SOME HOPF ALGEBRAS OF PHYSICAL INTEREST

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GRADED CONNECTED HOPF ALGEBRAS

$$A = \bigoplus_{n \geq 0} A_n$$

GRADED ALGEBRA OVER FIELD k (CHAR. 0) ; $A_0 = k1$.

COPRODUCT $\Delta : A \to A \otimes A$ RESPECTS GRADING, IS ALGEBRA MAP, AND HAS

$$\Delta(x) = x \otimes 1 + \sum_{|x'|,|x''|>0} x' \otimes x'' + 1 \otimes x \, , \quad |x|>0$$

x IS <u>PRIMITIVE</u> IF $\Delta(x) = 1 \otimes x + x \otimes 1$

EXISTENCE OF <u>ANTIPODE</u> $S : A \to A$ IS AUTOMATIC ; HAVE $S(1) = 1$ AND

$$S(x) = -\sum S(x')x'' - x$$

FOR $|x|>0$. IF A IS COMMUTATIVE OR COCOMMUTATIVE, $S^2 = id$. IN GENERAL S IS ALGEBRA ANTI-AUTOMORPHISM

DUALS

THE GRADED DUAL OF A IS ALSO A HOPF ALGEBRA, WITH PRODUCT

$$\langle m^*(u \otimes v), w \rangle = \langle u \otimes v, \Delta(w) \rangle$$

AND COPRODUCT

$$\langle \Delta^*(u), w_1 \otimes w_2 \rangle = \langle u, w_1 w_2 \rangle$$

A IS SELF-DUAL ($A^* \cong A$) IF IT ADMITS AN INNER PRODUCT $(,)$ WITH $(uv, w) = (u \otimes v, \Delta(w))$.

HERE IS A TABLE OF HOPF ALGEBRAS WE WILL DISCUSS:

	COMMUTATIVE?	CO-COMMUT.?	DUAL
Sym	YES	YES	Sym
QSym	YES	NO	NSym
$\mathfrak{I}$	NO	YES	$\mathcal{H}_K$
$P \cong \mathcal{H}_F$	NO	NO	$\mathcal{H}_F$

Sym AND QSym

LET $\mathcal{B} \subset k[[x_1, x_2, \ldots]]$ BE THE FORMAL POWER SERIES OF BOUNDED DEGREE. ($\mathcal{B}$ GRADED BY $|x_i| = 1 \ \forall i$.)

A SERIES $f \in \mathcal{B}$ IS IN Sym IF THE COEFFICIENT OF $x_{i_1}^{p_1} x_{i_2}^{p_2} \cdots x_{i_k}^{p_k}$ AGREES WITH THAT OF $x_{j_1}^{p_1} x_{j_2}^{p_2} \cdots x_{j_k}^{p_k}$ FOR ANY TWO SETS $\{i_1, \ldots, i_k\}$ AND $\{j_1, \ldots, j_k\}$ OF <u>DISTINCT</u> SUBSCRIPTS.

A SERIES $f \in \mathcal{B}$ IS IN QSym IF THE COEFFICIENT OF $x_{i_1}^{p_1} x_{i_2}^{p_2} \cdots x_{i_k}^{p_k}$ AGREES WITH THAT OF $x_{j_1}^{p_1} x_{j_2}^{p_2} \cdots x_{j_k}^{p_k}$ FOR $i_1 < i_2 < \cdots < i_k$ AND $j_1 < j_2 < \cdots < j_k$.

