Hopf Algebras - the basics

Algebra *A*

$x, y \in A$ $\quad x+y \in A$ $\quad xy \in A$

$k \in K$ $\quad kx \in A$

Bialgebra *also has*

Co-product $\Delta : A \to A \otimes A$

Co-unit $\varepsilon : A \to K$

Hopf Algebra *also has*

Antipode $S : A \to A$

Generic Example

Algebra *A* generated by symbols x, y

$xxyxyy \in A$ $\quad xxy + 2\ yx \in A$

Unit e $\quad e\,x = x = x\,e$

Coproduct Δ

On unit: $\Delta(e) = eXe$

On Generator: $\Delta(x) = xXe + eXx$

$\Delta(AB) = \Delta(A)\ \Delta(B)$ (algebra map)

Antipode S

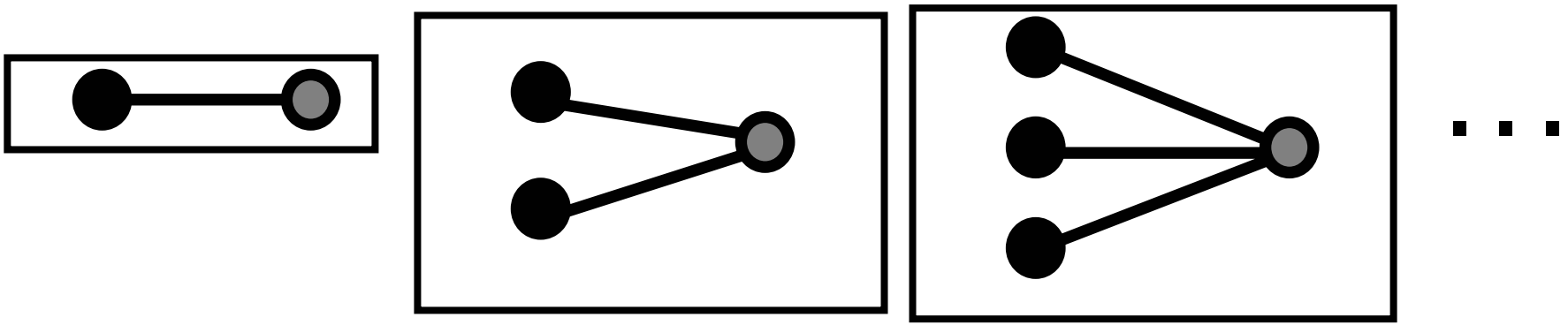
On unit: $S(e) = -e$

On Generator: $S(x) = -x$

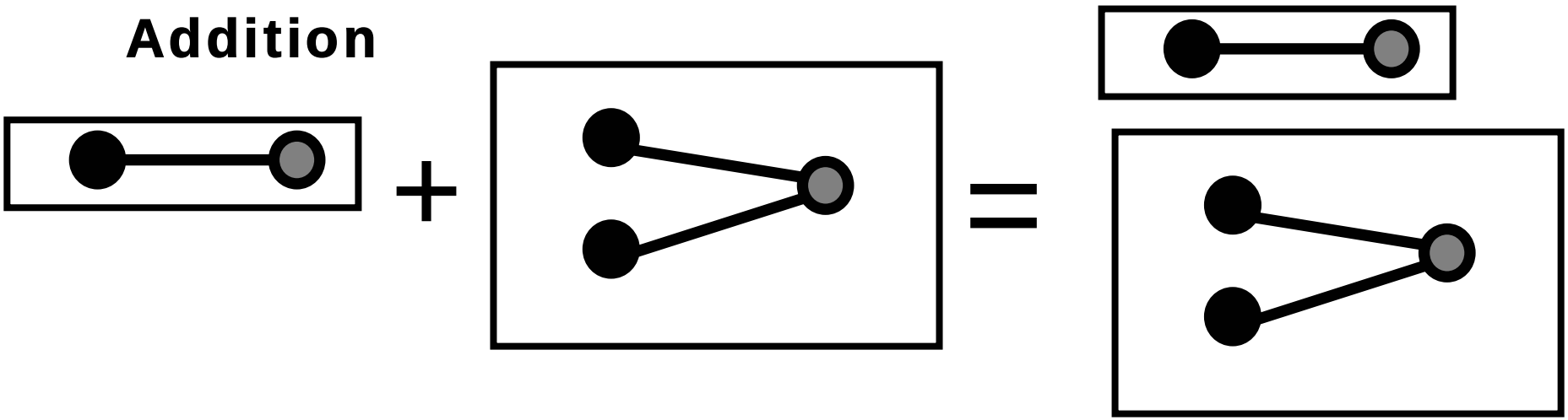
$S(AB) = S(B)\ S(A)$ (anti-algebra map)

