A Quantum Field Theoretical Representation of Euler-Zagier Sums

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Abstract

We establish a novel representation of arbitrary Euler-Zagier sums in terms of weighted vacuum graphs. This representation uses a toy quantum field theory with infinitely many propagators and interaction vertices. The propagators involve Bernoulli polynomials and Clausen functions to arbitrary orders. The Feynman integrals of this model can be decomposed in terms of a vertex algebra whose structure we investigate. We derive a large class of relations between multiple zeta values, of arbitrary lengths and weights, using only a certain set of graphical manipulations on Feynman diagrams. Further uses and possible generalizations of the model are pointed out.

1 Introduction

The perturbative evaluation of Green's functions in quantum field theory lead to a class of iterated parameter integrals whose explicit calculation becomes very difficult beyond the first few orders in the coupling constant expansion. Any progress in this area of work must be based on an intimate knowledge of the properties of various types of special functions such as polylogarithms, hypergeometric functions, and their generalizations (see e.g. [38]).

In recent years, some structure is seen to emerge from the seemingly haphazard occurence of those special functions as the values of individual Feynman diagrams. Kreimer's hypothesis [27], based on a rule of associating knots to Feynman diagrams, allows one to predict from knot-theoretical considerations the level of transcendentality which can possibly appear in the counterterm coefficients of an ultraviolet divergent diagram. Even though it has been verified for a large number of examples [13] the raison d'être for the correspondence between graphs and knots remains presently mysterious. More recently there are indications that knot-theoretical concepts may be of relevance even for the finite parts of Feynman diagrams [11].

The remarkably rich mathematical structures surfacing in this correspondence make Feynman diagrams increasingly interesting from the pure mathematician's point of view. The objects encountered in the calculation of UV divergences in perturbative quantum field theory, multiple harmonic sums, are of considerable relevance to number theory and other branches of mathematics (see e.g. [15, 43, 28, 19, 40, 41, 17, 33]).

Quantum field theory amplitudes can be calculated in coordinate space or in momentum space. In four-dimensional field theory the arising integrals are normally of a similar type and degree of difficulty. This is very different in the case of a one-dimensional quantum field theory compactified on a circle, which will be considered in the present paper. Such quantum field theories arise naturally if one represents one-loop amplitudes in D-dimensional field theory in terms of first-quantized path integrals. An approach to quantum field theory along these lines has gained some popularity in recent years after it was discovered that it allows one to reorganise ordinary field theory amplitudes in a manner similar to string theory amplitudes [32, 4, 39, 36, 37]. In this type of formalism the D-dimensional space-time enters as a target space, and amplitudes are calculated in terms of an auxiliary field theory in one-dimensional parameter space. Green's functions in parameter space are then used for the evaluation of Feynman diagrams in this one-dimensional 'worldloop' theory.

As a simple example, let us consider the one-loop effective action for a scalar field theory with a $\frac{\lambda}{3!}\phi^3$ interaction. This effective action can be expressed in terms of a first-quantized path integral as follows,

$$\Gamma[\phi] = \frac{1}{2}\int_0^\infty \frac{\mathrm{d}T}{T}\,\mathrm{e}^{-m^2 T}\int_{x(T)=x(0)} \mathcal{D}x(\tau)\,\mathrm{e}^{-\int_0^T \mathrm{d}\tau\left(\frac{1}{4}\dot{x}^2+\lambda\phi(x(\tau))\right)}. \tag{1.1}$$

Here T is the usual Schwinger proper-time for the particle circulating in the loop. At fixed T a path integral has to be calculated over the space of closed loops in spacetime with period T. This integral contains a zero mode which is removed by fixing the center-of-mass of the loop $x_0 \equiv \frac{1}{T}\int_0^T \mathrm{d}\tau x(\tau)$. The reduced path integral is evaluated perturbatively by expanding the interaction exponential and using the parameter space Green's function [1]

$$G(\tau_1,\tau_2) \quad\equiv\quad 2T\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{\mathrm{e}^{2\pi \mathrm{i} n\frac{\tau_1-\tau_2}{T}}}{(2\pi\mathrm{i}n)^2} = \mid \tau_1-\tau_2\mid - \frac{(\tau_1-\tau_2)^2}{T} - \frac{T}{6}. \tag{1.2}$$

Momentum space methods have also sometimes been used, which in this case lead to Fourier sum representations. In [3, 14] a number of such sums were calculated to provide a check on the resolution of certain ambiguities which in the first-quantized formalism can arise in curved backgrounds. In this comparison one finds that terms given by simple polynomial integrals in coordinate space may, in momentum space, correspond to non-trivial multiple sums of the Euler-Zagier type.

In the present work we will turn the logic around, and use this formalism as a tool for the systematic study of Euler-Zagier sums. Euler-Zagier sums, also called multiple ζ values or multiple harmonic series, are defined by

$$\zeta(k_1,\ldots,k_m) \quad=\quad \sum_{n_1>n_2>\cdots>n_m>0}\frac{1}{n_1^{k_1}\cdots n_m^{k_m}} = \mathrm{Li}_{k_1\ldots k_m}(1,\ldots,1). \tag{1.3}$$

They are special values of the multidimensional polylogarithms $\mathrm{Li}_{k_1,\ldots,k_m}$, defined as

$$\mathrm{Li}_{k_1\ldots k_m}(z_1,\ldots,z_m) \quad\equiv\quad \sum_{n_1>n_2>\cdots>n_m>0}\frac{z_1^{n_1}\cdots z_m^{n_m}}{n_1^{k_1}\cdots n_m^{k_m}}. \tag{1.4}$$

[1]The constant part of this Green's function is irrelevant for the final physical results and usually deleted from the beginning.

One calls m the length (or depth) of such a series, and $k_1 + k_2 + \cdots + k_m$ its level (or weight). Sums of the type (1.3) were first considered by Euler [16]. Euler himself noted that numerous relations exist between Euler-Zagier sums. Some simple examples are the following (all given by Euler)

$$\begin{aligned}
\zeta(2,1) &= \zeta(3), & (1.5)\\
\zeta(3,1) &= \frac{3}{2}\zeta(4) - \frac{1}{2}\zeta^2(2) = \frac{\pi^4}{360}, & (1.6)\\
\zeta(2,2) &= \frac{1}{2}\zeta^2(2) - \frac{1}{2}\zeta(4) = \frac{\pi^4}{120}, & (1.7)\\
\zeta(3,2) &= -\frac{11}{2}\zeta(5) + 3\zeta(2)\zeta(3), & (1.8)\\
\zeta(4,1) &= 2\zeta(5) - \zeta(2)\zeta(3). & (1.9)
\end{aligned}$$

Further results for the length two case can be found in [42, 1]. Systematic investigations of Euler-Zagier sums of length higher than two have been undertaken only in recent years [34, 21, 30, 24, 20, 9, 22, 6, 31, 2, 25]. From the point of view of physics the study of their relations is relevant for attempts at a classification of the possible ultraviolet divergences in quantum field theory [12, 13].

We would like to be able to represent arbitrary such sums in terms of one-dimensional Feynman diagrams. To achieve this goal we have to generalize the usual worldline path integral formalism in the following ways:

1. As explained above, the first-quantized loop path integral is defined to run over the space of all periodic functions, with the constant functions eliminated. Here we will restrict it to one "chiral half" spanned by the basis functions $f_n(u) = \mathrm{e}^{2\pi i n u}$, $n = 1, 2, \ldots$. (This amounts to a complexification of spacetime.)

2. We choose the kinetic term of our model in such a way that arbitrary inverse powers of derivatives will appear.

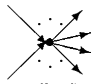
Those purely mathematical considerations lead us to define the "ζ-model", a one-dimensional quantum field theory given by the following partition function,

$$\begin{aligned}
Z(g,\lambda) &= \int_{\mathcal{H}} \mathcal{D}x(u)\, \mathrm{e}^{-S},\\
S &= \int_0^1 \mathrm{d}u_1 \int_0^1 \mathrm{d}u_2\, \frac{1}{2}\bar{x}(u_1)\left(1 - \lambda 2\pi \mathrm{i}\partial^{-1}\right)x(u_2) - \int_0^1 \mathrm{d}u\, \mathrm{e}^{gx(u)+\bar{g}\bar{x}(u)}.
\end{aligned} \tag{1.10}$$

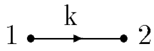
Here and in the following $\partial \equiv \frac{\mathrm{d}}{\mathrm{d}u}$ denotes the ordinary derivative. The path integral is to be performed over the Hilbert space

$$\mathcal{H} = \Big\{ x(u) \Big| x(u) = \sum_{n=1}^{\infty} a_n \mathrm{e}^{2\pi \mathrm{i} n u}, \quad \sum_{n=1}^{\infty} |a_n|^2 < \infty \Big\}. \tag{1.11}$$

The perturbative expansion of both the kinetic and the interaction terms for this toy model leads to the following Feynman rules,

Vertices $\mathcal{V}^{p,q}$:  $p, q = 0, 1, 2, \ldots$

$\bar{g}^p g^q$

Propagators: 

$k = 0, 1, 2, \ldots$

$\lambda^k (2\pi\mathrm{i})^k g_{12}^{(k)}$

Figure 1: Feynman rules of the ζ-model.