THE FORMER CONDITION IS MORE RESTRICTIVE, SO $\mathrm{Sym} \subset \mathrm{QSym}$

SYM AS HOPF ALGEBRA

SYM HAS THE BASIS

$$m_\lambda = \sum_{\substack{\text{DISTINCT} \\ i_1,\dots,i_k}} x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_k}^{\lambda_k}$$ (MONOMIAL S.F.'s)

WHERE $\lambda = \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$ IS ANY INTEGER PARTITION

AS AN <u>ALGEBRA</u>,

$$\text{Sym} \cong k[e_1, e_2, \dots]$$

WHERE e_i IS THE ELEMENTARY S.F.

$$e_i = m_{\underbrace{11\dots1}_{i}}$$

SO WE ALSO HAVE THE BASIS

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}$$

WE CAN MAKE SYM A HOPF ALGEBRA BY MAKING THE e_i <u>DIVIDED</u> <u>POWERS</u>:

$$\Delta(e_n) = \sum_{i=0}^{n} e_i \otimes e_{n-i}$$

IN GENERAL

$$\Delta(m_\lambda) = \sum_{\alpha \cup \beta = \lambda} m_\alpha \otimes m_\beta$$

IN FACT, Sym IS SELF-DUAL SINCE THERE IS AN INNER PRODUCT $(\cdot, \cdot)$ WITH $(e_\alpha, m_\beta) = \delta_{\alpha\beta}$; THEN

$$(e_\alpha \otimes e_\beta, \Delta(m_\gamma)) = (e_\alpha e_\beta, m_\gamma).$$

THE ANTIPODE $S: \text{Sym} \to \text{Sym}$ IS THE AUTOMORPHISM OF Sym THAT INTERCHANGES e_n WITH $(-1)^n h_n$, WHERE h_n ARE THE COMPLETE S.F.'S

$$h_n = \sum_{|\lambda| = n} m_\lambda .$$

QSym

A BASIS FOR QSym IS GIVEN BY THE MONOMIAL QUASISYMMETRIC FUNCTIONS

$$M_I = \sum_{j_1 < \cdots < j_k} x_{j_1}^{i_1} x_{j_2}^{i_2} \cdots x_{j_k}^{i_k}$$

INDEXED BY INTEGER <u>COMPOSITIONS</u> (ORDERED PARTITIONS). SO, E.G.,

$$M_{12} + M_{21} = m_{21} \quad , \quad M_{11} = m_{11}$$

AS AN ALGEBRA,

$$QSym \cong k\,[M_I \mid I \text{ LYNDON}]$$

WHERE I RUNS OVER "LYNDON WORDS" IN 1, 2, 3, ...

THE HOPF ALGEBRA STRUCTURE IS GIVEN BY

$$\Delta(M_I) = \sum M_{I_1} \otimes M_{I_2}$$

WHERE THE SUM IS OVER STRINGS I_1, I_2 WITH $I_1 I_2 = I$ (CONCATENATION)

THIS IS NON-COCOMMUTATIVE.

A FORMULA FOR THE ANTIPODE S CAN BE PROVED BY INDUCTION:

$$S(M_I) = (-1)^{\ell(I)} \sum_{J \preccurlyeq \bar{I}} M_J$$

WHERE $\ell(I)$ = LENGTH OF I

$\preccurlyeq$ IS REFINEMENT ORDER

$\bar{I}$ = REVERSE OF I ($\overline{12} = 21$)

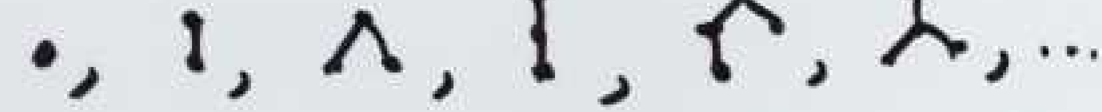
THE DUAL OF QSym IS A NONCOMMUTATIVE, COCOMMUTATIVE HOPF ALGEBRA CALLED NSym (NOCOMMUTATIVE SYMMETRIC FUNCTIONS). THEY WERE DISCUSSED BY GELFAND ET. AL. AS AN ALGEBRA,

$$NSym = k\langle\langle e_1, e_2, \ldots \rangle\rangle$$

WITH COPRODUCT

$$\Delta(e_i) = \sum_{j=0}^{i} e_j \otimes e_{i-j}$$

HOPF ALGEBRA $\mathcal{T}$ OF ROOTED TREES

TREES 

(DON'T DISTINGUISH BETWEEN  AND )

