One-parameter groups and combinatorial physics

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Abstract.

`à rajouter : Riordan matrices, divisibility property`

1. Introduction

...\...
In the preceding articles the cases $(a^+)^r a^s$ were studied....

2. Bonson string matrices

Let $w \in \{a, a^+\}^*$ be a word in the letters $\{a, a^+\}$, and define $e = |w|_{a^+} - |w|_a$ (the excess), then the normal form of w^n reads

$$\mathcal{NF}(w^n) = (a^+)^{ne} \left(\sum_{k=0}^{\infty} S_w(n,k)(a^+)^k a^k \right) \tag{1}$$

when e is positive (i.e. there is more creations than annihilations).
In the opposite case (i.e. there is more annihilations than creations) the normal form of w^n is

$$\mathcal{NF}(w^n) = \left(\sum_{k=0}^{\infty} S_w(n,k)(a^+)^k a^k \right) (a)^{n|e|} \tag{2}$$

in each case, the coefficients S_w are well defined by the corresponding equation (1 and 2).
Now, for any boson string u one has

$$\mathcal{NF}(u) = (a^+)^{|u|_{a^+}} a^{|u|_a} + \sum_{|v|<|u|} \lambda_v v. \tag{3}$$

It has been observed [12] that the numbers λ_v are indeed rook numbers.

Let us give, as examples, the upper-left corner of these (doubly infinite) matrices.

For $w = a^+a$, one gets the usual matrix of Stirling numbers of the second kind.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 7 & 6 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 15 & 25 & 10 & 1 & 0 & \cdots \\ 0 & 1 & 31 & 90 & 65 & 15 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{4}$$

For $w = a^+aa^+$, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 4 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 6 & 18 & 9 & 1 & 0 & 0 & 0 & \cdots \\ 24 & 96 & 72 & 16 & 1 & 0 & 0 & \cdots \\ 120 & 600 & 600 & 200 & 25 & 1 & 0 & \cdots \\ 720 & 4320 & 5400 & 2400 & 450 & 36 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{5}$$

For $w = a^+aaa^+a^+$, one gets

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 12 & 60 & 54 & 14 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 144 & 1296 & 2232 & 1296 & 306 & 30 & 1 & 0 & 0 & \cdots \\ 2880 & 40320 & 109440 & 105120 & 45000 & 9504 & 1016 & 52 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{6}$$

Remark that, in each case, the matrix S_w has a staircase form and the "step" depends of the number of a's in the word w. More precisely, due to equation (3) one can prove that the ones ending each row are have (n, nr) as addresses (where $r = |w|_a$). Thus all the matrices are row finite and unitriangular iff $r = 1$, which case will be of special interest in the following. Moreover, the first column is $(1, 0, 0 \cdots, 0, \cdots, 0, \cdots)$ iff w ends with an a (this means that $\mathcal{NF}(w^n)$ is free of constant for all $n > 0$).

3. The algebra $\mathcal{L}(\mathbf{C}^{\mathbf{N}})$ and sequence transformations

Let $\mathbf{C}^{\mathbf{N}}$ be the vector space of all complex sequences, endowed with the Frechet product topology. It is easy to check that the algebra $\mathcal{L}(\mathbf{C}^{\mathbf{N}})$ of all continuous operators $\mathbf{C}^{\mathbf{N}} \to \mathbf{C}^{\mathbf{N}}$ is the space of *row-finite* matrices with complex coefficients. Such a matrix M is indexed by $\mathbf{N} \times \mathbf{N}$ and has the property that, for every fixed row index n, the sequence the sequence $(M(n,k))_{k\geq 0}$ has finite support. For a sequence $A = (a_n)_{n\geq 0}$, the transformed sequence $B = MA$ is given by $B = (b_n)_{n\geq 0}$ with

$$b_n = \sum_{k\geq 0} M(n,k) a_k \tag{7}$$

The combinatorial coefficients S_w defined above are indeed row-finite matrices.

To a sequence $(a_n)_{n\in\mathbf{N}}$ can be associated (univariate) series. It's generating series, formal or not, with a sequence of prescribed denominators $(d_n)_{n\in\mathbf{N}}$ is

$$\sum_{n\geq 0} a_n \frac{z^n}{d_n}. \tag{8}$$

For example, with $d_n = 1$, we get the ordinary generating functions (OGF), with $d_n = n!$, we get the exponential generating functions (EGF) and with $d_n = (n!)^2$, the doubly exponential generating functions (DEGF) and so on. Thus, once the denominators have been choosen, to every (linear continuous) transformation of generating functions, one can associate it's matrix.

