

Hopf Algebras - the basics

Algebra A

$$x, y \in A \qquad x + y \in A \qquad xy \in A$$

$$k \in K \qquad kx \in A$$

Bialgebra *also has*

$$\text{Co-product } \Delta : A \rightarrow A \otimes A$$

$$\text{Co-unit } \varepsilon : A \rightarrow K$$

Hopf Algebra *also has*

$$\text{Antipode } S : A \rightarrow A$$

Generic Example

Algebra A generated by symbols x, y

$$xxyxyy \in A \quad xxy + 2yx \in A$$

Unit e $e x = x = x e$

Coproduct Δ

On unit: $\Delta(e) = eXe$

On Generator: $\Delta(x) = xXe + eXx$

$\Delta(AB) = \Delta(A) \Delta(B)$ (algebra map)

Antipode S

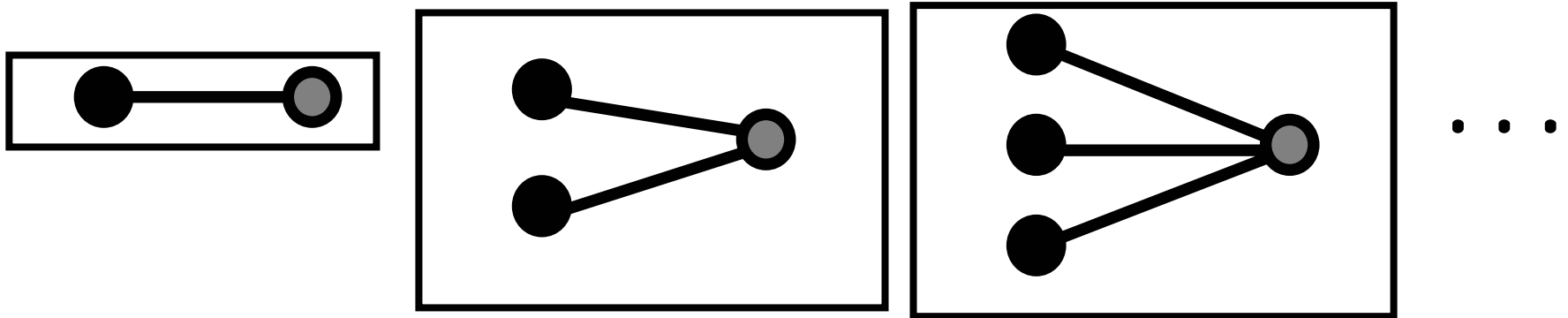
On unit: $S(e) = -e$

On Generator: $S(x) = -x$

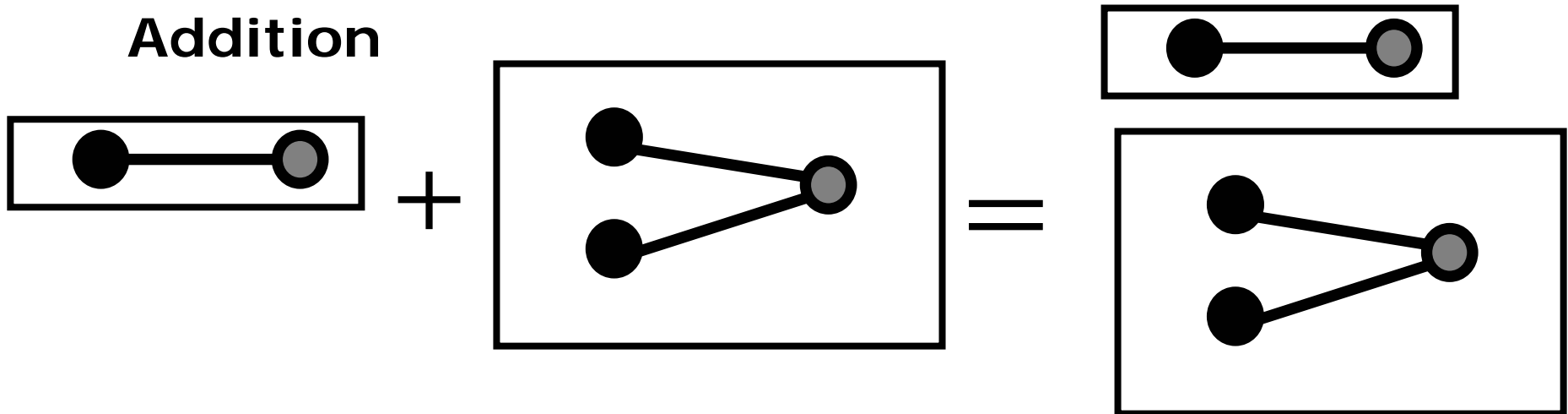
$S(AB) = S(B) S(A)$ (anti-algebra map)

Hopf Algebra of Model Diagrams

Generators are connected diagrams

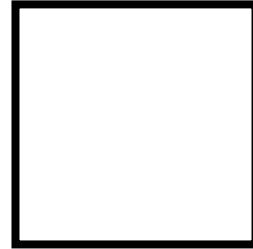


Addition



Hopf Algebra of Model Diagrams (continued)