FORESTS 

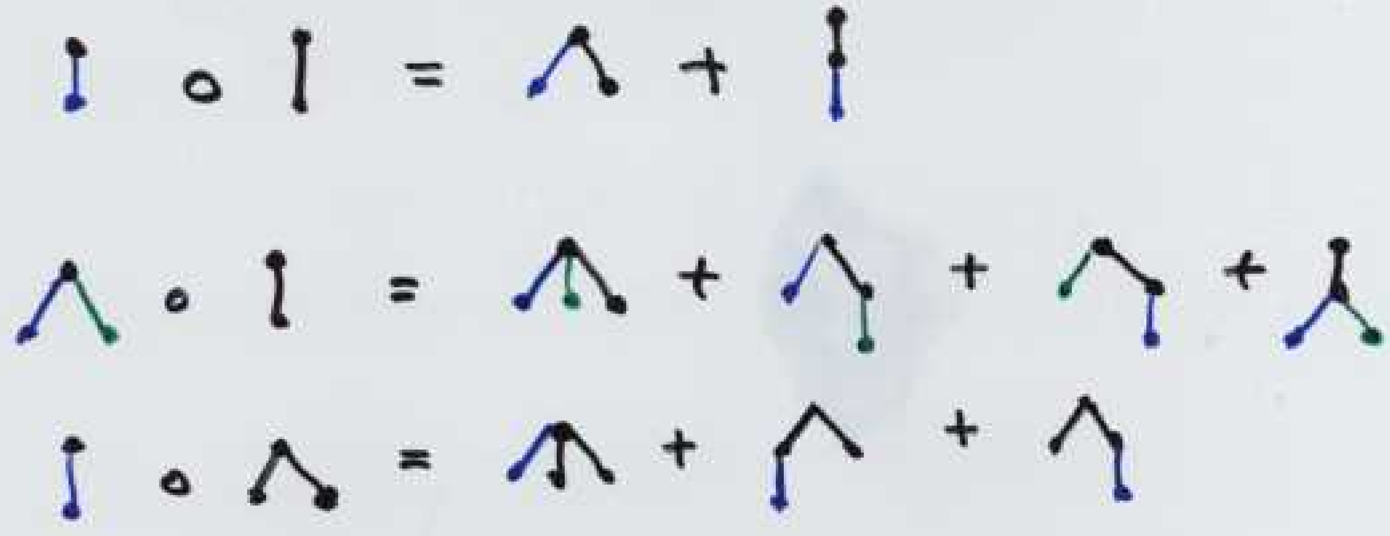
(ORDER IMMATERIAL)

B_+ : FORESTS $\rightarrow$ TREES 

GRADE TREES BY NUMBER OF NON-ROOT VERTICES: 

GROSSMAN-LARSON PRODUCT $t \circ t'$:

LET $t = B_+(t_1 \cdots t_k)$. THEN $t \circ t'$ IS THE SUM OF THE $(|t'|+1)^k$ TREES OBTAINED BY ATTACHING THE k BRANCHES OF t IN ALL POSSIBLE LOCATIONS AMONG THE $|t'|+1$ VERTICES OF t'



SO $\circ$ IS NON-COMMUTATIVE.

(THE ONE-VERTEX TREE $\bullet$ IS THE IDENTITY; THE PRODUCT IS ASSOCIATIVE)