In section 2 we write the propagators $g_{12}^{(k)}$ explicitly in terms of Bernoulli polynomials and Clausen functions. We show that any multiple ζ sum (1.3) can be represented as a Feynman diagram in this model. In section 3 we investigate the properties of the elementary tree-level n-point integrals with arbitrary external propagators. Section 4 demonstrates how one can use partial integrations and reality conditions to derive a large class of relations between multiple ζ sums. In section 5 we point out possible further uses of the model, as well as some generalizations.

2 Basic Properties

Inverting the kinetic part of our (non-local) Lagrangian (1.10), and writing ∂^{-k} in the defining basis of the Hilbert space $\mathcal{H}$, we find

$$g_{12}^{(k)} \;\; \equiv \;\; g^{(k)}(u_{12}) = \sum_{n=1}^{\infty} \frac{\mathrm{e}^{2\pi\mathrm{i} n u_{12}}}{(2\pi\mathrm{i} n)^k}. \tag{2.1}$$

If we represent the unit circle in the complex plane, this sum will turn into the k-th polylogarithm,

$$g_{12}^{(k)} \;\; = \;\; \frac{1}{(2\pi\mathrm{i})^k} \, \mathrm{Li}_k\Big(\frac{z_1}{z_2}\Big). \tag{2.2}$$

($u_{12} = u_1 - u_2$, $z_i = \mathrm{e}^{2\pi \mathrm{i} u_i}$). We note the following properties of $g^{(k)}$,

$$\frac{\partial}{\partial u_1} g_{12}^{(k)} = -\frac{\partial}{\partial u_2} g_{12}^{(k)} = g_{12}^{(k-1)}, \tag{2.3}$$

$$g_{21}^{(k)} = (-1)^k \bar{g}_{12}^{(k)} = g^{(k)}(1 - u_{12}), \tag{2.4}$$

$$g^{(k)}(0) = \frac{\zeta(k)}{(2\pi \mathrm{i})^k}, \tag{2.5}$$

$$\int_0^1 \mathrm{d}u_{1,2}\, g_{12}^{(k)} = 0. \tag{2.6}$$

(2.3) can be inverted using the explicitly known integral kernel for inverse derivatives in this space [36],

$$g_{12}^{(k+l)} = \int_0^1 \mathrm{d}u \,\langle u_1 \mid \partial^{-l} \mid u\rangle\, g^{(k)}(u - u_2), \tag{2.7}$$

$$\langle u_1 \mid \partial^{-n} \mid u_2\rangle = -\frac{1}{n!} B_n(|u_{12}|)\,\mathrm{sign}^n(u_{12}) \tag{2.8}$$

$$= -\frac{B_n(u_{12})}{n!} + \frac{u_{12}^{n-1}}{2(n-1)!}(\mathrm{sign}(u_{12}) - 1). \tag{2.9}$$

Here B_k denotes the k-th Bernoulli polynomial. To write $g_{12}^{(k)}$ more explicitly we split it into its real and imaginary parts,

$$g_{12}^{(k)} = \bar{g}_{12}^{(k)} + \mathrm{i}\,\hat{g}_{12}^{(k)}. \tag{2.10}$$

Using the standard integral representation of the polylogarithm

$$\mathrm{Li}_k(z) = \frac{(-1)^{k-1}}{(k-1)!}\, z \int_0^1 \mathrm{d}x\, \frac{\ln^{k-1}(x)}{1 - xz}, \tag{2.11}$$

it is then easy to show that

$$g^{(0)}_{12} = \frac{1}{2}\Big(\delta(u_{12}) - 1 + \mathrm{i}\,\cot(\pi u_{12})\Big), \tag{2.12}$$

$$g^{(1)}_{12} = \frac{1}{2}\Big(\frac{1}{2}\mathrm{sign}(u_{12}) - u_{12} + \frac{\mathrm{i}}{\pi}\ln|2\sin(\pi u_{12})|\Big), \tag{2.13}$$

$$g^{(2)}_{12} = \frac{1}{4}\Big(|u_{12}| - u_{12}^2 - \frac{1}{6} + \frac{\mathrm{i}}{\pi^2}\int_0^1 \mathrm{d}\xi\,\frac{\ln\xi\sin(2\pi u_{12})}{1-2\xi\cos(2\pi u_{12})+\xi^2}\Big), \tag{2.14}$$

$$\begin{aligned} g^{(3)}_{12} &= -\frac{1}{24}(\mathrm{sign}(u_{12}) - 2u_{12})(|u_{12}| - u_{12}^2) \\ &\quad + \frac{\mathrm{i}}{16\pi^3}\int_0^1 \frac{\mathrm{d}\xi}{\xi}\ln\xi\ln(1-2\xi\cos(2\pi u_{12})+\xi^2), \end{aligned} \tag{2.15}$$

$$\begin{aligned} g^{(k\neq 0\text{ even})}_{12} &= -\frac{1}{2}\frac{1}{k!}B_k(|u_{12}|) \\ &\quad + \frac{\mathrm{i}}{(2\pi)^k}\frac{(-1)^{\frac{k}{2}+1}}{(k-1)!}\int_0^1 \mathrm{d}\xi\,\ln^{k-1}\xi\frac{\sin(2\pi u_{12})}{1-2\xi\cos(2\pi u_{12})+\xi^2}, \end{aligned} \tag{2.16}$$

$$\begin{aligned} g^{(k\neq 1\text{ odd})}_{12} &= -\frac{1}{2}\frac{1}{k!}B_k(|u_{12}|)\mathrm{sign}(u_{12}) \\ &\quad + \frac{\mathrm{i}}{2(2\pi)^k}\frac{(-1)^{\frac{k+1}{2}}}{(k-2)!}\int_0^1 \frac{\mathrm{d}\xi}{\xi}\ln^{k-2}\xi\ln(1-2\xi\cos(2\pi u_{12})+\xi^2), \end{aligned} \tag{2.17}$$

($u_{12} \in [-1,1]$). Note also that the imaginary part of $g^{(k)}_{12}$ is, up to a normalization factor, identical with the k-th Clausen function $\mathrm{Cl}_k(2\pi u_{12})$ (see, e.g., [29]). Only the real parts are present in the calculation of worldline path integrals in standard field theory. In particular, note that $\bar{g}^{(2)}$ is, up to a conventional factor of 4, identical with the function G introduced in (1.2) (for $T=1$). Only the imaginary parts are capable of producing $\zeta(n)$'s with n odd.

A vacuum diagram in our model involves a multiple integral on the unit circle with an integrand which is a product of propagators (2.1). (One of the integrations is redundant due to translation invariance.) The result of the integrations can obviously be decomposed as a sum of multiple ζ values. Our model thus defines a map from the set of weighted vacuum graphs to the (vector space generated by) the Euler-Zagier sums. This map is, moreover, easily seen to be surjective; the reader will immediately convince herself that with the above Feynman rules the following "sea shell" diagram (fig.

2) evaluates to

$$\lambda^{\sum k_i}(g\bar{g})^{2m-1}\zeta(k_1,\ldots,k_m). \tag{2.18}$$

The u-integrations produce δ functions for "momentum conservation" at every vertex, while the Fourier sums of the inserted $g^{(0)}$ propagators yield Heaviside step functions leading to the desired ordering of the remaining sums.

