Hierarchical Dobiński-type relations via substitution and the moment problem

K A Penson[†], **P Blasiak**[†‡], **G Duchamp**[§§],
A Horzela[‡] **and A I Solomon**[†§]

[†]Université Pierre et Marie Curie,
Laboratoire de Physique Théorique des Liquides, CNRS UMR 7600
Tour 16, $5^{ième}$ étage, 4, place Jussieu, F 75252 Paris Cedex 05, France

[‡]H. Niewodniczański Institute of Nuclear Physics, Polish Academy of Sciences
Department of Theoretical Physics
ul. Radzikowskiego 152, PL 31-342 Kraków, Poland

[§]The Open University
Physics and Astronomy Department
Milton Keynes MK7 6AA, United Kingdom

[§§]Université de Rouen
Laboratoire d'Informatique Fondamentale et Appliquée de Rouen
76821 Mont-Saint Aignan, France

E-mail:
blasiak@lptl.jussieu.fr, gduchamp2@free.fr, andrzej.horzela@ifj.edu.pl,
penson@lptl.jussieu.fr, a.i.solomon@open.ac.uk

Abstract. We consider the transformation properties of integer sequences arising from the normal ordering of exponentiated boson ($[a, a^\dagger] = 1$) monomials of the form $\exp[\lambda(a^\dagger)^r a]$, $r = 1, 2, \ldots$, under the composition of their exponential generating functions (egf). They turn out to be of *Sheffer-type*. We demonstrate that two key properties of these sequences remain preserved under substitutional composition: *a*) the property of being the solution of the Stieltjes moment problem; and *b*) the representation of these sequences through infinite series (Dobiński-type relations). We present a number of examples of such composition satisfying properties *a*) and *b*). We obtain new Dobiński-type formulas and solve the associated moment problem for several hierarchically defined combinatorial families of sequences.

1. Introduction

In a recent series of articles [1],[2],[3],[4],[5],[6] we investigated the properties of integer sequences appearing in the process of the normal ordering of powers of boson monomials $[(a^\dagger)^r a^s]^n$, with n, r, s -integers, where a and $a^\dagger$ are the boson annihilation and creation operators respectively, satisfying $[a, a^\dagger] = 1$. They are extensions of earlier works [7],[8]. We observed that the normal form of $[(a^\dagger)^r a^s]^n$, with all the annihilation operators to the right, denoted by $\mathcal{N}\left([(a^\dagger)^r a^s]^n\right)$, can be written in the form $(r \geq s)$:

$$[(a^\dagger)^r a^s]^n \equiv \mathcal{N}\left([(a^\dagger)^r a^s]^n\right) = (a^\dagger)^{n(r-s)} \sum_{k=s}^{ns} S_{r,s}(n,k)(a^\dagger)^k a^k \tag{1}$$

where $S_{r,s}(n,k)$ are generalizations of the conventional $(r = s = 1)$ Stirling numbers of the second kind and

$$B_{r,s}(n) = \sum_{k=s}^{ns} S_{r,s}(n,k) \tag{2}$$

generalize the conventional $(r = s = 1)$ Bell numbers.

For general $r \geq s$ we have worked out a complete theory of the numbers $S_{r,s}(n,k)$ and $B_{r,s}(n)$, including their recurrence relations, generating functions and closed-form formulas. In particular, the generalized Bell numbers $B_{r,s}(n)$ can be expressed as infinite series, thereby extending the celebrated Dobiński relation valid for $r = s = 1$ [9]:

$$B_{1,1}(n) = \frac{1}{e}\sum_{k=0}^{\infty} \frac{k^n}{k!}, \qquad n = 0, 1, 2, \ldots \tag{3}$$