Hopf Algebra of Model Diagrams

Generators are connected diagrams

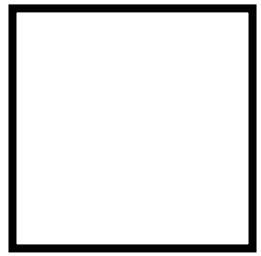


Addition



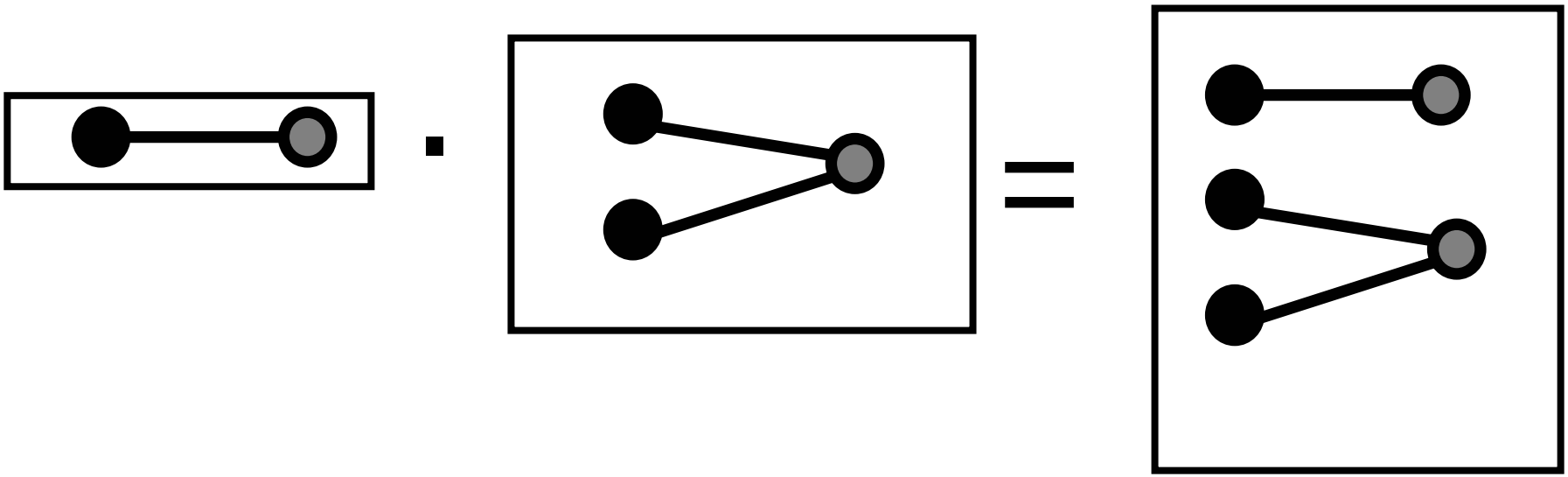
Hopf Algebra of Model Diagrams (continued)

Algebra Identity *e*



$(=\phi)$

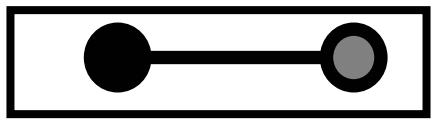
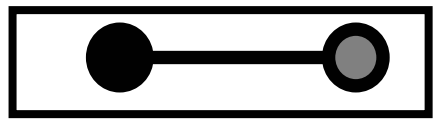
Multiplication

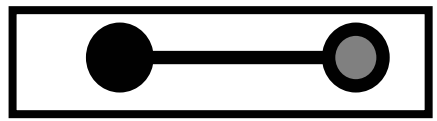


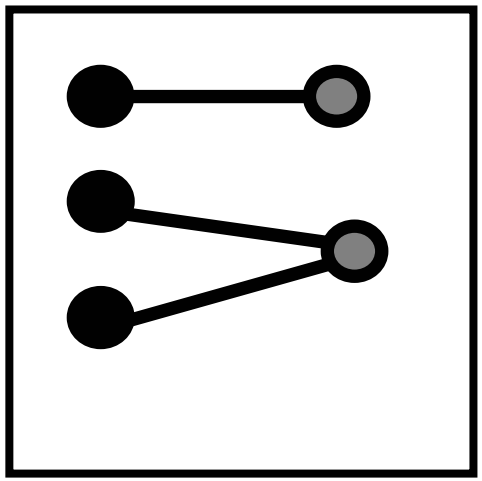
Hopf Algebra of Model Diagrams (continued)

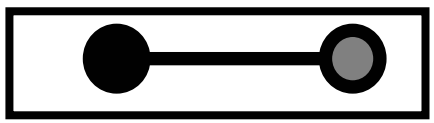
Coproduct Δ (Examples)

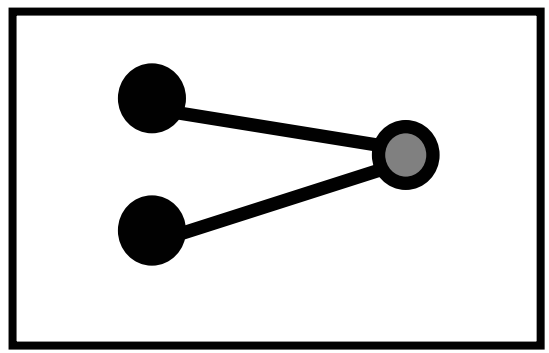
Unit $\Delta\,\Phi = \Phi \mathbf{X} \Phi$