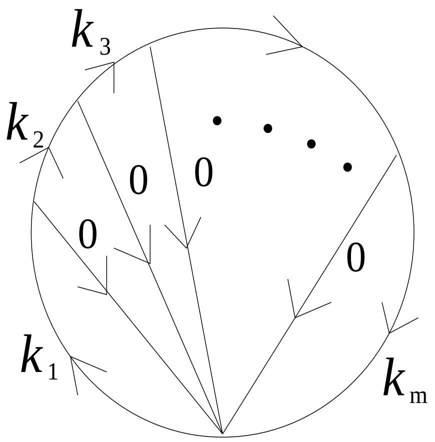


Figure 2: 'Sea shell' diagram representing the general Euler-Zagier sum.

3 The elementary vertex integrals

Let us now start on an investigation of the properties of the Feynman integrals in $x(=u)$-space. An obvious first step is to consider the folding of the elementary vertices with arbitrary sets of propagators. We denote the elementary vertex integral by

$$\begin{aligned} I^{l_1\ldots l_q}_{k_1\ldots k_p}(u_1,\ldots,u_{p+q}) \equiv & \int_0^1 \mathrm{d}u\, g^{(k_1)}(u_1-u)\cdots g^{(k_p)}(u_p-u) \\ & \times g^{(l_1)}(u-u_{p+1})\cdots g^{(l_q)}(u-u_{p+q}). \end{aligned} \tag{3.1}$$

We note that it has the following obvious properties,

$$I^{l_1\ldots l_q}_{k_1\ldots k_p} = 0 \qquad \text{if } p=0 \text{ or } q=0, \tag{3.2}$$

$$\bar{I}^{l_1\ldots l_q}_{k_1\ldots k_p} = (-1)^{\sum_{i=1}^p k_i+\sum_{j=1}^q l_j} I^{k_1\ldots k_p}_{l_1\ldots l_q}. \tag{3.3}$$

3.1 Two-vertex integral

By construction two-point vertices can be integrated out trivially,

$$\int_0^1 \mathrm{d}u_3\, g_{13}^{(k)} g_{32}^{(l)} = g_{12}^{(k+l)}. \tag{3.4}$$

3.2 Three-vertex integral

The evaluation of vertex integrals is complicated by singularities which can appear due to the presence of the cotangent function in $g^{(0)}$. Integrals involving $\cot(\pi u_{12})$ need to be performed using the principal value prescription. One way of calculating them is to transform them into complex contour integrals via the substitution $z = \exp(2\pi \mathrm{i} u)$,

$$\int_0^1 \mathrm{d}u \prod_{k=1}^{n} \cot(\pi(u-u_k)) = \mathrm{i}^n \oint \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \prod_{k=1}^{n} \frac{z+z_k}{z-z_k}. \tag{3.5}$$

Those are evaluated by means of residues (the poles on the contour give half values due to the principal value prescription) [2]. The first few integrals are

$$\begin{aligned}
\int_0^1 \mathrm{d}u\, \cot(\pi(u-u_1)) &= 0, &(3.6)\\
\int_0^1 \mathrm{d}u \prod_{k=1}^{2} \cot(\pi(u-u_k)) &= \delta(u_1-u_2) - 1, &(3.7)\\
\int_0^1 \mathrm{d}u \prod_{k=1}^{3} \cot(\pi(u-u_k)) &= \delta(u_1-u_2)\cot(\pi(u_2-u_3)) + \\
&+ \delta(u_1-u_3)\cot(\pi(u_1-u_2)) + \\
&+ \delta(u_2-u_3)\cot(\pi(u_2-u_1)). &(3.8)
\end{aligned}$$

Alternatively one can also calculate those integrals recursively using, under the integral, the following identity,

$$\begin{aligned}
&\cot(\pi u_{12})\cot(\pi u_{13}) + \cot(\pi u_{21})\cot(\pi u_{23}) + \\
&\quad + \cot(\pi u_{31})\cot(\pi u_{32}) = -1 + \delta(u_{12})\delta(u_{13}).
\end{aligned} \tag{3.9}$$

[2] We remark that, if one expresses the propagators of the sea shell diagram via (2.2) and (2.11), and then calculates the u_i-integrals by means of residues, then one arrives precisely at Kontsevich's integral representation [26] for the multiple ζ sum.

This identity will be of further use later on. With these results the vertex integral of three propagators $g^{(0)}$ is

$$
\begin{aligned}
I^0_{00}(u_1,u_2,u_3) &= \int_0^1 \mathrm{d}u\, g^{(0)}(u_1-u)g^{(0)}(u_2-u)g^{(0)}(u-u_3) \\
&= \frac{1}{4}\left[\delta(u_{13})\delta(u_{23}) - \delta(u_{13}) - \delta(u_{23}) + 1\right. \\
&\quad +(\delta(u_{13})-1)\mathrm{i}\cot(\pi u_{23}) \\
&\quad +(\delta(u_{23})-1)\mathrm{i}\cot(\pi u_{13}) \\
&\quad \left.-\cot(\pi u_{23})\cot(\pi u_{13})\right],
\end{aligned}
\tag{3.10}
$$

and can be identified as

$$
I^0_{00}(u_1,u_2,u_3) = g^{(0)}_{13}g^{(0)}_{23}. \tag{3.11}
$$

Applying the identity (3.9) gives also

$$
\begin{aligned}
I^0_{00}(u_1,u_2,u_3) &= g^{(0)}_{12}g^{(0)}_{23} - g^{(0)}_{12}g^{(0)}_{13} - \frac{1}{2}g^{(0)}_{13} \\
&= g^{(0)}_{21}g^{(0)}_{13} - g^{(0)}_{21}g^{(0)}_{23} - \frac{1}{2}g^{(0)}_{23} \\
&= g^{(0)}_{12}g^{(0)}_{23} + g^{(0)}_{21}g^{(0)}_{13} - \frac{1}{2}\delta_{12}g^{(0)}_{13}.
\end{aligned}
\tag{3.12}
$$

Folding of eq. (3.11) with $\langle u_{1'}|\partial^{-n}|u_1\rangle$ and $\langle u_{2'}|\partial^{-m}|u_2\rangle$ leads to

$$
\begin{aligned}
I^0_{nm}(u_1,u_2,u_3) &= \int_0^1 \mathrm{d}u\, g^{(n)}(u_1-u)g^{(m)}(u_2-u)g^{(0)}(u-u_3) \\
&= g^{(n)}_{13}g^{(m)}_{23},
\end{aligned}
\tag{3.13}
$$

whereas the forms (3.12) can be folded with $\langle u_3|\partial^{-k}|u_{3'}\rangle$

$$
\begin{aligned}
I^k_{00}(u_1,u_2,u_3) &= \int_0^1 \mathrm{d}u\, g^{(0)}(u_1-u)g^{(0)}(u_2-u)g^{(k)}(u-u_3) \\
&= g^{(0)}_{12}g^{(k)}_{23} - g^{(0)}_{12}g^{(k)}_{13} - \frac{1}{2}g^{(k)}_{13} \\
&= g^{(0)}_{21}g^{(k)}_{13} - g^{(0)}_{21}g^{(k)}_{23} - \frac{1}{2}g^{(k)}_{23} \\
&= g^{(0)}_{12}g^{(k)}_{23} + g^{(0)}_{21}g^{(k)}_{13} - \frac{1}{2}\delta_{12}g^{(k)}_{13}.
\end{aligned}
\tag{3.14}
$$

Nontrivial is the case

$$\begin{aligned} I^l_{0k}(u_1,u_2,u_3) &= \int_0^1 \mathrm{d}u\, g^{(0)}(u-u_1)g^{(k)}(u-u_2)g^{(l)}(u_3-u) \\ &= \mathrm{Li}_{lk}(z_{31},z_{12}). \end{aligned} \tag{3.15}$$

The general case of all three indices different from zero can be reduced to this case by partial integrations, e.g.

$$\begin{aligned} I^1_{11}(u_1,u_2,u_3) &= \int_0^1 \mathrm{d}u\, g^{(1)}(u_{41})g^{(1)}(u_{42})g^{(1)}(u_{34}) \\ &= \int_0^1 \mathrm{d}u\, g^{(1)}(u_{41})g^{(0)}(u_{42})g^{(2)}(u_{34}) \\ &\quad + \int_0^1 \mathrm{d}u\, g^{(0)}(u_{41})g^{(1)}(u_{42})g^{(2)}(u_{34}) \\ &= \mathrm{Li}_{21}(z_{31},z_{12}) + \mathrm{Li}_{21}(z_{32},z_{21}). \end{aligned} \tag{3.16}$$

3.3 A three – point relation

Beyond the three-point case the systematic investigation of the elementary vertex integrals becomes cumbersome. We will be satisfied here to note that, from the representation (2.12) for $g^{(0)}$ and the identity (3.9) one easily derives the following pair of (complex conjugate) three-point identities,

$$\begin{aligned} g^{(0)}_{21}g^{(0)}_{31} + g^{(0)}_{12}g^{(0)}_{32} + g^{(0)}_{13}g^{(0)}_{23} &= 1 + \delta_{12}g^{(0)}_{32} + \delta_{31}g^{(0)}_{21} + \delta_{23}g^{(0)}_{13} - \delta_{12}\delta_{13}, \\ g^{(0)}_{12}g^{(0)}_{13} + g^{(0)}_{21}g^{(0)}_{23} + g^{(0)}_{31}g^{(0)}_{32} &= 1 + \delta_{12}g^{(0)}_{23} + \delta_{31}g^{(0)}_{12} + \delta_{23}g^{(0)}_{31} - \delta_{12}\delta_{13}. \end{aligned} \tag{3.17}$$

We may represent the first identity graphically as in fig. 3.