The algebra $\mathcal{L}(\mathbf{C}^{\mathbf{N}})$ possesses many interesting subalgebras and groups as the algebra of lower triangular transformations $\mathcal{T}(\mathbf{N},\mathbf{C})$, the group of inversible elements of the latter $\mathcal{T}_{inv}(\mathbf{N},\mathbf{C})$ (which is the set of infinite lower triangular matrices with non-zero elements on the diagonal), the subgroup of unipotent transformations $\mathcal{UT}(\mathbf{N},\mathbf{C})$ (i.e. the set of infinite lower triangular matrices with elements on the diagonal alla equal to 1) and it's Lie algebra $\mathcal{NT}(\mathbf{N},\mathbf{C})$, the algebra of locally nilpotent transformations (with zeroes on the diagonal). One has the inclusions (with $\mathcal{D}(\mathbf{N},\mathbf{C})$, the set of diagonal matrices).

$$\begin{array}{l}\mathcal{UT}(\mathbf{N},\mathbf{C}) \subset \mathcal{T}_{inv}(\mathbf{N},\mathbf{C}) \subset \mathcal{T}(\mathbf{N},\mathbf{C}) \subset \mathcal{L}(\mathbf{C}^{\mathbf{N}}) \\ \mathcal{D}(\mathbf{N},\mathbf{C}) \subset \mathcal{T}(\mathbf{N},\mathbf{C}) \text{ and } \mathcal{NT}(\mathbf{N},\mathbf{C}) \subset \mathcal{L}(\mathbf{C}^{\mathbf{N}})\end{array} \tag{9}$$

We can remark that $\mathcal{T}_{inv}(\mathbf{N},\mathbf{C}) = \mathcal{D}(\mathbf{N},\mathbf{C}) \bowtie \mathcal{UT}(\mathbf{N},\mathbf{C})$ because $\mathcal{UT}$ is normalized by $\mathcal{D}$ and $\mathcal{T}_{inv} = \mathcal{D}.\mathcal{UT}$ (every invertible transformation is the product of it's diagonal by a unipotent trasformation).

We will examine now an important class of transformations of $\mathcal{T}$ as well as it's diagonal : the substitutions with prefunctions.

3.1. Substitutions with prefunctions

Let $(d_n)_{n\geq 0}$ bet a fixed set of denominators. We consider, for a generating function f, the transformation

$$\Phi_{g,\phi}[f](x) = g(x)f(\phi(x)) \tag{10}$$

the matrix of this transformation $M_{g,\phi}$ is given by the transforms of the monomials $\frac{x^k}{d_k}$ hence

$$\sum_{n\geq 0} M_{g,\phi}(n,k)\frac{x^n}{d_n} = \Phi_{g,\phi}\left[\frac{x^k}{d_k}\right] = g(x)\frac{(\phi(x))^k}{d_k} \tag{11}$$

if $g,\phi \neq 0$ (otherwise the trasformation is trivial), we can write

$$g(x) = a_l\frac{x^l}{d_l} + \sum_{r>l} a_r\frac{x^r}{d_r},\ \ \phi(x) = \alpha_m\frac{x^m}{d_m} + \sum_{s>m}\alpha_s\frac{x^s}{d_s} \tag{12}$$

with $a_l, \alpha_m \neq 0$ and then, by (11)

$$\Phi_{g,\phi}\left[\frac{x^k}{d_k}\right] = a_l(\alpha_m)^k\frac{x^{l+mk}}{d_l d_m^k d_k} + \sum_{t>l+mk} b_t\frac{x^t}{d_t} \tag{13}$$

one has

$$M_{g,\phi} \text{ is row} - \text{finite} \Longleftrightarrow \phi \text{ has no constant term} \tag{14}$$

and, in this case, it is always lower triangular. From now on, we will suppose that ϕ has non constant term ($\alpha_0 = 0$).