Generator Δ  =  **X**Φ

$+\Phi\mathbf{X}$ 

Δ  = $\Delta(AB) = \Delta(A)\,\Delta(B)$

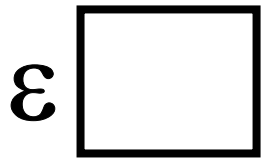
where A = 

and B = 

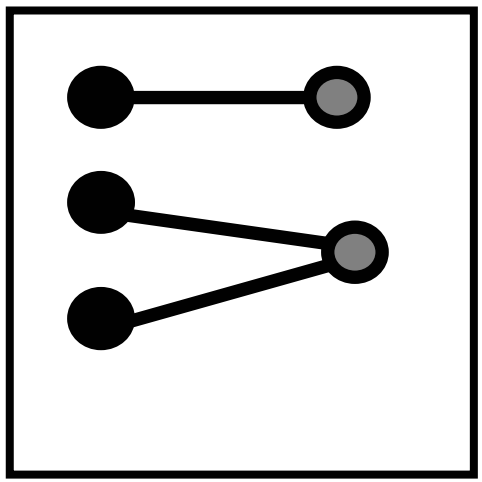
Hopf Algebra of Model Diagrams (continued)

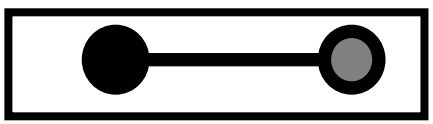
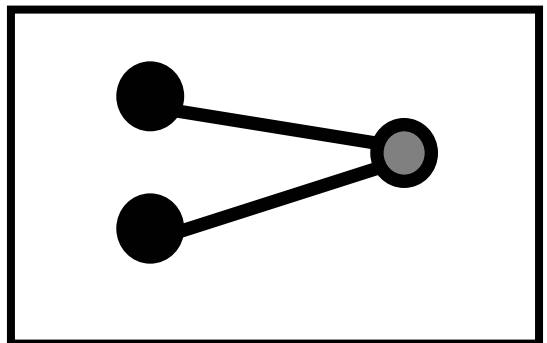
Counit ε (Examples)

on Unit $\varepsilon\,\Phi = 1$

ε  $= 1$

otherwise $\varepsilon(A)=0$

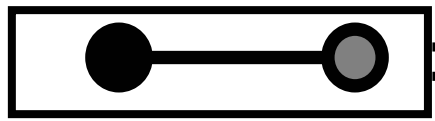
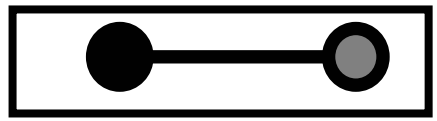
ε  $=$ $\varepsilon(AB)= \varepsilon(A)\,\varepsilon(B)$

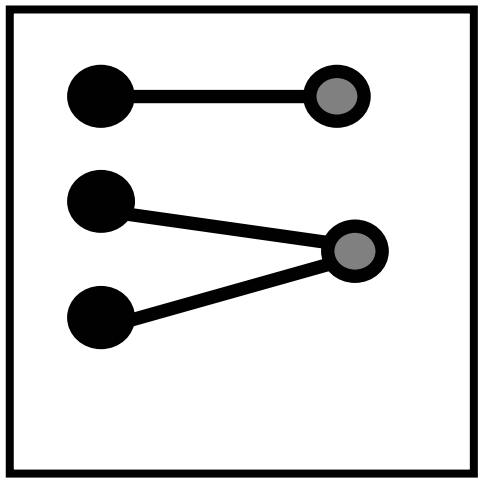
where A =  and B = 

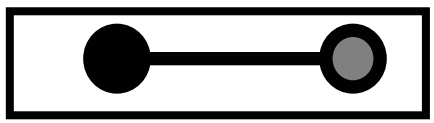
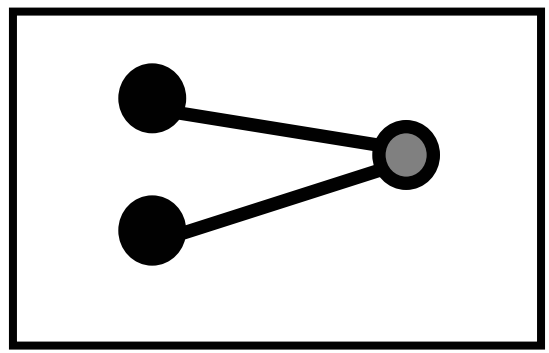
Hopf Algebra of Model Diagrams (continued)

Antipode S (Examples)

Unit $S\,\Phi = -\Phi$

Generator S  $=$ $-$ 

S  $= S(AB) = S(B)\,S(A)$

where A = 

and B =