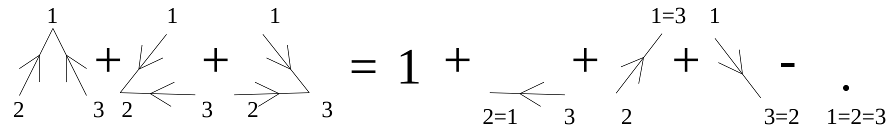


Figure 3: Three-point relation.

The second identity has all arrows reversed. Those identities can be used to transform any vertex integral involving two $g^{(0)}$'s which are either both ingoing or both outgoing.

By iteration of this three-point identity one can contruct an analogous identity for an arbitrary number of points. The formulas are rather cumbersome and will not be given here.

4 Derivation of Multiple ζ Relations

We will now show how can use the formalism developed above for deriving a large class of multiple ζ relations by simple manipulations on graphs, with no need to ever explicitly write down sums. In those manipulations we will make use of the following elements:

1. The triviality of the real part of $g^{(0)}$, $\bar{g}^{(0)}_{12} = \frac{1}{2}(\delta_{12} - 1)$ (see (2.12)).
2. Partial integrations.
3. The three-point relations eq. (3.17).
4. The vanishing of diagrams containing a vertex with only ingoing or only outgoing propagators.
5. The two-vertex integration formula eq. (3.4).

At intermediate steps sometimes ill-defined sums will appear such as $\zeta(1)$. Those will always cancel out in the final results. The appearance of divergent sums could be easily avoided using a regularization such as in [9] but we will not do so here.

4.1 Length two

Let us begin at length two. The simplest way of arriving at identities is provided by the first one of the points listed above. Consider the sea shell diagram at length two (fig. 4a), representing $\zeta(a,b)$, and the same diagram with the middle propagator reversed (fig. 4b), representing $\zeta(b,a)$. (We will generally disregard the coupling constant factors in the following.)
Adding up both diagrams we can replace the middle propagator by twice its real part (fig. 4). Since $\bar{g}^{(0)}_{12} = \frac{1}{2}(\delta_{12} - 1)$ this diagram can then be replaced by the sum of the two diagrams shown in the right hand side of fig. 4. Using eq.(3.4) on the rightmost one of those we obtain the identity

$$\zeta(a,b) + \zeta(b,a) \quad = \quad \zeta(a)\zeta(b) - \zeta(a+b). \tag{4.1}$$

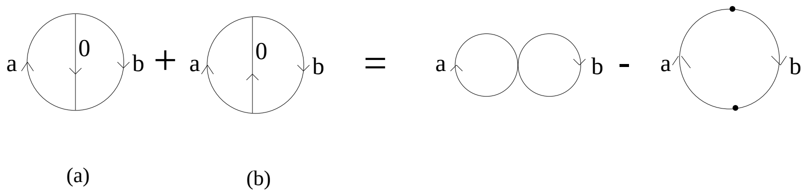


Figure 4: Diagrammatic representation of the reflection identity.

This identity is well-known [21, 9], and has been named "reflection formula" in [9].

Another way of obtaining identities is partial integration. Instead of considering the sea shell diagram (fig. 4a), which represents $\zeta(a,b)$, let us consider the more general diagram (fig. 5) that represents a number $G_{a,b,c} = G_{a,c,b}$. For $a = 0$, $b = 0$ or $c = 0$, it can be represented by multiple ζ

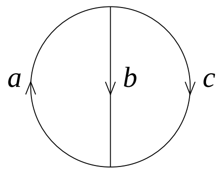


Figure 5: Auxiliary diagrams for multiple ζ relations of length 2.

functions,

$$G_{0,b,c} = \zeta(b)\zeta(c), \quad G_{a,b,0} = G_{a,0,b} = \zeta(a,b). \tag{4.2}$$

To see the first of these identities one adds the same diagram with the left propagator reversed, and uses elements one and four from the above list. For the later discussion we refer to these diagrams as zeta diagrams in contrast to the other, non-zeta diagrams. If both a and c are greater than 1, we use integration by parts at the upper vertex according to fig. 6 and obtain

$$G_{a,b,c} = G_{a-1,b+1,c} - G_{a,b+1,c-1}. \tag{4.3}$$

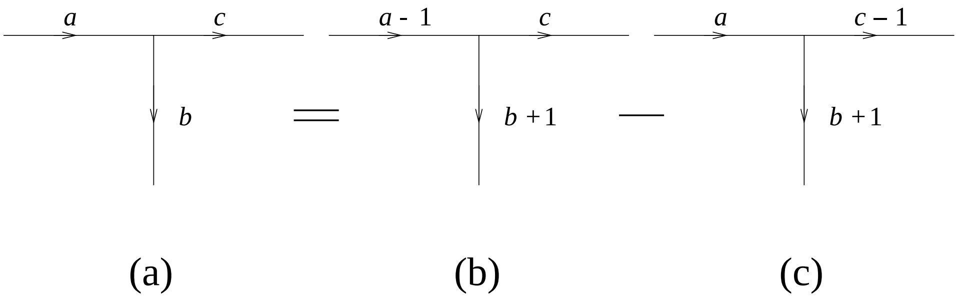


Figure 6: Integration by parts.

This can be repeated until either $a = 0$ or $c = 0$

$$\begin{aligned} G_{a,b,c} &= \sum_{n=1}^{c} (-1)^{c+n} \binom{a+c-n-1}{a-1} G_{0,a+b+c-n,n} + \\ &\quad + \sum_{n=1}^{a} (-1)^{c} \binom{a+c-n-1}{c-1} G_{n,a+b+c-n,0}. \end{aligned} \tag{4.4}$$

The right side of this equation can be translated immediately into ζ values by (4.2). For $b = 0$, we obtain the relation

$$\begin{aligned} \zeta(a,b) = (-1)^b \quad \Big[& \sum_{n=1}^{b} (-1)^n \binom{a+b-n-1}{a-1} \zeta(n)\zeta(a+b-n) + \\ & + \sum_{n=1}^{a} \binom{a+b-n-1}{b-1} \zeta(n, a+b-n) \Big]. \end{aligned} \tag{4.5}$$

The divergent $\zeta(1)$ and $\zeta(1,a)$ appear here always in the combination

$$\zeta(1)\zeta(a+b-1) - \zeta(1, a+b-1), \tag{4.6}$$

which can be reexpressed in terms of convergent sums by the reflection identity (4.1). In this way we arrive at

$$\begin{aligned} \zeta(a,b) = (-1)^b \quad \Big[& \sum_{n=2}^{b} (-1)^n \binom{a+b-n-1}{a-1} \zeta(n)\zeta(a+b-n) + \\ & + \sum_{n=2}^{a} \binom{a+b-n-1}{b-1} \zeta(n, a+b-n) - \\ & - \binom{a+b-2}{a-1} \left(\zeta(a+b) + \zeta(a+b-1, 1) \right) \Big]. \end{aligned} \tag{4.7}$$

Here on the left side we must assume $a > 1$ if regularisation is to be avoided.

4.2 Length three

Proceeding to sums of length three, let us again begin by exploiting the triviality of $\bar{g}^{(0)}$. Since there are two $g^{(0)}$ propagators we have now several possibilities. Fig. 7 shows the "standard" sea shell diagram, representing $\zeta(a, b, c)$, as well as the diagrams related to it by a change of direction of one or both of the $g^{(0)}$ propagators.

