A multipurpose Hopf deformation of the Algebra of Feynman-like Diagrams

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Abstract. We construct a three parameter deformation of the Hopf algebra **LDIAG**. This new algebra is a true Hopf deformation which reduces to **LDIAG** on one hand and to **MQSym** on the other, relating **LDIAG** to other Hopf algebras of interest in contemporary physics. Further, its product law reproduces that of the algebra of polyzeta functions.

1. Introduction

The complete journey between the first appearance of a product formula by Bender et al. [2] and their related Feynman-like diagrams to the discovery of a Hopf algebra structure [11] on the diagrams themselves, goes roughly as follows.
Firstly, Bender, Brody, and Meister [2] introduced a special field theory which proved to be particularly rich in combinatorial links and by-products (not to mention the link with vector fields and one-parameter groups [12, 15]).
Secondly, the Feynman-like diagrams of this theory label monomials which combine naturally in a way compatible with monomial multiplication and co-addition. This is the Hopf algebra **DIAG**.
Thirdly, the natural noncommutative pull-back of this algebra, **LDIAG**, has a basis (the *labelled* diagrams) which is in one-to-one correspondence with that of the *Matrix*

Quasi-Symmetric Functions (the *packed matrices* of **MQSym**), but their algebra and co-algebra structures are completely different. In particular, multiplication in **MQSym** involves a sort of shifted shuffle with overlappings reminiscent of Hoffmann's shuffle used in the theory of polyzeta functions [3]. The superpositions and overlappings involved there are not present in (non-deformed) **LDIAG** and, moreover, the coproduct of **LDIAG** is co-commutative while that of **MQSym** is not.

The aim of this paper is to announce the existence of a Hopf algebra deformation which connects **LDIAG** to other Hopf algebras relevant to physics (Connes-Kreimer, Connes-Moscovici, Brouder-Frabetti, [7]) and other fields (noncommutative symmetric functions, Euler-Zaguier sums).

2. Labelled Diagrams and Diagrams

Labelled diagrams can be identified with their weight functions which are mappings $\omega : \mathbb{N}^+ \times \mathbb{N}^+ \longrightarrow \mathbb{N}$ such that the supporting subgraph

$$\Gamma_\omega = \{(i,j) \in \mathbb{N}^+ \times \mathbb{N}^+ \mid w(i,j) \neq 0\} \tag{1}$$

has specific projections i.e. $pr_1(\Gamma_\omega) = [1..p]$; $pr_2(\Gamma_\omega) = [1..q]$ for some $p, q \in \mathbb{N}$ (notice that when one of p, q is zero so too is the other).
These graphs are represented by labelled diagrams as follows

The labelled diagrams form the set **ldiag** and prescribe monomials through the formula $\mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)}$ where $\alpha(d)$ (resp. $\beta(d)$) is the "white spot type" (resp. the "black spot type") i.e. the multi-index $(\alpha_i)_{i\in\mathbb{N}^+}$ $((\beta_i)_{i\in\mathbb{N}^+})$ such that α_i (resp. β_i) is the number of white spots (resp. black spots) of degree i (i lines connected to the spot).
There is a (graphically) natural multiplicative structure on **ldiag** such that the arrow

$$m(?, \mathbb{L}, \mathbb{V}) : d \mapsto \mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)} \tag{2}$$

is a morphism.
It is clear that one can permute black spots, or white spots, of d without changing the monomial $\mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)}$. The classes of (labelled) diagrams up to this equivalence (permutations of white - or black - spots among themselves) are naturally represented by unlabelled diagrams and will be denoted **diag** (including the empty one).

For both types of diagram the product consists of concatenating the diagrams i.e. placing d_2 on the right of d_1 [11] (the result, for d_1, d_2, will be denoted $[d_1|d_2]_D$ in **diag** and $[d_1|d_2]_L$ in **ldiag**). These products endow **diag** and **ldiag** with the structure of monoids, with the empty diagram as neutral element. The corresponding commutative diagram is as followa.

$$\begin{array}{ccccc}
\text{Labelled diagrams}^2 & \xrightarrow{\textit{Unlabelling}} & \text{Diagrams}^2 & \xrightarrow{m(?,\mathbb{L},\mathbb{V})\times m(?,\mathbb{L},\mathbb{V})} & \text{Monomials}^2 \\
{\scriptstyle\text{product}}\downarrow & & {\scriptstyle\text{product}}\downarrow & & {\scriptstyle\text{product}}\downarrow \\
\text{Labelled diagrams} & \xrightarrow{\textit{Unlabelling}} & \text{Diagrams} & \xrightarrow{m(?,\mathbb{L},\mathbb{V})} & \text{Monomials}
\end{array} \tag{3}$$

It is easy to see that the labelled diagram (resp. diagrams) form free monoids. We denote by **DIAG** and **LDIAG** the K-algebras of these monoids [**?**, 1] (K is a field). One can shuffle the product in **ldiag**, counting crossing and superpositions. The definition of the deformed product is expressed by the descriptive formula and diagram below.

$$[d_1|d_2]_{L(q_c,q_s)} = \sum_{\substack{cs(?)\ all\ crossing\ and\\ superpositions\ of\ black\ spots}} q_c^{nc \times weight_2} q_s^{weight_1 \times weight_2} cs([d_1|d_2]_L) \tag{4}$$

where

- q_c, q_s are coefficients in K
- the exponent of $q_c^{nc \times weight_2}$ is the number of crossings of "what crosses" times its weight
- the exponent of $q_s^{weight_1 \times weight_2}$ is the product of the weights of "what is overlapped"
- $cs(?)$ are the diagrams obtained from $[d_1|d_2]_L$ by the process of crossing and superposing the black spots of d_2 on those of d_1, the order and identity of the black spots of d_1 (i.e. d_2) being preserved.

$$= \qquad + q_s^2 \qquad + q_c^2$$

$$+ q_c^2 q_s^6 \qquad + q_c^8$$

What is striking is that this law is associative. Moreover, it can be shown [5, 4] that this process decomposes into two transformations: twisting and shifting. In fact, specialized to certain parameters, this law is reminiscent of others [3].

Parameters	$(0,0)$ (shifted)	$(1,1)$ (shifted)	$(1,1)$ (unshifted)
Laws	**LDIAG**	**MQSym**	Hoffmann & Euler-Zagier

3. Hopf Deformation

Using a total order on the monomials of **ldiag**, it can be shown that the algebra $\mathbf{LDIAG}(q_c, q_s)$ is free. Thus one may construct a coproduct Δ_t such that $(\mathbf{LDIAG}(q_c, q_s), 1_{\mathbf{ldiag}}, \Delta_t, \varepsilon, S)$ is a Hopf algebra. We have the following specializations

$(q_c, q_s, t) =$	$(0,0,0)$	$(1,1,1)$
$\simeq$	**LDIAG**	**MQSym**

We finally obtain the following general picture

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