Here are some examples of such relations:

$$B_{r,1}(n) = \frac{1}{e}\sum_{k=1}^{\infty} \frac{1}{k!}\prod_{j=1}^{n}[k + (j-1)(r-1)] \tag{4}$$

$$B_{r,r}(n) = \frac{1}{e}\sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{(k+r)!}{k!}\right]^{n-1} \tag{5}$$

they are all derived from the general polynomial-type formula $(n = 1, 2, \ldots)$

$$\begin{aligned} B_{r,s}(n,y) &= \sum_{k=s}^{ns} S_{r,s}(n,k) y^k \\ &= e^{-y}\sum_{k=s}^{\infty}\frac{1}{k!}\prod_{j=1}^{n}\left[(k+(j-1)(r-s))\cdot(k+(j-1)(r-s)-1)\cdot\right. \\ &\qquad \left.\ldots\cdot(k+(j-1)(r-s)-s+1)\right] y^k. \end{aligned} \tag{6}$$

We may associate a *Generating Function* $C(x)$ with a given sequence $\{c_n\}$ by [9]

$$C(x) = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}. \tag{7}$$

This particular form of Generating Function is known as a *Generating Function of Exponential Type* or *egf* for short, due to the $n!$ denominators. Of particular interest for us here are those sequences $\{B_{r,s}(n)\}$ for which the *egf* can in fact be expressed as an exponential function; they include $B_{r,1}(n)$, $r = 1, 2, ...$ for which

$$e^{e^x-1} = \sum_{n=0}^{\infty} B_{1,1}(n)\frac{x^n}{n!} \tag{8}$$

and [1],[2],[10]

$$\exp\left(\frac{1}{\sqrt[r-1]{1-(r-1)x^{r-1}}} - 1\right) = \sum_{n=0}^{\infty} B_{r,1}(n)\frac{x^n}{n!}. \qquad r = 2, 3, \ldots \tag{9}$$

The numbers $S_{r,s}(n, k)$ appear when in Eqs.(8) and (9) an indeterminate y is introduced through

$$e^{y(e^x-1)} = \sum_{n=0}^{\infty}\left(\sum_{k=1}^{n} S_{1,1}(n, k)y^k\right)\frac{x^n}{n!} \tag{10}$$

and

$$\exp\left[y\left(\frac{1}{\sqrt[r-1]{1-(r-1)x^{r-1}}} - 1\right)\right] = \sum_{n=0}^{\infty}\left(\sum_{k=1}^{n} S_{r,1}(n, k)y^k\right)\frac{x^n}{n!}, \qquad r = 2, 3, \ldots \tag{11}$$

Eqs.(10) and (11) define polynomials of order n:

$$B_{r,1}(n, y) = \sum_{k=1}^{n} S_{r,1}(n, k)y^k. \qquad r = 1, 2, \ldots. \tag{12}$$

Evidently, $B_{r,1}(n) = B_{r,1}(n, 1)$. The polynomials of Eqs.(12) share another characteristic property: they can be written as ratios of two infinite series in y. These are the so-called Dobiński-type relations [1],[2], which for $r = 1$ and $r > 1$ respectively are:

$$\frac{1}{e^y}\sum_{k=1}^{\infty}\frac{k^n}{k!}y^k = \sum_{k=1}^{n} S_{1,1}(n, k)y^k, \qquad n = 0, 1, \ldots \tag{13}$$

and

$$\frac{(r-1)^n}{e^y}\sum_{k=1}^{\infty}\frac{\Gamma(n+\frac{k}{r-1})}{k!\Gamma(\frac{k}{r-1})}y^k = \sum_{k=1}^{n} S_{r,1}(n, k)y^k, \qquad n = 1, 2, \ldots. \tag{14}$$