Moreover $M_{g,\phi} \in \mathcal{T}_{inv}$ iff $a_0, \alpha_1 \neq 0$ and then the diagonal term with address (n,n) is $\frac{a_0}{d_0}\left(\frac{\alpha_1}{d_1}\right)^n$. We get

$$M_{g,\phi} \in \mathcal{UT} \Longleftrightarrow \frac{a_0}{d_0} = \frac{\alpha_1}{d_1} = 1 \tag{15}$$

In particular for the EGF and the OGF, we have the condition that

$$g(x) = 1 + \text{higher terms } \textit{and } \phi(x) = x + \text{higher terms} \tag{16}$$

4. Unipotent transformations

4.1. Lie group structure

We first remark that $n \times n$ truncations (i.e. the fact of taking the $[0..n] \times [0..n]$ submatrix of a matrix) are algebra morphisms

$$\tau_n : \mathcal{T}(\mathbf{N}, \mathbf{C}) \to \mathcal{M}([0..n] \times [0..n], \mathbf{C}) \tag{17}$$

we can endow $\mathcal{T}(\mathbf{N}, \mathbf{C})$ with the Frechet topology giiven by these morphisms. We will not develop this point in details, but this topology is metrisable and given by the following convergence criterium :

$$\begin{aligned} &a\ sequence\ (M_k)\ of\ matrices\ in\ \mathcal{T}(\mathbf{N}, \mathbf{C})\ converges\ iff \\ &for\ all\ fixed\ n \in \mathbf{N} \\ &the\ sequence\ of\ truncated\ matrices\ \ (\tau_n(M_k))\ \ converges. \end{aligned} \tag{18}$$

This topology is compatible with the structure of $\mathbf{C}$-algebra of $\mathcal{T}(\mathbf{N}, \mathbf{C})$.

The two maps $\exp : \mathcal{NT}(\mathbf{N}, \mathbf{C}) \to \mathcal{UT}(\mathbf{N}, \mathbf{C})$ and $\log : \mathcal{UT}(\mathbf{N}, \mathbf{C}) \to \mathcal{NT}(\mathbf{N}, \mathbf{C})$ are continous and mutually inverse.

4.2. Examples

4.2.1. Provided by the exponential formula

We first recall the “classical exponential formula” (see appendix A for a precise categorical - and general - version of this formula).
For a class of objects $\mathcal{C}$ with some technical restrictions (see appendix), we denote $EGF(\mathcal{C})$ the exponential generating series of $\mathcal{C}$. Denoting $\mathcal{C}^c$ the connected objects of $\mathcal{C}$, we have

$$EGF(\mathcal{C}) = e^{EGF(\mathcal{C}^c)} \tag{19}$$

The reader is invited to check, using the appendix, to convince himself that the use of the exponantial formula in the following examples is quite legal.

Example 1 : Stirling numbers.

We here use the graphs of equivalence relations. Then using the statistics $x^{(\text{number of points})}y^{(\text{number of connected components})}$ we get

$$\sum_{n,k\geq 0} S(n,k)\frac{x^n}{n!}y^k =$$
$$\sum_{\text{all equivalence graphs }\Gamma} \frac{x^{(\text{number of points of }\Gamma)}}{(\text{number of points of }\Gamma)!}y^{(\text{number of connected components of }\Gamma)} =$$
$$\exp\left(\sum_{\Gamma\ connected} \frac{x^{(\text{number of vertices of }\Gamma)}}{(\text{number of points of }\Gamma)!}y^{(\text{number of connected components of }\Gamma)}\right) =$$
$$\exp\left(\sum_{n\geq 1} y\frac{x^n}{n!}\right) = e^{y(e^x-1)} \tag{20}$$

we will see that the transformation associated with the matrix $S(n,k)$ is $f \to f(e^x - 1)$.

Example 2 : Idempotent numbers.
We consider the graphs of endofunctions (i.e. functions from a finite set to itself). Then using the statistics $x^{(\text{number of points of the set})}y^{(\text{number of connected components of the graph})}$ and denoting $I(n,k)$ the number of endofunctions of a given set with n elements having k connected components, we get

$$\sum_{n,k\geq 0} I(n,k)\frac{x^n}{n!}y^k =$$
$$\sum_{\text{all graphs of endofunctions }\Gamma} \frac{x^{(\text{number of vertices of }\Gamma)}}{(\text{number of vertices of }\Gamma)!}y^{(\text{number of connected components of }\Gamma)} =$$
$$\exp\left(\sum_{\Gamma\ connected} \frac{x^{(\text{number of vertices of }\Gamma)}}{(\text{number of vertices of }\Gamma)!}y^{(\text{number of connected components of }\Gamma)}\right) =$$
$$\exp\left(\sum_{n\geq 1} y\frac{nx^n}{n!}\right) = e^{yxe^x} \tag{21}$$

for these numbers, we get the (doubly) infinite matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 3 & 6 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 4 & 24 & 12 & 1 & 0 & 0 & \cdots \\ 0 & 5 & 80 & 90 & 20 & 1 & 0 & \cdots \\ 0 & 6 & 240 & 540 & 240 & 30 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{22}$$

we will see that the transformation associated with this matrix is $f \to f(xe^x)$