IF $\mathcal{J}$ IS GIVEN THE COPRODUCT

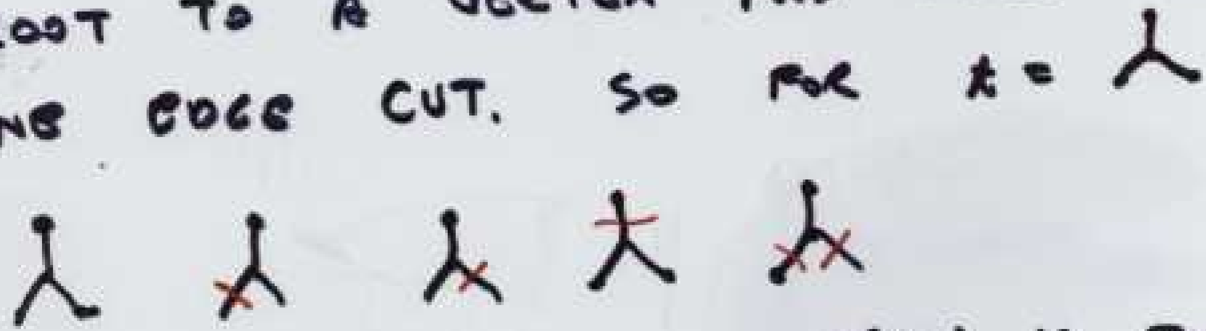
$$\Delta(B_+(A_1 \cdots A_k)) = \sum_{I \cup J = \{1, \dots, k\}} B_+(A_I) \otimes B_+(A_J)$$

THEN $\mathcal{J}$ IS A HOPF ALGEBRA, DESCRIBED BY GROSSMAN AND LARSON

KREIMER HOPF ALGEBRA $\mathcal{H}_K$

OUR BASIC OBJECTS ARE NOW FORESTS; MULTIPLICATION IS BY CONCATENATION. THE IDENTITY IS THE EMPTY FOREST $\emptyset$.

FOR A TREE t, A CUT IS THE REMOVAL OF SOME EDGES. A CUT IS ADMISSIBLE IF NO PATH FROM ROOT TO A VERTEX HAS MORE THAN ONE EDGE CUT. SO FOR $t =$



ARE ADMISSIBLE CUTS. $P^c(t)$ IS THE FOREST PARTED FROM t BY THE CUT c; $R^c(t)$ IS WHAT'S LEFT.

THE COPRODUCT IN $\mathcal{H}_K$ IS

$$\Delta(t) = t \otimes \emptyset + \sum_{\text{ADMISSIBLE } c} P^c(t) \otimes R^c(t)$$

AND EXTENDED TO FORESTS MULTIPLICATIVELY.

THE ANTIPODE IS GIVEN ON TREES BY

$$S(t) = - \sum_{\text{ALL CUTS } c} (-1)^{|c|} P^c(t) R^c(t)$$

WHERE THE SUM IS OVER <u>ALL</u> CUTS c OF t

IF WE DEFINE AN INNER PRODUCT ON TREES BY

$$(t, t') = \begin{cases} 0, & t \neq t' \\ |\mathrm{Sym}\, t|, & t = t' \end{cases}$$

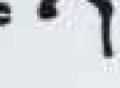
THEN THERE IS AN ISOMORPHISM $\chi : \mathcal{T} \longrightarrow \mathcal{H}_K^*$ GIVEN BY