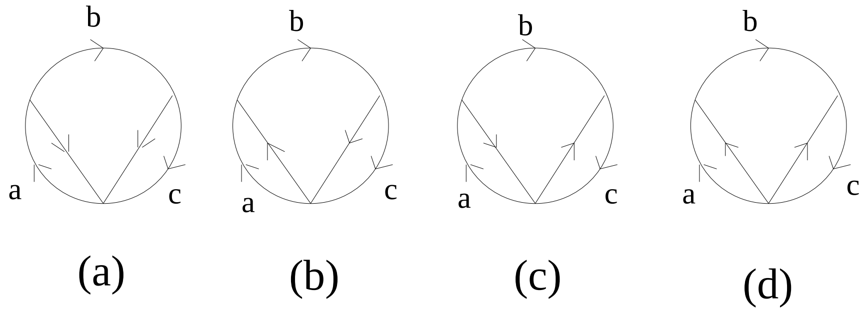


Figure 7: Diagrams related to $\zeta(a, b, c)$. The propagators without label are $g^{(0)}$ propagators.

Those diagrams represent the quantities

$$
\begin{aligned}
(a) &: \zeta(a, b, c), \\
(b) &: \zeta(b, a, c) + \zeta(b, c, a) + \zeta(b, a + c), \\
(c) &: \zeta(a, c, b) + \zeta(c, a, b) + \zeta(a + c, b), \\
(d) &: \zeta(c, b, a).
\end{aligned}
\tag{4.8}
$$

Adding those diagrams in pairs to create $\bar{g}^{(0)}$'s one obtains the following four identities,

$$
\zeta(a, b, c) + \zeta(b, a, c) + \zeta(b, c, a) + \zeta(b, a + c) + \zeta(a + b, c) - \zeta(a)\zeta(b, c) = 0, \tag{4.9}
$$

$$\zeta(a,b,c)+\zeta(a,c,b)+\zeta(c,a,b)+\zeta(a+c,b)+\zeta(a,b+c)-\zeta(c)\zeta(a,b) = 0, \tag{4.10}$$
$$\zeta(b,a,c)+\zeta(b,c,a)+\zeta(c,b,a)+\zeta(b,a+c)+\zeta(b+c,a)-\zeta(c)\zeta(b,a) = 0, \tag{4.11}$$
$$\zeta(a,c,b)+\zeta(c,a,b)+\zeta(c,b,a)+\zeta(a+c,b)+\zeta(c,a+b)-\zeta(a)\zeta(c,b) = 0. \tag{4.12}$$

These identites generalize the reflection identity (4.5). Let us call them "permutation identities".

At length three we can also already make use of the three-point identities (3.17). We consider again diagram 7a, and apply the first one of the three-point relations to the two $g^{(0)}$ propagators running into the "root" vertex. The result is the diagrammatic identity shown in fig. 8.

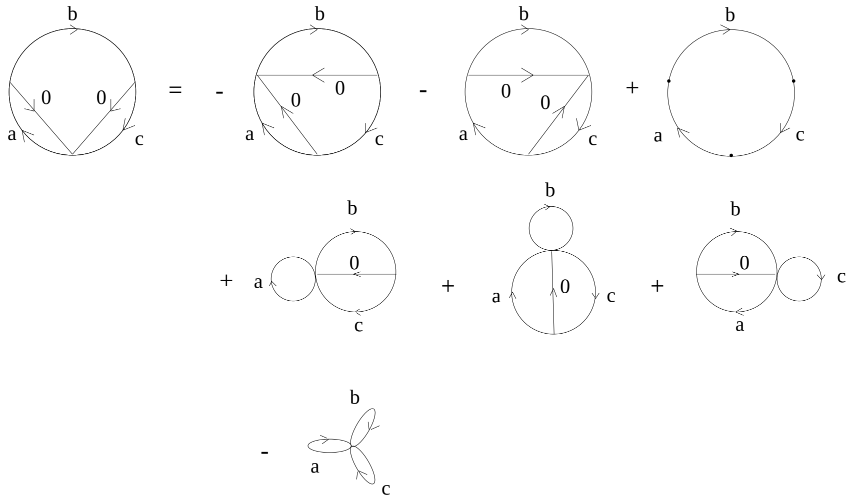


Figure 8: Diagrammatic identity derived from the three-point identity.

It can be translated term by term into the following ζ identity,

$$\begin{aligned}\zeta(a,b,c) &= -\zeta(b,c,a)-\zeta(c,a,b)+\zeta(a+b+c)+\zeta(a)\zeta(b,c)\\ &\quad +\zeta(b)\zeta(c,a)+\zeta(c)\zeta(a,b)-\zeta(a)\zeta(b)\zeta(c).\end{aligned} \tag{4.13}$$

The identities (4.10) through (4.12) can be obtained by applying the permutations $b \to a \to c \to b$, $a \leftrightarrow c$, $b \leftrightarrow c$ on (4.9). Taking the latter identity and applying all permutations, we obtain 6 identities which can be written as

$$M\vec{z} = \vec{a}, \tag{4.14}$$

where $\vec{z} = (\zeta(a,b,c), \zeta(a,c,b), \ldots, \zeta(c,b,a))$ (all permutations of the arguments) and $\vec{a}$ is a vector which contains only ζ values of length 1 and 2. The rank of the coefficient matrix M is 4, i.e. four zeta values in $\vec{z}$ can be expressed by the other two (and lower-length zeta values), e.g. by $\zeta(a,b,c)$ and $\zeta(a,c,b)$. Taking into account also relation (4.13) the coefficient matrix M gets more rows but the rank does not change. Therefore this relation is up to lower-length identities not independent from (4.9). When two of the arguments a, b, c coincide, say $b = c$, $\vec{z}$ becomes $(\zeta(a,b,b), \zeta(b,a,b), \zeta(b,b,a))$ and the rank of M reduces to 2. For all arguments coinciding the rank of M is 1.

There is another possibility to derive an identity from diagram (7d) which generalizes immediately to ζ values of larger length. Every $g^{(0)}$ propagator in (7d) can be replaced by the reverted propagator according to

$$g^{(0)}_{12} = -g^{(0)}_{21} + \delta_{12} - 1. \tag{4.15}$$

This leads to a sum of diagrams where all ocurring $g^{(0)}$ propagators are directed towards the "root" vertex. The resulting identity

$$\begin{aligned} \zeta(c,b,a) \;\; = \;\; & \zeta(a,b,c) - \zeta(a,b)\zeta(c) + \zeta(a,b+c) + \\ & + \zeta(a)\left(-\zeta(b,c) + \zeta(b)\zeta(c) - \zeta(b+c)\right) + \\ & + \zeta(a+b,c) - \zeta(a+b)\zeta(c) + \zeta(a+b+c) \end{aligned} \tag{4.16}$$

can be obtained also by subtracting (4.9) from (4.12) and applying appropriately the length-2 identity (4.1), thus it is not a new identity.

Partial integrations yield additional identities. As for ζ values of length 2, we consider the more general diagrams fig. 9 which evaluate to the numbers $G_{a,k,b,l,c} = G_{a,k,b,c,l}$. Some of these numbers can be identified with ζ values:

$$G_{a,0,b,0,c} = G_{a,0,b,c,0} = \zeta(a,b,c), \quad G_{0,k,b,0,c} = G_{0,k,b,c,0} = \zeta(k)\zeta(b,c). \tag{4.17}$$

We start with $\zeta(a,b,c) = G_{a,0,b,0,c}$. First, we integrate by parts at the upper right vertex until $b = 0$ or $c = 0$. The combinatorics is the same as in eq. (4.4). The terms with $c = 0$ can be identified with ζ values. In the terms where $b = 0$, the two inner propagators can be exchanged according to the

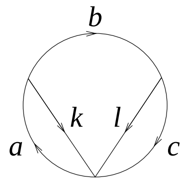


Figure 9: Auxiliary diagrams for partial integrations at length 3.