4.2.2. Provided by linearization of powers of boson strings

To get unipotent matrices, one takes a bonson string with only one derivation. The string then reads $w = (a^+)^{r-p}a(a^+)^p$. We have given the examples with
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We will see in a moment that

- if $p = 0$, $S_w(n,k)$ is the matrix of a unipotent substitution
- if > 0, $S_w(n,k)$ is the matrix of a unipotent substitution with prefunction

4.3. A criterium

In fact, we have the general proposition.

proposition Let $T = (T(n,k))_{n,k\geq 0}$ be the matrix of a unipotent transformation, then the following properties are equivalent :

(i) T is the matrix of the transformation $f \to g(x)f(\phi(x))$ with $g(0) = 1$ and $\phi(x) = x +$ higher terms

(ii) For every k one has

$$\sum_{n\geq 0} T(n,k)\frac{x^n}{n!} = g(x)\frac{(\phi(x))^k}{k!}$$

with $g(x) = \sum_{n\geq 0} T(n,0)\frac{x^n}{n!}$ and $\phi(x) = \sum_{n\geq 1} T(n,1)\frac{x^n}{n!}$
*(Sheffer-type condition see [**?**])*

(iii)
$$\sum_{n,k\geq 0} T(n,k)\frac{x^n}{n!}y^k = g(x)e^{y\phi(x)}$$

which gives immediately the solution for the matrices of "graph-type".

To cope with the matrices coming from the linearization of boson strings let us do a small excursion to analysis and formal groups.

5. One-parameter subgroups of $UT(\mathbf{N},\mathbf{C})$

5.1. Exponential of elements of $NT(\mathbf{N},\mathbf{C})$

Let $M = I + N \in UT(\mathbf{N},\mathbf{C})$ ($I = I_{\mathbf{N}}$ is the diagonal, hence the indentity matrix). One has

$$M^t = \sum_{k\geq 0} \begin{pmatrix} t \\ k \end{pmatrix} N^k \tag{23}$$

where $\begin{pmatrix} t \\ k \end{pmatrix}$ is the generalized binomial coefficient defined by

$$\begin{pmatrix} t \\ k \end{pmatrix} = \frac{(t)(t-1)\cdots(t-k+1)}{k!} \tag{24}$$

one can see that, due to the local nilpotency of N, the matrix coefficient $M^t(n,k)$ is well defined and, in fact, a polynomial of degree $n-k$ in t. We have the additive property $M^{t_1+t_2} = M^{t_1}M^{t_2}$ and the correspondence $t \to M^t$ is continuous. Conversely, using the projections and the theorem about continous one-parameter groups in Lie groups (see [**?**], for example) one can prove that, if $t \to M_t$ is a continous local one-parameter group in $UT(\mathbf{N},\mathbf{C})$ that is, for some real $\epsilon > 0$

$$t_1, t_2 < \epsilon \Longrightarrow M_{t_1}M_{t_2} = M_{t_1+t_2} \tag{25}$$

then there exists a unique matrix $H \in NT(\mathbf{N},\mathbf{C})$ such that $M_t = exp(tH)$. In case $M_t = M^t$ is defined by formula (23) we have $H = log(I+N) = \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} N^k$.

The mapping $t \to M^t$ will be called a *one parameter group* of $UT(\mathbf{N},\mathbf{C})$.

proposition 3 Let M be the matrix of a substitution, then so is M^t for all $t \in \mathbf{C}$.

The proof will be detailed in a forthcoming paper [**?**] and uses the fact that "to be the matrix of a substitution" is a property of polynomial type. But, using composition, it is straightforward that M^t is the matrix of a substitution far all $t \in \mathbf{N}$. Thus, using an argument in the style of Zariski, we get the fact that the property is true for all $t \in \mathbf{C}$.

