*Paris, le 15 Mai 2003, 21h03*

**1) Endofunctions by number of cycles. —**

For $f : F \to F$, we define $f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ factors}}$. Let $p$ be a prime and set

$$\mathcal{J}_p(n) = \#\{f : [[n]] \to [[n]] s.t.\ f^{1+p} = f\} \tag{1}$$

where $[[n]] := [1..n]$. One has the following characterization $f^{1+p} = f$ iff every connected component of the graph has

- a 1- or a $p$-cycle
- for all $x$, $f(x)$ is in the cycle

then the EGF for connected components with a 1-cycle is

$$\sum_{n\geq 1} \frac{nt^n}{n!} = te^t \tag{2}$$

the EGF for connected components with a $p$-cycle is

$$\sum_{n\geq p} \binom{n}{p} (p-1)! p^{(n-p)} \frac{t^n}{n!} t^n = \frac{t^p}{p} e^{pt} \tag{3}$$

Hence, having analyzed the connected components, one gets, with the exponential formula,

$$\sum_{n\geq 0} \mathcal{J}_p(n) \frac{t^n}{n!} = e^{te^t + \frac{t^p}{p} e^{pt}} \tag{4}$$

If, one wants to count these endofonctions more finely, we can get the EGF again with the exponential formula. with

$$\sum_{n\geq 1} \frac{xnt^n}{n!} = xte^t \tag{5}$$

instead of (2) and

$$\sum_{n\geq p} y \binom{n}{p} (p-1)! p^{(n-p)} \frac{t^n}{n!} t^n = y \frac{t^p}{p} e^{pt} \tag{6}$$

instead of (3), we get

$$\sum_{n\geq 0} \alpha_{k,l}^{(n)} x^k y^l \frac{t^n}{n!} = \sum_{n\geq 0} H_n^{(p)}(x,y) \frac{t^n}{n!} = e^{xte^t + y\frac{t^p}{p} e^{pt}} \tag{7}$$

**2) Élémentary computation of the $\alpha_{k,l}^{(n)}$. —**

By elementary counting, we get

$$\alpha_{k,l}^{(n)} = \binom{n}{k} \binom{n-k}{pl} \frac{(pl)!}{p^l l!} (k+pl)^{n-k-pl} \tag{8}$$

and, from (7)

$$H_n^{(p)}(x,y) = \sum_{k+3l\leq n} \alpha_{k,l}^{(n)} x^k y^l \tag{9}$$

if we glue the two types of cycles, one gets the numbers $\mathcal{J}_p(n,k)$ which is the number of such endofunctions with $k$ cycles. These numbers fill a lower triangular matrix ($\mathcal{J}_p(n,k) = 0$ for $k > n$). We have

$$\sum_{n\geq 0} \mathcal{J}_p n,k^{(n)} x^k \frac{t^n}{n!} = e^{x(te^t + \frac{t^p}{p} e^{pt})} \tag{10}$$

and this proves that we are in presence of a *substitution matrix*.