$$\langle \chi(t), u \rangle = (t, B_+ u)$$

HOPF ALGEBRA $\mathcal{P}$ OF PLANAR ROOTED TREES

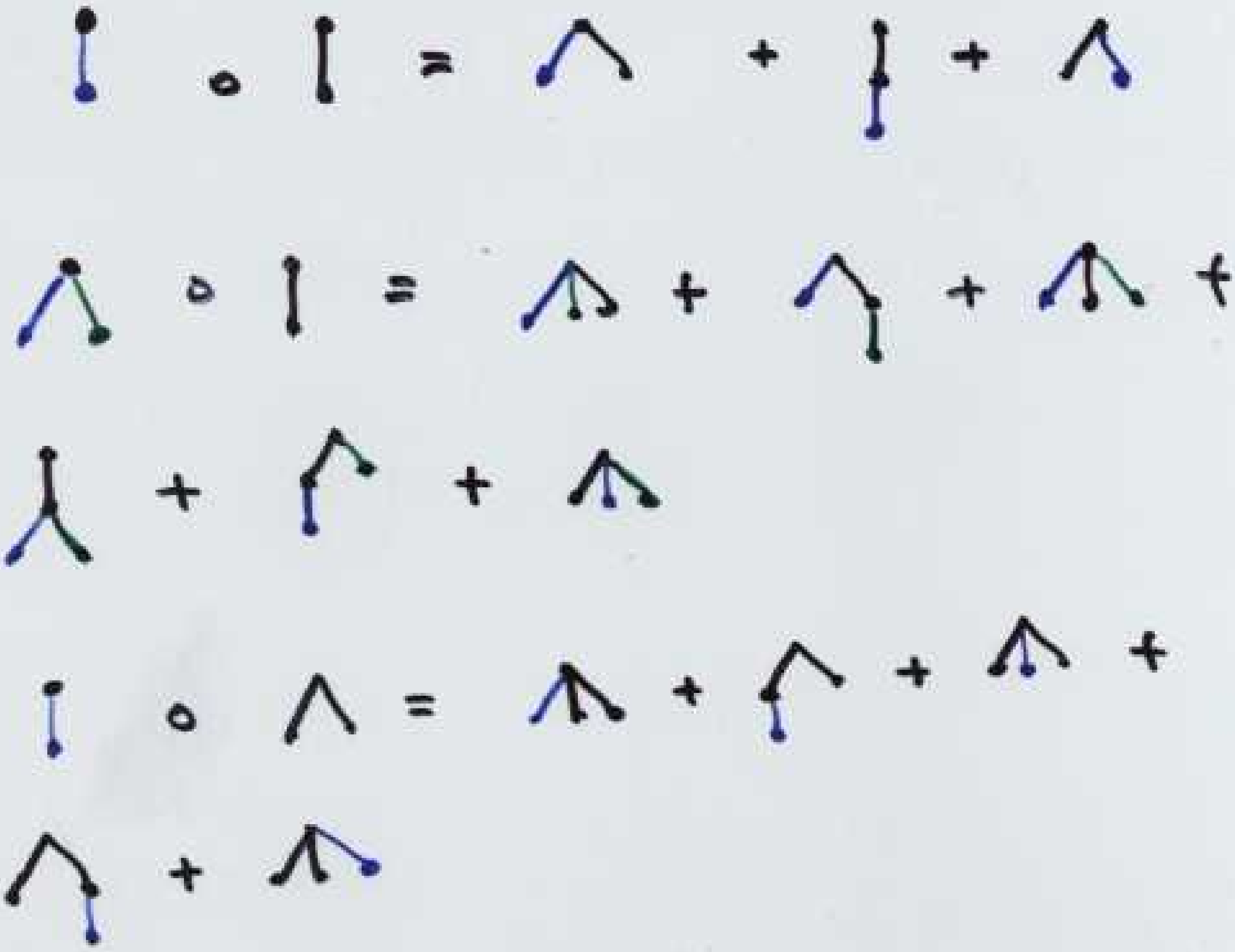
PLANAR TREES 

ORDERED FORESTS 

$B_+ :$ ORDERED FORESTS $\longrightarrow$ PLANAR TREES $\quad B_+(\cdot\,\mathfrak{l}) =$ 

PRODUCT OF PLANAR ROOTED TREES:

LET $t = B_+(t_1 \cdots t_k)$. THEN $t \circ t'$ IS THE SUM OF THE $\binom{2|t'|-1}{k}$ TREES OBTAINED BY ATTACHING, <u>IN</u> <u>ORDER</u>, THE k BRANCHES OF t TO VERTICES OF t', RESPECTING THE NATURAL ORDER ON THE VERTICES OF t'.



ALTHOUGH NOT COMMUTATIVE, THIS HAS THE CHARACTER OF A SHUFFLE PRODUCT. IN FACT, IT IS AN ASYMMETRIC SHUFFLE PRODUCT IN THE FOLLOWING SENSE.

REPRESENT PLANAR ROOTED TREES BY BALANCED BRACKET ARRANGEMENTS (BBA'S):

$\longleftrightarrow \quad \langle\langle\langle\rangle\rangle\langle\rangle\rangle$

CALL SUBSTRINGS OF A BBA THAT ARE THEMSELVES BBA'S COMPONENTS (CORRESPOND TO BRANCHES OF THE PLANAR ROOTED TREE).

