Renormalization in Perturbative Quantum Field Theory via Hopf Algebras

(4 Lectures)

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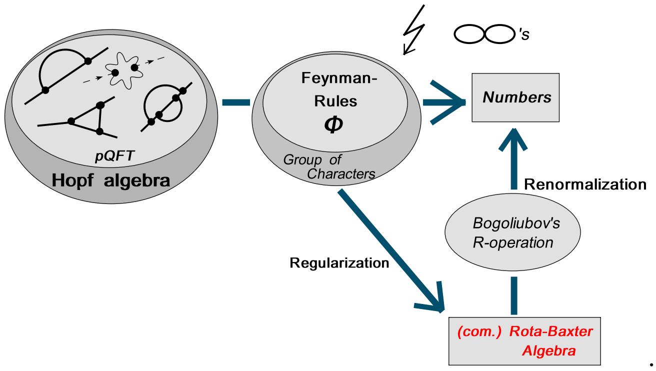
Introduction: Hopf Algebra of Feynman Graphs

> Few physicists object nowadays to the idea that diagrams contain more truth than the underlying formalism, [. . .]
>
> G. 't Hooft and M. Veltman, "Diagrammar"[25]

The perturbative approach to quantum field theory (QFT) is the most successful. Theoretical predictions match experimental results with a vertiginous high precision. Nevertheless, in most, if not all, of the interesting and relevant 4-dimensional quantum field theories, performing even simple perturbative calculations, one can not avoid facing ill-defined integrals. The removal of these divergencies in a sound way is the process of renormalization. The basic idea of perturbative renormalization in QFT goes back to Kramer [4], and was successfully applied for the first time in a 1947 paper by Bethe [3], dealing with the concrete problem of the self energy contribution for the Lamb shift in perturbative quantum electrodynamics (QED). The latter, nowadays can be regarded as one of the best tested theories in physics. Maintaining the physical principles of locality, unitarity, and Lorentz invariance, renormalization theory may be summarized by the statement, that to all orders in perturbation theory the (ultraviolet) divergencies can be absorbed in a redefinition of the parameters defining the QFT. Here two distinct concepts enter, that of the process of renormalization and the notion of renormalizability.

Unfortunately, despite its accomplishments, renormalization was stigmatized, especially for its lack of a firm mathematical underpinning. One reason for this weakness might have been the fact that its building blocks, the (one particle irreducible) Feynman graphs in itself appeared to be unrelated to any mathematical structure that may possibly underlie the renormalization prescription.

Almost five decades after Bethe's paper, this changed to a great extend with the original paper by Kreimer [17], followed by the work of Kreimer [18], and Connes and Kreimer [7, 8, 9]. In this approach the combinatorial and algebraic side of the process for renormalization is captured via a combinatorial Hopf algebra of Feynman graphs, $\mathcal{H}_{\mathcal{F}}$, which is essentially characterized by its coproduct map. Connes' and Kreimer's Hopf algebra formalism gives rise to an elegant and useful disentanglement of analytic and algebraic aspects of perturbative renormalization.



The restricted dual of the Hopf algebra of Feynman graphs, denoted by $\mathcal{H}^*_{\mathcal{F}}$, contains the group $G := char(\mathcal{H}_{\mathcal{F}}, \mathbb{C})$ of characters, that is, algebra homomorphisms from $\mathcal{H}_{\mathcal{F}}$ to the underlying base field $\mathbb{C}$. Feynman rules are understood as such linear and multiplicative maps. To this group of characters corresponds a Lie algebra of derivations, or infinitesimal characters, $g := \partial char(\mathcal{H}_{\mathcal{F}}, \mathbb{C})$, which comes from a fundamental pre-Lie algebra structure on Feynman graphs.