5.2. Link with local Lie groups : Straightening vector fields on the line

Let us treat first the case of $p = 0$. The string $(a^+)^r a$ corrresponds, in the Bargmann-Fock representation, to the vector field $x^r \frac{d}{dx}$ defined on the whole line.
Now, we can try (at least locally) to straighten this vector field by a diffeomorphism u to get the constant vector field (this procedure has been introduced by G. Goldin in the context of algebras of currents [5]). As the one-parameter group generated by a constant field is the shift, the one-parameter (local) group of transformations should read, on a suited domain)

$$U_\lambda[f](x) = f\left(u^{-1}\left(u(x)+\lambda\right)\right) \tag{26}$$

Now, we know from the theory that if two one-parameter have the same tangent vector at the origin, then they coincide *(tangent paradigm)*.
Direct computation gives this tangent vector :

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} f\left(u^{-1}\left(u(x)+\lambda\right)\right) = \frac{1}{u'(x)} f'(x) \tag{27}$$

so the local one-parameter group U_λ has $\frac{1}{u'(x)}\frac{d}{dx}$ as tangent vector field.
Here, we have to solve $\frac{1}{u'(x)} = x^r$ in order to get the diffeomorphism u.
In the case $r \neq 1$, we have (with $\mathcal{D} =]0,+\infty[$ as domain

$$u(x) = \frac{x^{1-r}}{1-r} = y;\ u^{-1}(y) = ((1-r)y)^{\frac{1}{1-r}} \tag{28}$$

and

$$e^{\lambda x^r \frac{d}{dx}}[f](x) = f\left(\frac{x}{(1-\lambda(r-1)x^{r-1})^{\frac{1}{r-1}}}\right) \tag{29}$$

The substitution factor $s_\lambda(x) = \frac{x}{(1-\lambda(r-1)x^{r-1})^{\frac{1}{r-1}}}$ has been obtained by other means in [**?**]. The computation is similar for the case when $r=1$ and, for this case, we get

$$e^{\lambda x\frac{d}{dx}}[f](x) = f\left(e^\lambda x\right) \tag{30}$$

with $s_\lambda(x) = e^\lambda x$ as substitution factor.
First examples are summarized in the following table

$r =$	$s_\lambda(x) =$	Name
0	$x+\lambda$	Shift
1	$e^\lambda x$	Dilation
2	$\frac{x}{1-\lambda x}$	Homography
3	$\frac{x}{\sqrt{1-2\lambda x^2}}$	-

Comment If one uses classical analysis (i.e. convergent Taylor series), one must be careful about the domain where the substitution are defined and the one-parameter groups are defined only locally.
On each of these examples, one can check by hand that, for suitable (and small) values of λ, μ, one has $s_\lambda(s_\mu(x)) = s_{\lambda+\mu}(x)$ (one-parameter group property).
It is possible to get rid of the discusion on the domains by considering λ, μ as new variables and applying the “substitution principle” saying that it is possiible to substitute a series without constant term in a series (within $\mathbf{C}[[x, \lambda, \mu]]$).

Using the same method, one can start wiith more complicated operators. Examples and substitution factors are given below

Operator	Substitution Factor	Description
$(1+(a^+)^2)\,a$	$s_\lambda(x) = \frac{xcos(\lambda)+sin(\lambda)}{cos(\lambda)-xsin(\lambda)}$	Composition of a rotation with an homography
$\frac{\sqrt{1+(a^+)^2}}{a^+}$	$s_\lambda(x) = \sqrt{x^2+2\lambda\sqrt{1+x^2}+\lambda^2}$	Composition of quadratic direct and inverse functions

5.3. Case $p > 0$: another conjugacy trick and a shocking formula

Now, seing vector fields as infinitesimal generators of one-parameter groups, leads to conjugacy as, if U_λ is a one-parameter group of transformation, so is $VU_\lambda V^{-1}$, for any continous invertible operator V. The case $(a^+)^{r-p}a(a^+)^p$; $p > 0$ belongs to this setting as $(a^+)^{-p}\left((a^+)^r a\right)(a^+)^p$. More generally, supposing all the terms be defined, with

$$\Omega = u_1(x)\frac{d}{dx}u_2(x) = \frac{1}{u_2(x)}\left(u_1(x)u_2(x)\frac{d}{dx}\right)u_2(x)$$

one has

$$e^{\lambda\Omega} = \frac{1}{u_2(x)}\left(e^{\lambda u_1(x)u_2(x)\frac{d}{dx}}\right)u_2(x) \tag{31}$$

This shocking formula (31) may be understood as an operator equality.
Now, the tangent paradigm tels us that, if we adjust tha tangent vector to coincide with $x^{r-p}\frac{d}{dx}x^p$ (recall that the original problem was the integration of the operator $\Omega = (a^+)^{r-p}a(a^+)^p;\ p > 0$), then we get the right one-parameter group. Using this "conjugacy trick" we get

$$e^{\lambda\Omega}[f](x) = \left(\frac{s_\lambda(x)}{x}\right) f(s_\lambda(x)) \text{ with } s_\lambda(x) = \frac{x}{(1-\lambda(r-1)x^{r-1})^{\frac{1}{r-1}}} \quad (32)$$