SAY t, t' HAVE CORRESPONDING BBA'S c, c'. LET $c_1 c_2 \cdots c_k$ BE COMPONENTS OF t. SHUFFLE THE SYMBOLS $c_1, \ldots, c_k$ INTO THE BBA c', THEN REPLACE $c_1, \ldots, c_k$ WITH CORRESPONDING BBA'S

FOR $c = \langle\rangle\langle\rangle$, $c' = \langle\rangle$ WE HAVE

$$c \circ c' = c_1 c_2 \langle\rangle + c_1 \langle c_2 \rangle + c_1 \langle\rangle c_2 + \langle c_1 c_2 \rangle + \langle c_1 \rangle c_2 + \langle\rangle c_1 c_2$$

$$= \langle\rangle\langle\rangle\langle\rangle + \langle\rangle\langle\langle\rangle\rangle + \langle\rangle\langle\rangle\langle\rangle + \langle\langle\rangle\langle\rangle\rangle + \langle\langle\rangle\rangle\langle\rangle + \langle\rangle\langle\rangle\langle\rangle$$

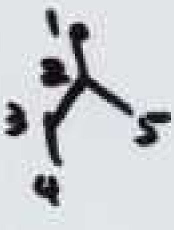
(ASYMMETRIC SINCE ONLY COMPONENTS OF LEFT-HAND FACTOR ARE KEPT TOGETHER WHILE SHUFFLING)

FOISSY HOPF ALGEBRA $\mathcal{H}_F$

OUR BASIC OBJECTS ARE ORDERED FORESTS OF PLANAR ROOTED TREES; JUXTAPOSITION PRODUCT IS NOW NON-COMMUTATIVE. COPRODUCT IS THAT FOR KREIMER'S HOPF ALGEBRA

$$\Delta(t) = t \otimes 1 + \sum_{\text{ADMISSIBLE } c} P^c(t) \otimes R^c(t)$$

NOTE THAT $P^c(t)$ CAN BE ORDERED BY USING THE ORDER OF THE "NEW" ROOT VERTICES; ALL PLANAR ROOTED TREES HAVE A NATURAL NUMBERING OF VERTICES:



ANTIPODE IS

$$S(t) = -\sum (-1)^{|c|} \overline{P^c(t)} \otimes R^c(t)$$

BY SAME PROOF AS BEFORE, $\mathcal{P} \cong \mathcal{H}_F^*$

BUT IN FACT $\mathcal{H}_F^* \cong \mathcal{H}_F$; FOISSY GAVE AN INNER PRODUCT ON $\mathcal{H}_F$ WITH $(FG, H) = (F \otimes G, \Delta(H))$.

IF WE DEFINE eF BY

$$(eF, G) = \delta_{FG}$$

THEN THE FUNCTION $x \mapsto eB_-(x)$ (WHERE $B_- = B_+^{-1}$) IS AN ISOMORPHISM. FOR EXAMPLE,

$$e_\bullet e_\bullet = \bullet\bullet = e_{\mathbf{I}} + 2e_{\bullet\bullet}$$

SINCE $e_\bullet = \bullet$, $e_{\bullet\bullet} = \mathbf{I}$ AND $e_{\mathbf{I}} = \bullet\bullet - 2\,\mathbf{I}$; THIS CORRESPONDS TO

$$\mathbf{I} \circ \mathbf{I} = 2\Lambda + \vdots \, .$$

A COMMUTATIVE DIAGRAM

LET ℓ_k BE THE "LADDER" $B_+^{k-1}(\bullet)$

THE KREIMER COPRODUCT FORMULA SHOWS THE ℓ_k ARE DIVIDED POWERS, SO THERE IS A HOPF ALGEBRA MAP

$$\mathrm{Sym} \xrightarrow{\ell} \mathcal{H}_K$$

SENDING e_k TO ℓ_k. IN FACT, THE DIAGRAM

$$\begin{array}{ccc} \mathrm{NSym} & \xrightarrow{\ell} & \mathcal{H}_F \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathrm{Sym} & \xrightarrow{\ell} & \mathcal{H}_K \end{array}$$

COMMUTES, WHERE $\pi : \mathrm{NSym} \to \mathrm{Sym}$ ABELIANIZES AND $\varphi : \mathcal{H}_F \to \mathcal{H}_K$ FORGETS ORDER.

