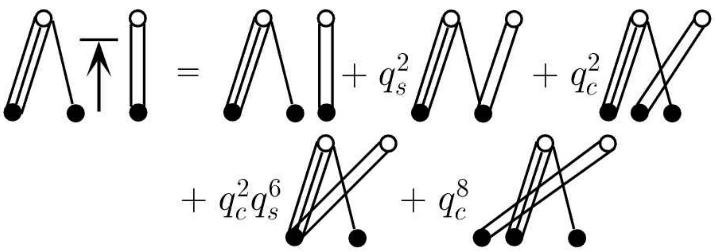
is a morphism.
It is clear that one can permute black spots, or white spots, of d without changing the monomial $\mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)}$. The classes of (labelled) diagrams up to this equivalence (permutations of white - or black - spots among themselves) are naturally represented by unlabelled diagrams and will be denoted **diag** (including the empty one).



For both types of diagram the product consists of concatenating the diagrams i.e. placing d_2 on the right of d_1 [8] (the result, for d_1, d_2, will be denoted $[d_1|d_2]_D$ in **diag** and $[d_1|d_2]_L$ in **ldiag**). These products endow **diag** and **ldiag** with the structure of monoids, with the empty diagram as neutral element. The corresponding commutative diagram is as follows (where X^2 means the cartesian square of the set X).

$$\begin{array}{ccccc} \text{Labelled diagrams}^2 & \xrightarrow{Unlabelling^2} & \text{Diagrams}^2 & \xrightarrow{m_{(\mathbb{L},\mathbb{V})}\times m_{(\mathbb{L},\mathbb{V})}} & \text{Monomials}^2 \\ \text{product}\downarrow & & \text{product}\downarrow & & \text{product}\downarrow \\ \text{Labelled diagrams} & \xrightarrow{Unlabelling} & \text{Diagrams} & \xrightarrow{m_{(\mathbb{L},\mathbb{V})}} & \text{Monomials} \end{array} \qquad (3)$$

It is easy to see that the labelled diagram (resp. diagrams) form free monoids. We denote by **DIAG** and **LDIAG** the K-algebras of these monoids [8] (K is a field).
One can shuffle the product in **ldiag**, counting crossing and superpositions. The definition of the deformed product is expressed by the diagrammatic formula



and the descriptive formula below.

$$[d_1|d_2]_{L(q_c,q_s)} = \sum_{\substack{cs([d_1|d_2]_L)\ are\ all\ crossing\ and \\ superpositions\ of\ black\ spots}} q_c^{nc\times weight_2} q_s^{weight_1\times weight_2} cs([d_1|d_2]_L) \qquad (4)$$

where

- q_c, q_s are coefficients in K

- the exponent of $q_c^{nc \times weight_2}$ is the number of crossings of "what crosses" times its weight
- the exponent of $q_s^{weight_1 \times weight_2}$ is the product of the weights of "what is overlapped"
- terms $cs([d_1|d_2]_L)$ are the diagrams obtained from $[d_1|d_2]_L$ by the process of crossing and superposing the black spots of d_2 on those of d_1, the order and identity of the black spots of d_1 (resp. d_2) being preserved.

What is striking is that this law (denoted above $\overline{\uparrow}$) is associative. Moreover, it can be shown [3, 4] that this process decomposes into two transformations: twisting and shifting. In fact, specialized to certain parameters, this law is reminiscent of others [2].

Parameters	$(0,0)$ (shifted)	$(1,1)$ (shifted)	$(1,1)$ (unshifted)
Laws	**LDIAG**	**MQSym**	Hoffmann & Euler-Zagier

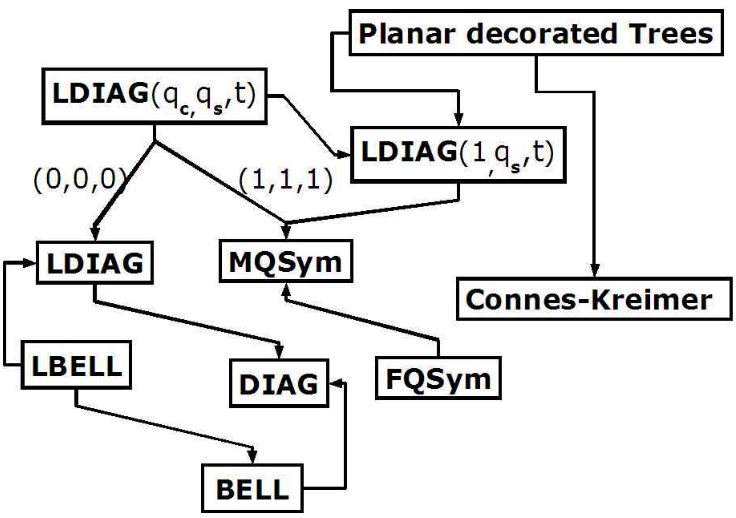
3 Hopf Deformation

Using a total order on the monomials of **ldiag**, it can be shown that the algebra $\mathbf{LDIAG}(q_c,q_s)$ is free. Thus one may construct a coproduct Δ_t; $t \in K$ such that $\mathbf{LDIAG}(q_c,q_s,t) = \big(\mathbf{LDIAG}(q_c,q_s), 1_{\mathbf{ldiag}}, \Delta_t, \varepsilon, S\big)$ is a Hopf algebra. We have the following specializations

$(q_c,q_s,t) =$	$(0,0,0)$	$(1,1,1)$
$\simeq$	**LDIAG**	**MQSym**

4 Conclusion

The results which we announced here in this note can be illustrated by the following picture. All details will be given in forthcoming papers [3, 4].



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