Remark It can be checked that, if $s_\lambda(x)$ is a substitution factor (i.e. at least locally $s_\lambda(s_\mu(x)) = s_{\lambda+\mu}(x)$) such that $s_\lambda(0) = 0$ for evary λ (which is the case in most of our examples) then the transformations defined by $U_\lambda[f](x) = \left(\frac{s_\lambda(x)}{x}\right) f(s_\lambda(x))$ form a one-parameter (possibly local) group.

The algebra generated by $a^+, (a^+)^{-1}, a$ is graded by

$$\text{weight}(a^+) = 1,\ \text{weight}\left((a^+)^{-1}\right) = \text{weight}(a) = -1 \quad (33)$$

and every homogeneous operator of this algebra which is of the form

$$\Omega = \sum_{|w|_a=1,\ \text{weight}(w)=e} \alpha_w w \quad (34)$$

(there is only one derivative in each monomial) can be integrated in the preceding manner. So one would like to reconstruct the characteristic series

$$\sum_{n,k} S_\Omega(n,k)\frac{x^n}{n!}y^k \quad (35)$$

from the knowledge of the one-parameter subgroup $e^{\lambda\Omega}$.
This is the aim of the following paragraph.

5.4. Characteristic series ↔ one parameter group correspondence

For every homogeneous operator as above with $e \geq 0$, one defines the coefficients $S_\Omega(n,k)$ as in **??** by

$$\mathcal{N}(\Omega^n) = (a^+)^{ne}\sum_{k=0}^{\infty} S_\Omega(n,k)(a^+)^k a^k \quad (36)$$

One has the following proposition

Proposition 3 With the preceding denotations, the following conditions are equivalent:

$$\sum_{n,k\geq 0} S_\Omega(n,k)\frac{x^n}{n!}y^k = g(x)e^{y\phi(x)} \quad (37)$$

$$U_\lambda[f](x) = g(\lambda x^e)f\left(x\left(1+\phi(\lambda x^e)\right)\right) \quad (38)$$

which solves the problem.

6. Conclusion and remaining problems

Acknowledgments

We thank .... for important discussions.

7. Appendix : A statistical (categorical) version of the exponential formula

Throughout the paper, we will be interested to compute various examples of EGF for combinatorial objects having (a finite set of) nodes (their set-theoretical support) so we use as central concept the mapping σ which associates to every structure, its set of nodes.

We need to draw what could be called "square-free decomposable objects" (SFD). This version is suited to our needs for the "exponential formula". It is sufficiently general to contain, as a particular case, the case of multivariate series. For other points of view, see [6, 8, 9]

Let $\mathcal{C}$ be a class of (combinatorial) objects and **FSt** the catgory of finite sets, $\mathcal{C}$ will be called (SFD) if it fulfills the two following conditions.

(DS) *Direct sum.* — There is a (partial) binary law $\oplus$ on $\mathcal{C}$, defined for couples of objects (ω_1, ω_2) such that $\sigma(\omega_1) \cap \sigma(\omega_2) = \emptyset$, which is associative, commutative and such that

$$\mathcal{C}_{F_1} \times \mathcal{C}_{F_2} \stackrel{\oplus}{\rightarrow} \mathcal{C}_{F_1 \cup F_2} \tag{39}$$

is into.

Moreover, $\mathcal{C}_\emptyset$ consists in a single element $\{\epsilon\}$ which is neutral in the sense that, identically

$$\epsilon \oplus \omega = \omega \oplus \epsilon = \omega \tag{40}$$

(LP) *Levi's property.* — Let $\omega = \omega_1 \oplus \omega_2 = \omega^1 \oplus \omega^2$ be two decompositions. Then it can be found a four terms decomposition $\omega = \oplus_{i,j=1,2}\omega_j^i$ refining the original data in the sense that the maginal sums give the factors of the decompositions i.e.

$$\omega_j = \omega_j^1 \oplus \omega_j^2 \text{ and } \omega^i = \omega_1^i \oplus \omega_2^i;\ i,j = 1,2 \tag{41}$$

Note that condition (39) implies that $\sigma(\omega_1 \oplus \omega_2) = \sigma(\omega_1) \sqcup \sigma(\omega_2)$.