Ultraviolet divergencies in general demand for a regularization prescription, where we replace the base field $\mathbb{C}$ by a (commutative and unital) algebra A of Feynman amplitudes, and we consider the space of regularized linear maps $\mathrm{Hom}(\mathcal{H}_F, A)$, which contains $G_A := char(\mathcal{H}_F, A)$, the group of regularized characters, respectively its associated Lie algebra $g_A := \partial char(\mathcal{H}_F, A)$. As a principal example serves dimensional regularization, where $A = \mathbb{C}[\epsilon^{-1}, \epsilon]]$, the field of Laurent series. In this context perturbative renormalization may be formulated as a factorization problem, to wit, the algebraic Birkhoff decomposition of Feynman rules [7, 8, 9]. The proof of the Connes–Kreimer factorization of regularized Feynman rules uses the property that Laurent series actually form a Rota–Baxter algebra [2, 22] with the pole part projection, $R := R_{ms}$, as Rota–Baxter operator (minimal subtraction scheme map) fulfilling the Rota–Baxter relation (of weight 1)

$$R(x)R(y) + R(xy) = R\big(R(x)y + xR(y)\big),\ \forall x, y \in A.$$

Using the generalization of Spitzer's classical identity [12, 13, 23, 24] to non-commutative Rota–Baxter algebras, together with Atkinson's factorization theorem [1] for Rota–Baxter algebras, one can show that the multiplicative decomposition of Connes–Kreimer follows from an additive factorization through the exponential map. Hereby we derive Bogoliubov's $\bar{\mathrm{R}}$-map as a special case of Spitzer's identity.

It is the goal of these lecture series to introduce the necessary background, both in physics and mathematics, for understanding the recent work of Connes and Kreimer on renormalization theory in perturbative quantum field theory. In the first part we will outline the basics of QFT and its perturbative treatment providing the setting in which the problem of divergencies appear, and hence renormalization theory lives. The second will start with some modest calculations of renormalizations in QFT toy model examples and QED. Thereafter, the Hopf-algebraic structure of Feynman graphs is introduced with the necessary algebraic-combinatorial background. In the third and fourth part we continue with the Hopf-algebraic description of the process of renormalization, formulating it as an factorization problem. After constructing the Hopf algebra of Feynman graphs, providing the sought-after combinatorial-algebraic setting, the process of renormalization in perturbative QFT can be described in almost one line, that is to say, the Birkhoff decomposition of regularized Feynman rules characters, discovered by Alain Connes and Dirk Kreimer. We will provide the algebraic underpinning for this factorization in terms of complete filtered Rota–Baxter algebras, Spitzer's identity and Atkinson's theorem. In the fourth lecture we will finish with some aspects related to the renormalizatioin group in the Connes-Kreimer setting. We hope to find some time at the end of the last lecture to address some of the more recent results [10, 20].

Each talk will start with a brief review of the results of the forgoing lecture.
We would like to point the reader to [5, 6, 11, 14, 19, 21] for more introductory references, both with respect to classical renormalization theory in perturbative QFT, and its recently discovered Hopf-algebraic aspects.

1 Lecture I: Perturbative Quantum Field Theory in a Nutshell

- *introduction to perturbative quantum field theory, Feynman rules and diagrams, the problem of divergencies*

2 Lecture II: Introduction to Renormalization Theory

- *calculations of Feynman graphs in toy model examples and QED, regularization prescription I, renormalization schemes, Bogoliubov's* $\bar{\mathrm{R}}$*-map, introduction of Feynman graphs Hopf algebra*

3 Lecture III: Hopf Algebra of Renormalization: Feynman Graphs Combinatorics

- *continuation of the presentation of Connes' and Kreimer's Hopf algebra of renormalization, regularization prescription II, complete filtered Rota–Baxter algebras, Spitzer's classical identity and its generalization to non-commutative Rota–Baxter algebras, Atkinson's factorization theorem for Rota–Baxter algebras*

4 Lecture IV: Hopf Algebra of Renormalization: Birkhoff decomposition

- *continuation of lect. III: Connes' and Kreimer's algebraic Birkhoff decomposition of regularized Hopf algebra characters, renormalization group aspects*
  miscellanies...

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