THIS DUALIZES TO

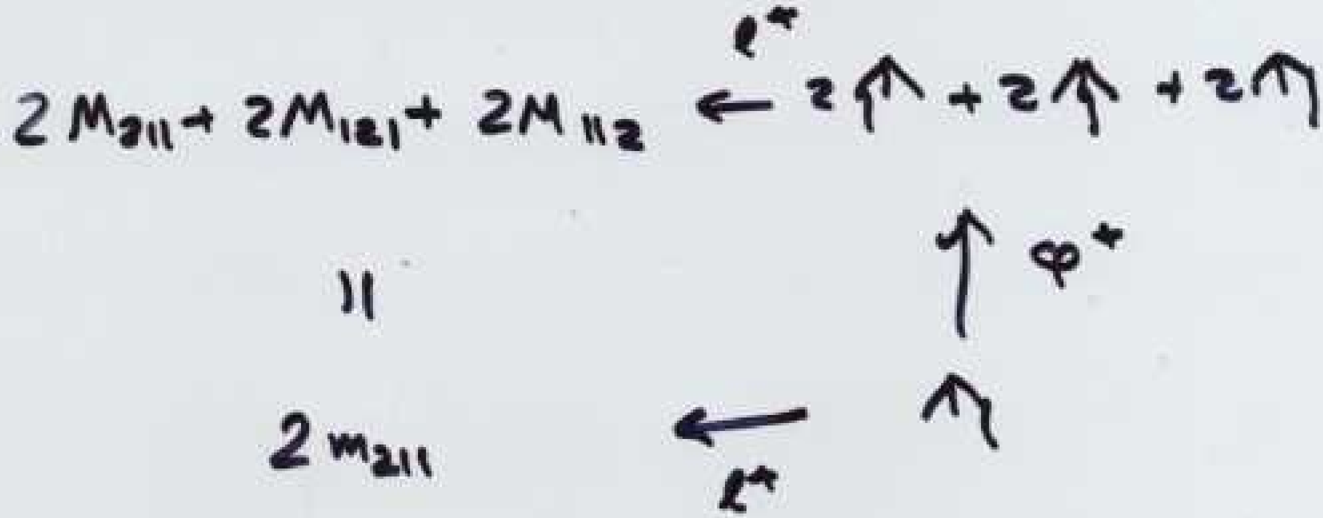
$$
\begin{array}{ccc}
QSym & \overset{\ell^*}{\longleftarrow} & \mathcal{P} \cong \mathcal{H}_K \\
\cup & & \uparrow \varphi^* \\
Sym & \overset{\ell^*}{\longleftarrow} & \mathcal{J}
\end{array}
$$

HERE, ℓ^* SENDS $B_+(\ell_{i_1} \cdots \ell_{i_k}) \in \mathcal{P}$ TO THE MONOMIAL QUASISYMMETRIC FUNCTION $M_{i_1 \cdots i_k}$, AND ALL OTHER TREES TO 0; φ^* SENDS A ROOTED TREE TO A SUM OF PLANAR ROOTED TREES GIVEN BY "EXERCISING" $*$ BY PERMUTING BRANCHES OUT OF ALL ITS VERTICES:

AND ℓ^* SENDS $B_+(\ell_{i_1} \cdots \ell_{i_k}) \in \mathcal{J}$

TO $\displaystyle\sum_{\substack{\text{COMPOSITIONS} \\ (j_1 \cdots j_k) \text{ OF} \\ \text{DISTINCT INTS.}}} x_{j_1}^{i_1} x_{j_2}^{i_2} \cdots x_{j_k}^{i_k}$, AND ALL OTHER

TREES TO 0.


2 M211 + 2M121 + 2M112
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