By setting $y = 1$ in Eqs.(10) and (11) we obtain a representation of the integers $B_{r,1}(n)$ as an infinite series (compare Eqs.(3)-(5)); this constitutes a fertile ground for their probabilistic interpretation [11],[12]. The numbers $B_{r,1}(n)$ can also be given various combinatorial intepretations [13],[14]. The second consequence of Eqs.(13) and (14) (and of the more general formulas for $s > 1$, see [1],[2]) is the fact, that $B_{r,1}(n, y)$ for $y > 0$ is the n-th Stieltjes moment of a non-negative probability distribution, which is either discrete (for $r = 1$, giving a so called *Dirac comb* [3]) or continuous (for $r > 1$). This fact permits one to use the $B_{r,s}(n, y)$ to construct various quantum collective states called coherent states [4],[15]. The interpretation of combinatorial sequences as moments [5] has led to new calculational approaches to hyperdeterminants [16]. Another aspect of Eqs.(13) and (14) which deserves mention here is that the numbers $S_{r,1}(n, k)$ $(1 \le k \le n)$

form a non-singular lower-triangular matrix with ones on the diagonal. Such matrices form a group, called the *Riordan group*, which has important applications in enumerative combinatorics [17],[18].

The purpose of this note is to place Eqs.(10)-(14) in the more general context of *Sheffer-type* polynomials and to address the question of compositional substitution and its implication for the existence of Dobiński-type relations as solutions of the Stieltjes moment problem.

We first recall the known fact [19],[20],[21] that a compositional substitution corresponds to multiplication of the matrices $S_{r,1}(n,k)$. Then we go on to demonstrate that if two polynomial sequences $B_F(n,y)$ and $B_G(n,y)$ generated by $e^{yF(x)}$ and $e^{yG(x)}$ respectively are solutions of the associated Stieltjes moment problems, then the sequence $B_{F(G)}(n,y)$ is also a solution of another, closely related, Stieltjes moment problem. We further prove that if $B_F(n,y)$ and $B_G(n,y)$ are both given by Dobiński-type relations, see Eqs.(13) and (14), then the sequence $B_{F(G)}(n,y)$ is also given by an analogous formula. We then illustrate these *reproducing* properties of Dobiński-type relations and moment problem solutions by some specific examples. They comprise multiple compositions of standard Bell numbers with themselves (composing discrete with discrete distributions), compositions of Lah numbers (related to Laguerre polynomials) with themselves and finally composing discrete with continuous distributions and vice versa.

2. Sheffer-type polynomials

A polynomial $B(n,y)$ of order n in the variable y is of Sheffer-type if the associated egf can be written in the form [22]

$$1+\sum_{n=1}^{\infty} B(n,y)\frac{x^n}{n!} = A(x)e^{yF(x)} \tag{15}$$

with $A(0)=1$ and $F(0)=0$. Many such families of polynomials have been thoroughly investigated. Among the polynomials encountered in Quantum Mechanics, the Hermite and Laguerre polynomials are of Sheffer-type, whereas the Legendre and Gegenbauer are not. Comparing Eq.(15) with Eqs.(10) and (11) we observe that $B_{r,1}(n,y)$ are Sheffer-type polynomials with $A(x)=1$. In fact $B_{1,1}(n,y)$ are the so-called Bell (or exponential) polynomials [22] and $B_{2,1}(n,y)$ are the generalized Laguerre polynomials. The numbers $S_{1,1}(n,k)$ are the conventional Stirling numbers of the second kind and the numbers

$$S_{2,1}(n,k) = \frac{n!}{k!}\binom{n-1}{k-1} \tag{16}$$

are the so-called unsigned Lah numbers [1],[10].

More generally, consider two families of Sheffer-type polynomials $B_F(n,y)$ and $B_G(n,y)$ generated by

$$e^{yF(x)} = 1+\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} S_F(n,k)y^k\right)\frac{x^n}{n!} \tag{17}$$

and

$$e^{yG(x)} = 1 + \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} S_G(n,k)y^k\right)\frac{x^n}{n!}, \tag{18}$$

respectively, where $F(0) = G(0) = 0$ and

$$B_F(n,y) = \sum_{k=1}^{n} S_F(n,k)y^k \quad \text{and} \quad B_G(n,y) = \sum_{k=1}^{n} S_G(n,k)y^k. \tag{19}$$