Now, an *atom* is any object $\omega \neq \epsilon$ which cannot be split, formally

$$\omega = \omega_1 \oplus \omega_2 \Longrightarrow \epsilon \in \{\omega_1, \omega_2\} \tag{42}$$

As example of this setting we have:

(i) the positive square-free integers $\sigma(n)$ being the set of primes which divide n, the atoms being the primes.

(ii) the positive integers $\sigma(n)$ being the set of primes which divide n, the atoms being the primes.

(iii) all graphs, hypergraphs and weighted graphs, $\sigma(G)$ being the set of nodes and $\oplus$ the juxtaposition, here the atoms are connected graphs.

(iv) the class of endofunctions f with $\sigma(f) = dom(f)$

(v) the (multivariate) polynomials in $\mathbf{N}[X]$ with $\sigma = Alph$ and $\oplus = +$.

(vi) the square-free monic (for a given order on the variables) polynomials ; $\sigma(P)$ being the set of irreducible monic divisors of P and $\oplus$ being the multiplication.

(vii) complex algebraic curves ; $\sigma(V)$ being the set of monic irreducible bivariate polynomials vanishing on an infinite subset of V.

The prescriptions (DS,LP) imply that decomposition of objects into atoms always exists and is unique.

proposition Let $\omega \in \mathcal{C}$ then $\omega = \omega_1 \oplus \omega_2 \oplus \cdots \oplus \omega_l$ the ω_i being (dinstinct) atoms and the set $\{\omega_1, \omega_2 \cdots \omega_l\}$ depends only on ω.

In the class $\mathcal{C}$, objects are conceived to be "measured" by different parameters (data in statistical language). So, to get a general purpose tool, we suppose that the statistics takes it's values iin a ring K which contains $\mathbf{Q}$ (as, to write EGFs it is convenient to have no trouble with the fractions $\frac{1}{n!}$). Let then $c : \mathcal{C} \to K$ be the given statistic. In order to write generating series, we need

(i) that the sum $c_F = \sum_{\omega \in \mathcal{C}_F} c(w)$ exists

(ii) that $F \to c_F$ should depend only of the cardinality of F.

(iii) $c(\omega_1 \oplus \omega_2) = c(\omega_1).c(\omega_2)$

We formalize it in

(LF) *Local finiteness.* — For each finite set F, the subclass

$$\mathcal{C}_F = \{\omega \in \mathcal{C} | \sigma(\omega) = F\} \tag{43}$$

is a finite set.

(Eq) *Equivariance.* —

$$|F_1| = |F_2| \Longrightarrow c_{F_1} = c_{F_2} \tag{44}$$

(Mu) *Multiplicativity.* —

$$c(\omega_1 \oplus \omega_2) = c(\omega_1).c(\omega_2) \tag{45}$$

Note. —

a) In fact, (LF) is a property of the class $\mathcal{C}$, while (Eq) is a property of the statistics. In practice, we choose $\mathcal{C}$ which is locally finite and choose equivariant statistics for instance

$$c(\omega) = x^{(\text{number of cycles})} y^{(\text{number of fixed points})}$$

for some variables x, y.

b) More generally, it is typical to take integer-valued partial (additive) statistics $c_1, \cdots c_i, \cdots, c_r$ (for every $\omega \in \mathcal{C}$, $c_i(\omega) \in \mathbf{N}$) and set $c(\omega) = x_1^{c_1(\omega)} x_2^{c_2(\omega)} \cdots x_r^{c_r(\omega)}$.

c) The class of examples 7.ii is not locally finite, but other examples satisfy (LF): 7.iii if one asks that the number of arrows and weight is finite, 7.i and 7.v to 7.vii in any case.

Now, we are in position to state the exponential formula as it will be used throughout the paper.