Algebra Identity e

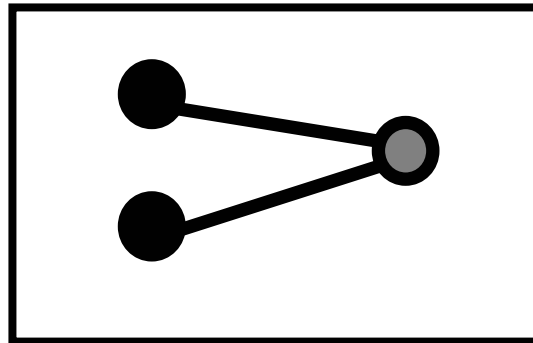


$(= \phi)$

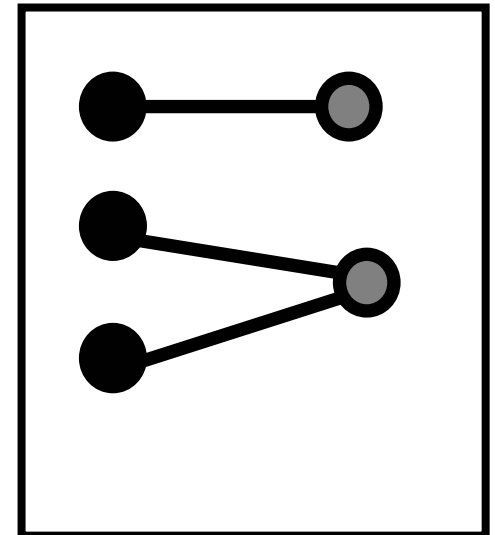
Multiplication



\cdot



$=$



Hopf Algebra of Model Diagrams (continued)

Coproduct Δ (Examples)

Unit $\Delta \Phi = \Phi X \Phi$

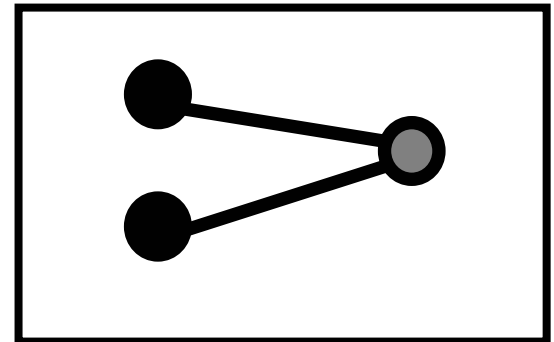
Generator

$$\Delta \left[\text{Diagram: box with two nodes (black and grey) connected by a horizontal line} \right] = \text{Diagram: box with two nodes (black and grey) connected by a horizontal line} X \Phi + \Phi X \left[\text{Diagram: box with two nodes (black and grey) connected by a horizontal line} \right]$$

$$\Delta \left[\text{Diagram: box with three nodes (two black, one grey) connected by lines forming a fan shape} \right] = \Delta(A) \Delta(B)$$

where $A = \left[\text{Diagram: box with two nodes (black and grey) connected by a horizontal line} \right]$

and $B = \left[\text{Diagram: box with two nodes (black and grey) connected by lines forming a fan shape} \right]$



Hopf Algebra of Model Diagrams (continued)

Counit ε (Examples)

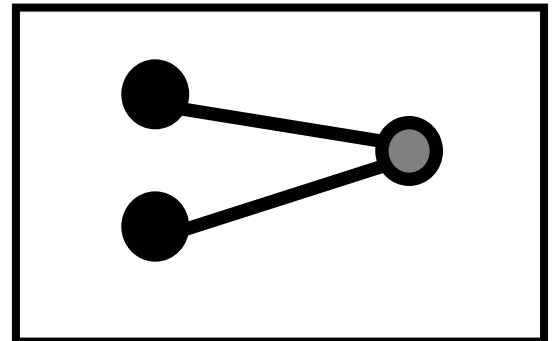
on Unit $\varepsilon \Phi = 1$

$$\varepsilon \square = 1$$

otherwise $\varepsilon(A) = 0$

$$\varepsilon \left[\begin{array}{c} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \end{array} \right] = \varepsilon(AB) = \varepsilon(A) \varepsilon(B)$$

where $A = \left[\bullet \text{---} \bullet \right]$ and $B =$



Hopf Algebra of Model Diagrams (continued)


Antipode S (Examples)

Unit $S \Phi = -\Phi$

Generator

$$S \left[\text{Diagram: box with two nodes (black and grey) connected by a horizontal line} \right] = - \left[\text{Diagram: box with two nodes (black and grey) connected by a horizontal line} \right]$$

$$S \left[\text{Diagram: box with three nodes (two black, one grey) connected by lines} \right] = S(AB) = S(B) S(A)$$

where $A =$ 

and $B =$