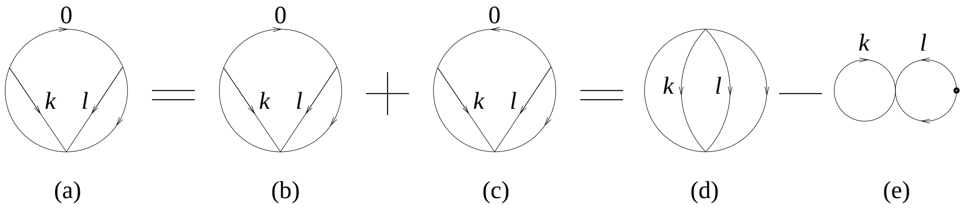


Figure 10: Identity which proves that k and l can be exchanged when the upper propagator is zero. Diagram (c) is zero because no propagator goes into the upper right vertex. Diagram (e) is zero for the same reason at the right vertex.

identity depicted in fig. 10. After this exchange we can integrate by parts at the upper left vertex until terms with $a = 0$ or $k = 0$ arise. In terms of ζ values, we obtain

$$
\begin{aligned}
\zeta(a,b,c) = & \\
= \quad (-1)^b \sum_{n=1}^{c} \Bigg[& \sum_{m=1}^{a} \binom{a+b+c-m-n-1}{b-1,\, a-m,\, c-n} \zeta(m, a+b+c-m-n, n) + \\
& + \sum_{m=1}^{b+c-n} (-1)^m \binom{b+c-n-1}{b-1} \binom{a+b+c-m-n-1}{a-1} \times \\
& \qquad \times \zeta(m)\zeta(a+b+c-m-n, n) \Bigg] + \\
& + \sum_{n=1}^{b} (-1)^c \binom{b+c-n-1}{c-1} \zeta(a, n, b+c-n), \qquad (4.18)
\end{aligned}
$$

where on the left hand side we have again to require that $a > 1$. The divergent terms for $m = 1$ appear always in a combination which can be eliminated by identity (4.10),

$$\zeta(1)\zeta(a,b)-\zeta(1,a,b)=\zeta(a,b,1)+\zeta(a,1,b)+\zeta(a+1,b)+\zeta(a,b+1). \quad (4.19)$$

Alternatively one may after the first step, in the terms with $b=0$, rather than interchanging the two inner propagators, those labelled "k" and "l" in fig. 9, instead interchange propagators "k" and "c". Proceeding in the same way as before one obtains the following identity,

$$\begin{aligned}
\zeta(a,b,c) \;=\;& (-1)^c \sum_{n=1}^{b} \binom{b+c-n-1}{c-1} \zeta(a,n,b+c-n) \\
&+(-1)^c \sum_{n=1}^{c}\sum_{m=1}^{a} \binom{b+c-n-1}{b-1}\binom{a-m+n-1}{n-1} \\
&\times\zeta(m,a-m+n,b-n+c) \\
&+\sum_{n=1}^{c}\sum_{m=1}^{n} (-1)^{c-m} \binom{a-m+n-1}{a-1}\binom{b-n+c-1}{b-1} \\
&\times\zeta(m)\zeta(a-m+n,b-n+c). \qquad (4.20)
\end{aligned}$$

Here again the terms involving a $\zeta(1,\ldots)$ appear only in the combination (4.10) and thus can be removed without the need for regularisation.

4.3 Arbitrary length

All three different procedures which we have used for constructing multiple ζ identities – partial integrations, reversion of $g^{(0)}$ propagators, and the use of the three-point identity – can be generalized to the arbitrary length case without difficulty.

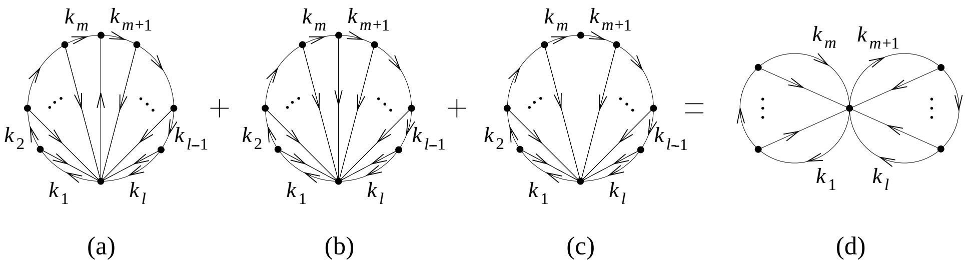


Figure 11: Diagrammatic representation of permutation identities for arbitrary length. The propagators without label are $g^{(0)}$ propagators.

The generalisation of the permutation identities to an arbitrary length is based on finding representations of sea shell diagrams with reverted inner $g^{(0)}$ propagators, which are added in order to use the two–point relation (4.15). Let us consider first the simplest case with one reverted propagator (fig. 11). Diagrams 11.b–d can be expressed immediately by ζ values. Diagrams 11.a–c represent the series

$$\sum_{\substack{n_1,\ldots,n_l \\ n_1>\ldots>n_m>0, \\ n_{m+1}>\ldots>n_l>0}}^{\infty} \frac{1}{n_1^{k_1} n_2^{k_2} \ldots n_l^{k_l}}. \tag{4.21}$$

By decomposing the summation range into regions where the n_λ's are completely ordered, we obtain a sum of ζ values which can be constructed as follows.

Let us denote by $M_{a_1,a_2,a_{12}} \subset \{\{1\},\{2\},\{1,2\}\}^{a_1+a_2+a_{12}}$ the set of all $(a_1+a_2+a_{12})$-tuples consisting of a_1 elements $\{1\}$, a_2 elements $\{2\}$ and a_{12} elements $\{1,2\}$. Further, we define a map ($l \equiv m+m'$)

$$\begin{aligned} \rho^a_{m,m'}\colon \mathbb{N}^l \times M_{m-a,m'-a,a} &\to \mathbb{N}^{l-a}, \\ (k_1,\ldots,k_m;k_{m+1},\ldots,k_l;m_1,\ldots,m_{l-a}) &\mapsto \\ &(b_1+b'_1,\ldots,b_{l-a}+b'_{l-a}), \end{aligned} \tag{4.22}$$

where $(b_1,\ldots,b_{l-a})$ results from $(m_1,\ldots,m_{l-a})$ by replacing $\{2\}$ with 0, and $\{1\}$ and $\{1,2\}$ with k_1, $\ldots$, k_m (in this order); $(b'_1,\ldots,b'_{l-a})$ results from $(m_1,\ldots,m_{l-a})$ by replacing $\{1\}$ with 0, and $\{2\}$ and $\{1,2\}$ with k_{m+1}, $\ldots$, k_l (in this order). For example,

$$\rho^1_{2,2}(k_1,k_2;k_3,k_4;\{1\},\{1,2\},\{2\}) = (k_1,k_2+k_3,k_4). \tag{4.23}$$

With these definitions, the identity depicted in fig. 11 can be written as

$$\begin{aligned} \sum_{a=0}^{\min(m,l-m)} \sum_{\xi\in M_{m-a,l-m-a,a}} \zeta(\rho^a_{m,l-m}(k_1,\ldots,k_m;k_{m+1},\ldots,k_l;\xi)) &= \\ \zeta(k_1,\ldots,k_m)\,\zeta(k_{m+1},\ldots,k_l).& \end{aligned} \tag{4.24}$$

The generalization to the cases with more than one reverted propagator should be obvious. But we note, that in these cases the ζ values of maximal length appear in combinations which can be constructed also from the identities (4.24). Thus we conjecture that all identities which are based on the reordering of summation ranges are generated by (4.24).

Imitating the considerations on page 16, we can write all permutation identities (4.24) for fixed l (but varying m), where the arguments of the length-l ζ values are taken from the set $\{k_1, \ldots, k_l\}$ in all possible orderings, in the form (4.14), where now the components of $\vec{z}$ consist of all different ζ values which result from $\zeta(k_1, \ldots, k_l)$ by permutations of the arguments. $\vec{a}$ contains only ζ values of length less than l. Assuming $k_1, \ldots, k_l$ mutually different, we found that the coefficient matrix M has rank 18 for $l = 4$ and rank 96 for $l = 5$. These results suggest that for length l the rank is $l! - (l-1)!$ and that the permutation identities suffice to express the $l!$ considered ζ values of length l by the subset of $(l-1)!$ ζ values where one of the arguments is held fixed at a certain position.

To generalize the partial integration procedure from length three to length m, we can proceed in various ways. For example, we can simply iterate the above exchange of inner propagators. In the first step one applies the same partial integration as in the length three case to the rightmost vertex of the sea shell diagram (fig. 2). For those terms in the result where $k_{m-1} = 0$ one makes use of this new $g^{(0)}$ propagator to interchange the adjacent inner propagators. Then one performs partial integrations on the next-to-rightmost vertex until either $k_{m-2} = 0$ or the next inner propagator becomes the zero propagator. In the first case the procedure continues with another interchange of inner propagators. After maximally $m-1$ such propagator interchanges, proceeding from right to left, the final step is reached, which is again the same as in the length three case.