Proposition Let $\mathcal{C}$ be a locally finite (SFD) and $c : \mathcal{C} \to K$ an equivariant statistics on $\mathcal{C}$. For every subclass $\mathcal{F}$ one sets the following exponential generating series

$$E(\mathcal{F}) = \sum_{n=0}^{\infty} c(\mathcal{F}_{[1..n]}) \frac{z^n}{n!} \tag{46}$$

Let $\mathcal{C}^a$ be the set of atoms of $\mathcal{C}$. Then, one has

$$E(\mathcal{C}) = e^{E(\mathcal{C}^a)} \tag{47}$$

Proof — (First Step). — We consider the subclasses of objects the atoms of which have a support of cardinality n i.e.

$$\mathcal{C}[n] = \{\omega \in \mathcal{C} | \omega = \omega_1 \oplus \omega_2 \oplus \cdots \oplus \omega_s \text{ with } \omega_i \in \mathcal{C}^a, \text{ and } |\sigma(\omega_i)| = n\} \tag{48}$$

□

These subclasses are closed under compositions (i.e. under $\oplus$) and decompositions and their atoms $\mathcal{C}^a[n] = \{\omega \in \mathcal{C}[n] \cap \mathcal{C}^a\}$. Now, one has, thanks to the partitions of $[1..n]$

$$\mathcal{C}_{[1..n]} = \bigsqcup_{\substack{k\geq 0,\ 0<n_1<n_2<\cdots<n_k \\ n_1+n_2+\cdots n_k=n}} \ \bigsqcup_{\substack{|P_j|=n_j \\ P_1\sqcup P_2\sqcup\cdots P_k=[1..n]}} \mathcal{C}_{P_1} \oplus \mathcal{C}_{P_2} \oplus \cdots \oplus \mathcal{C}_{P_k} \tag{49}$$

$$c(\mathcal{C}_{[1..n]}) = \sum_{k\geq 0} \sum_{\substack{0<n_1<n_2<\cdots<n_k \\ n_1+n_2+\cdots n_k=n}} \ \sum_{\substack{|P_j|=n_j \\ P_1\sqcup P_2\sqcup\cdots P_k=[1..n]}} c(\mathcal{C}_{P_1})c(\mathcal{C}_{P_2})\cdots c(\mathcal{C}_{P_k}) \tag{50}$$

as, for disjoint sets, it is easy to check that $c(\mathcal{C}_X \oplus \mathcal{C}_Y) = c(\mathcal{C}_X)c(\mathcal{C}_Y)$. Now, due to the equivariance of c and to the fact that partitions $(P_1, P_2, \cdots, P_k)$ such that $P_j = n_j$ and $P_1 \sqcup P_2 \sqcup \cdots P_k = [1..n]$ are in number

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

we get

$$c(\mathcal{C}_{[1..n]}) = \sum_{k\geq 0} \sum_{\substack{0<n_1<n_2<\cdots<n_k \\ n_1+n_2+\cdots n_k=n}} \frac{n!}{n_1!n_2!\cdots n_k!} c(\mathcal{C}_{[1..n_1]})c(\mathcal{C}_{[1..n_2]})\cdots c(\mathcal{C}_{[1..n_k]}) \tag{51}$$

and then

$$E(\mathcal{C}) = \prod_{n>0} E(\mathcal{C}[n]) \tag{52}$$

We then compute the factors.

$$E(\mathcal{C}[n]) = \sum_{k\geq 0} c(\mathcal{C}[n]_{[1..nk]}) \frac{z^{nk}}{(nk)!} \tag{53}$$

but

$$E(\mathcal{C}^a[n]) = c(\mathcal{C}^a_{[1..n]}) \frac{z^n}{n!} \tag{54}$$

(one monomial) and

$$e^{E(\mathcal{C}^a[n])} = \sum_{k\geq 0} c(\mathcal{C}^a_{[1..n]})^k \frac{z^{nk}}{(n!)^k k!} = \sum_{k\geq 0} c(\mathcal{C}^a_{[1..n]})^k \frac{z^{nk}}{(nk!)} \frac{(nk)!}{(n!)^k k!} =$$

$$\sum_{k\geq 0} c(\mathcal{C}[n]_{[1..nk]}) \frac{z^{nk}}{(nk)!} = E(\mathcal{C}[n]) \tag{55}$$

due to the fact that the number of (unordered) partitions of $[1..nk]$ into k blocs of cardinality n is $\frac{(nk)!}{(n!)^k k!}$. To end the proof, it suffices to remark that $\mathcal{C}^a = \sqcap_{n>0} \mathcal{C}^a[n]$ and then

$$E(\mathcal{C}) = \prod_{n>0} E(\mathcal{C}[n]) = \prod_{n>0} e^{E(\mathcal{C}^a[n])} = e^{\sum_{n>0} E(\mathcal{C}^a[n])} = e^{E(\mathcal{C}^a)} \tag{56}$$

□

Note. —

The proof suggests us that it is combinatorially fruitful to factor a class $\mathcal{C}$ into (full) subclasses i.e. that are generated by a partition of the atoms.

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