In the first step we have the same ambiguity as before. Depending on its resolution we arrive at a generalization of either (4.19) or (4.20). We give here the formula generalizing eq.(4.20),

$$
\begin{aligned}
\zeta(k_1, \ldots, k_m) \quad = \quad & (-1)^{k_m} \sum_{n_{m-1}=1}^{k_{m-1}} \binom{k_{m-1} - n_{m-1} + n_m - 1}{k_m - 1} \\
& \times \zeta(k_1, \ldots, k_{m-2}, n_{m-1}, k_{m-1} - n_{m-1} + n_m) \\
+(-1)^{k_m} & \sum_{n_{m-1}=1}^{n_m} \sum_{n_{m-2}=1}^{k_{m-2}} \binom{k_{m-1} - n_{m-1} + n_m - 1}{k_{m-1} - 1} \binom{k_{m-2} - n_{m-2} + n_{m-1} - 1}{n_{m-1} - 1} \\
& \times \zeta(k_1, \ldots, k_{m-3}, n_{m-2}, k_{m-2} - n_{m-2} + n_{m-1}, k_{m-1} - n_{m-1} + n_m)
\end{aligned}
$$

$$
\begin{aligned}
&+(-1)^{k_m} \sum_{n_{m-1}=1}^{n_m} \sum_{n_{m-2}=1}^{n_{m-1}} \sum_{n_{m-3}=1}^{k_{m-3}} \binom{k_{m-1}-n_{m-1}+n_m-1}{k_{m-1}-1} \binom{k_{m-2}-n_{m-2}+n_{m-1}-1}{k_{m-2}-1} \\
&\quad \times \binom{k_{m-3}-n_{m-3}+n_{m-2}-1}{n_{m-2}-1} \zeta(k_1,\ldots,k_{m-4},n_{m-3},k_{m-3}-n_{m-3}+n_{m-2},\ldots) \\
&\quad \vdots \\
&+(-1)^{k_m} \sum_{n_{m-1}=1}^{n_m} \sum_{n_{m-2}=1}^{n_{m-1}} \cdots \sum_{n_2=1}^{n_3} \sum_{n_1=1}^{k_1} \prod_{n=2}^{m-1} \binom{k_n-n_n+n_{n+1}-1}{k_n-1} \binom{k_1-n_1+n_2-1}{n_2-1} \\
&\quad \times \zeta(n_1,k_1-n_1+n_2,\ldots,k_{m-1}-n_{m-1}+n_m) \\
&+ \sum_{n_{m-1}=1}^{n_m} \sum_{n_{m-2}=1}^{n_{m-1}} \cdots \sum_{n_1=1}^{n_2} (-1)^{k_m-n_1} \prod_{n=1}^{m-1} \binom{k_n-n_n+n_{n+1}-1}{k_n-1} \\
&\quad \times \zeta(n_1)\zeta(k_1-n_1+n_2,k_2-n_2+n_3,\ldots,k_{m-1}-n_{m-1}+n_m),
\end{aligned}
\tag{4.25}
$$

where $n_m \equiv k_m$. Let us also give the special case $k_m = 1$ of this formula which is particularly simple,

$$
\begin{aligned}
\zeta(k_1,\ldots,k_{m-1},1) \;&=\; \zeta(1)\,\zeta(k_1,\ldots,k_{m-1}) - \\
&- \sum_{\kappa=1}^{m-1} \sum_{n_\kappa=1}^{k_\kappa} \zeta(k_1,\ldots,k_{\kappa-1},k_\kappa+1-n_\kappa,n_\kappa,k_{\kappa+1},\ldots,k_{m-1}).
\end{aligned}
\tag{4.26}
$$

The terms involving $\zeta(1,\ldots)$ can be removed by means of a special case of eq. (4.24), namely

$$
\begin{aligned}
&\zeta(1)\,\zeta(k_1,\ldots,k_{m-1}) - \zeta(1,k_1,\ldots,k_{m-1}) = \\
&\qquad \sum_{\kappa=1}^{m-1} [\zeta(k_1,\ldots,k_{\kappa-1},k_\kappa+1,k_{\kappa+1},\ldots,k_{m-1}) + \\
&\qquad + \zeta(k_1,\ldots,k_\kappa,1,k_{\kappa+1},\ldots,k_{m-1})].
\end{aligned}
\tag{4.27}
$$

Up to here, the partial integrations were sequentially applied starting at the rightmost vertex and ending at the leftmost one. An interesting alternative is to do the opposite. Consider fig. 12 (page 22). As in the derivation of the other partial-integration identities, this diagram represents $\zeta(k_1)\zeta(k_2,\ldots,k_m)$. On the other hand, we can use partial integrations at the leftmost vertex until we have only terms where either k_1 or k_2 became 0. If $k_1 = 0$ then the resulting diagram represents a multiple ζ value. If $k_2 = 0$ then we exchange the adjacent inner propagators and repeat the whole procedure at the next-to-leftmost vertex. This continues until the rightmost vertex is reached, where after the partial integrations all terms represent multiple ζ values. The result is

$$
\begin{aligned}
&\zeta(k_1)\zeta(k_2,\ldots,k_m) = \\
&\quad \sum_{n_1=1}^{k_2} \binom{k_2+k_1-n_1-1}{k_1-1} \zeta(k_2+k_1-n_1,n_1,k_3,\ldots,k_m) + \\
&\quad \sum_{\kappa=2}^{m-1} \sum_{n_1=1}^{n_0} \sum_{n_2=1}^{n_1} \cdots \sum_{n_{\kappa-1}=1}^{n_{\kappa-2}} \sum_{n_\kappa=1}^{k_{\kappa+1}} \prod_{\lambda=2}^{\kappa} \binom{k_\lambda+n_{\lambda-2}-n_{\lambda-1}-1}{k_\lambda-1} \times \\
&\quad \binom{k_{\kappa+1}+n_{\kappa-1}-n_\kappa-1}{n_{\kappa-1}-1} \times \\
&\quad \zeta(k_2+n_0-n_1,k_3+n_1-n_2,\ldots,k_{\kappa+1}+n_{\kappa-1}-n_\kappa,n_\kappa,k_{\kappa+2},\ldots,k_m) + \\
&\quad \sum_{n_1=1}^{n_0} \sum_{n_2=1}^{n_1} \cdots \sum_{n_{m-1}=1}^{n_{m-2}} \prod_{\lambda=2}^{m} \binom{k_\lambda+n_{\lambda-2}-n_{\lambda-1}-1}{k_\lambda-1} \times \\
&\quad \zeta(k_2+n_0-n_1,k_3+n_1-n_2,\ldots,k_m+n_{m-2}-n_{m-1},n_{m-1}), \qquad (4.28)
\end{aligned}
$$

($n_0 \equiv k_1$). The right hand side contains no divergent terms when $k_1, k_2 \geq 2$. For $k_1 = 1, k_2 \geq 2$ it becomes finite when combined with (4.27).

From our derivation clearly one would expect eqs. (4.25) and (4.28) to be equivalent. And indeed, it is a matter of pure combinatorics to show that (4.25) becomes trivially fulfilled if (4.28) is used on the right hand side. Nevertheless, we suspect that the form (4.28) may be more useful for the application of these formulas to the problem of constructing a minimal basis of independent multiple ζ sums.

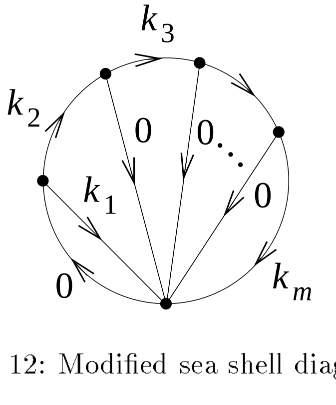


Figure 12: Modified sea shell diagram.

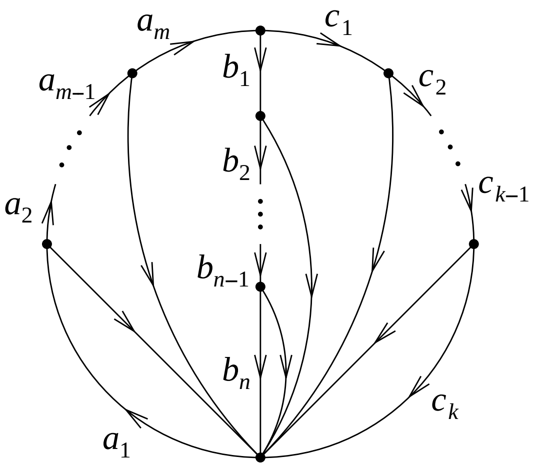


Figure 13: "Peacock" diagram. The propagators without label are $g^{(0)}$ propagators.

Eq. (4.28) is a special case of a class of identities derived from (weight-length) shuffle algebras. In order to derive that whole class, consider Fig. 13. It evaluates to a number which we denote by

$$Z(a_1,\ldots,a_m|b_1,\ldots,b_n|c_1,\ldots,c_k) = Z(a_1,\ldots,a_m|c_1,\ldots,c_k|b_1,\ldots,b_n). \tag{4.29}$$

Partial integrations at the top vertex yield, similarly to (4.4),

$$\begin{aligned} &Z(\ldots,a_m,0|b_1,b_2,\ldots|c_1,c_2,\ldots) = \\ &\quad \sum_{\nu=1}^{b_1} \binom{b_1+c_1-\nu-1}{c_1-1} Z(\ldots,a_m,b_1+c_1-\nu|\nu,b_2,\ldots|0,c_2,\ldots) \\ &\quad + \sum_{\nu=1}^{c_1} \binom{b_1+c_1-\nu-1}{b_1-1} Z(\ldots,a_m,b_1+c_1-\nu|0,b_2,\ldots|\nu,c_2,\ldots). \end{aligned} \tag{4.30}$$

Considerations like in Fig. 10 show

$$\begin{aligned} Z(a_1,\ldots,a_m|0,b_1,\ldots|c_1,\ldots) &= Z(a_1,\ldots,a_m|c_1,\ldots|0,b_1,\ldots) \\ &= Z(a_1,\ldots,a_m,0|b_1,\ldots|c_1,\ldots). \end{aligned} \tag{4.31}$$

Starting with

$$\zeta(a_1, \ldots, a_m)\, \zeta(b_1, \ldots, b_n) = Z(0|a_1, \ldots, a_m|b_1, \ldots, b_n) \tag{4.32}$$

and applying continuedly eqs. (4.30) and (4.31), we end up with terms of the form

$$\begin{aligned} Z(a_1, \ldots, a_m|0|b_1, \ldots, b_n) &= Z(a_1, \ldots, a_m|b_1, \ldots, b_n|0) \\ &= \zeta(a_1, \ldots, a_m, b_1, \ldots, b_n). \end{aligned} \tag{4.33}$$

The resulting identities have the same form as the shuffle identities in the literature, but here advantageously the binomial coefficients in (4.30) explicitly encode (in part) the combinatorics implicit in the shuffle algebra.

Additional multiple ζ identities can be derived using the three-point identity (3.17). At low lengths and levels it turns out that the multiple ζ identities obtained in this way are not independent from the set of equations generated by the propagator reversions and partial integrations. Whether this property holds true in general we do not know.

5 Discussion

In the present work we have established a novel representation of multiple ζ sums in terms of Feynman diagrams in a $1+0$ dimensional quantum field theory. We demonstrated the usefulness of this representation for the derivation of identities between such sums. The encoding into Feynman diagrams proposed here provides a very convenient book-keeping device for certain formal manipulations performed on such sums, as our examples should have amply demonstrated.

Concerning the novelty of the identities derived here, the length-two identities presented in section 4.1 are, of course, well-known. At length three, the identities derived by the partial integration procedure, (4.19), (4.20), are similar, and presumably equivalent, to the 'decomposition equations' derived in [9] by explicit series manipulations (their eq.(1)). Similarly, their 'permutation equations' (2) coincide with our eq.(4.10). Eq.(4.13) is contained as a special case in Theorem 2.2 of [21]. However, we have not been able to locate in the literature an exact equivalent of our length m identities (4.25),(4.28) [3]. The only identities available for arbitrary lengths and levels are those based on the 'shuffle algebra' [6, 26, 10, 7, 8] and its generalizations [23]. Of those the 'depth-length' shuffle identities (which

[3] The special case obtained by combining eqs.(4.26) and (4.27) is Theorem 5.1 of [21].

are also called 'stuffle identities' or '$*$ products' [22]) are obviously related, and in fact equivalent to our 'permutation' identities, as we have convinced ourselves. Similarly the 'weight-length' shuffle identities are clearly related to our various 'partial integration' identities. In this case the question of equivalence is more difficult and will require further investigation.

The reader will have noted that we did not make use at all of the precise form of the path integral action. Our considerations required the presence of all propagators $g^{(k)}$, as well as of all the vertices $\mathcal{V}^{p,q}$, however they did not determine the statistical weights with which they should appear in the Feynman diagrams. Our choice of the weights for the propagators is mainly motivated by the fact that it leads to a suggestive form for the free path integral determinant. Namely, a simple application of the "$\ln\det = \mathrm{tr}\ln$" identity shows that, formally, [4]

$$\ln Z(0,\lambda) \;=\; const. + \sum_{n=1}^{\infty}\lambda^n\frac{\zeta(n)}{n}. \tag{5.1}$$

Comparing this expression with the well-known formula for the logarithm of the Γ function

$$\Gamma(1+x) \;=\; \exp\Biggl[-\gamma x + \sum_{n=2}^{\infty}\frac{(-1)^n}{n}\zeta(n)x^n\Biggr], \tag{5.2}$$

we see that we can identify the free partition function with the Γ function under the assumption that the ill-defined $\zeta(1)$ appearing in (5.1) is renormalized to Euler's constant γ,[5]

$$Z_{\mathrm{renorm}}(0,\lambda) \;=\; const. \times \Gamma(1-\lambda). \tag{5.3}$$

Similarly the total propagator becomes relatively simply. Using the integral representation eq. (2.11) of the polylogarithm it is easily shown that

$$p_{12} \;\equiv\; \sum_{k=0}^{\infty}\lambda^k(2\pi\mathrm{i})^k g_{12}^{(k)} = g_{12}^{(0)} + \frac{z_{12}\lambda}{1-\lambda}\,{}_2F_1(1,1-\lambda;2-\lambda;z_{12}). \tag{5.4}$$

[4]This calculation may be seen as a "chiral" generalization of the calculation of the Scalar QED Euler-Heisenberg Lagrangian performed in [35].

[5]Considering the identities $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right)$ and $\sum_{k=1}^{\infty}\frac{1}{k^{\epsilon}} = \frac{1}{\epsilon} + \gamma + \mathrm{O}(\epsilon)$ this assumption seems quite natural.

For the interaction term there seems to be no such preferred choice. A question of obvious interest (but equally obvious difficulty) is whether non-trivial interaction potentials $V(g,\bar{g})$ exist such that the ζ-model would be exactly solvable.

Recently the following generalization of Euler-Zagier sums (1.3) has attracted some attention,

$$\zeta(k_1,\ldots,k_m;\sigma_1,\ldots,\sigma_m) \quad = \sum_{n_1>n_2>\cdots>n_m>0} \frac{\sigma_1^{n_1}\cdots\sigma_m^{n_m}}{n_1^{k_1}\cdots n_m^{k_m}}, \tag{5.5}$$

where $\sigma_j = \pm 1$. Those "alternating" Euler-Zagier sums arise naturally in the calculation of ultraviolet divergences in renormalizable quantum field theories, and in the application of knot theory to the classification of those divergences [13]. At the same time the inclusion of alternating Euler-Zagier sums seems to simplify the problem of reducing the set of all such sums to a basic set via multiple ζ identities [10]. (For a tabulation of alternating series see [5].)

More generally, arbitrary N-th roots of unity in place of the σ_j have been considered in connection with the study of mixed Tate motives over Spec $\mathbb{Z}$ [18, 33]. Those phase factors can be easily accomodated in the ζ-model. To generate a phase factor $\sigma = \mathrm{e}^{2\pi\mathrm{i}s}$ the propagator (2.1) has to be simply replaced by

$$g_\sigma^{(k)}(u_{12}) \equiv \sum_{n=1}^{\infty} \frac{\mathrm{e}^{2\pi\mathrm{i}(n+s)u_{12}}}{(2\pi\mathrm{i}n)^k}. \tag{5.6}$$

On the path integral level this can be achieved by changing from periodic to twisted boundary conditions, $x(1) = \sigma x(0)$, and replacing ∂ by $\partial - 2\pi\mathrm{i}s$.

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