We now consider the polynomials generated by $F(G(x))$, i.e.

$$e^{yF(G(x))} = 1 + \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} S_{F(G)}(n,k)y^k\right)\frac{x^n}{n!} \tag{20}$$

Before we calculate this sum we note the relation resulting from the change of summation in Eq.(18):

$$e^{yG(x)} = 1 + \sum_{k=1}^{\infty}\left(\sum_{n=k}^{\infty} S_G(n,k)\frac{x^n}{n!}\right)y^k. \tag{21}$$

Now comparison with the direct expansion of the left hand side of Eq.(21)

$$e^{yG(x)} = 1 + \sum_{k=1}^{\infty}(G(x))^k\ y^k/k! \tag{22}$$

yields

$$\frac{(G(x))^k}{k!} = \sum_{n=k}^{\infty} S_G(n,k)\frac{x^n}{n!}. \tag{23}$$

Proceeding to the direct calculation of Eq.(20) we recall that the matrices $S_F(n,k)$ and $S_G(n,k)$ are lower triangular (i.e. the entries for $k > n$ are zero):

$$e^{yF(G(x))} = 1 + \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} S_F(n,k)y^k\right)\frac{(G(x))^n}{n!} \tag{24}$$

$$= 1 + \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} S_F(n,k)y^k\right)\sum_{p=n}^{\infty} S_G(p,n)\frac{x^p}{p!} \tag{25}$$

$$= 1 + \sum_{p=1}^{\infty}\left(\sum_{k=1}^{p}\left(\sum_{n=1}^{p} S_G(p,n)S_F(n,k)\right)y^k\right)\frac{x^p}{p!}. \tag{26}$$

Comparison with Eq.(20) yields

$$S_{F(G)}(n,k) = \sum_{p=1}^{n} S_G(n,p)S_F(p,k) \tag{27}$$

This last equality means that compositional substitution within the Sheffer-type polynomial families is equivalent to the matrix product of the corresponding Stirling matrices [19],[20]

$$\mathbb{S}_{F(G)} = \mathbb{S}_G \cdot \mathbb{S}_F \quad . \tag{28}$$

A direct consequence of Eq.(27) is the formula

$$B_{F(G)}(n,y) = \sum_{p=1}^{n} S_{F(G)}(n,p)y^p = \sum_{p=1}^{n} y^p \sum_{k=1}^{n} S_G(n,k)S_F(k,p) \tag{29}$$

$$= \sum_{k=1}^{n} S_G(n,k) \sum_{p=1}^{k} S_F(k,p)y^p = \sum_{k=1}^{n} S_G(n,k)B_F(k,y). \tag{30}$$

The last equation can be seen as the generalized Stirling transform [23] of the polynomials $B_F(k,y)$ which for $y = 1$ reduces to the generalized Stirling transform of the sequence $B_F(k)$:

$$B_{F(G)}(n) = \sum_{k=1}^{n} S_G(n,k)B_F(k). \tag{31}$$

3. Compositional moment problem

The formulas (27) and (30) lead to important consequences if the initial Sheffer-type polynomials are solutions of the Stieltjes moment problems, i.e. if for $x, y > 0$ there exist positive weight functions $W_F(x,y)$ and $W_G(x,y)$ such that

$$B_F(n,y) = \int_0^\infty x^n W_F(x,y)dx, \tag{32}$$

$$B_G(n,y) = \int_0^\infty x^n W_G(x,y)dx\ . \tag{33}$$

Then the following equalities follow:

$$\begin{aligned}
B_{F(G)}(n,y) &= \sum_{k=1}^{n} S_G(n,k)B_F(k,y) \\
&= \sum_{k=1}^{n} S_G(n,k)\int_0^\infty x^k W_F(x,y)dx = \int_0^\infty W_F(x,y)\sum_{k=1}^{n} S_G(n,k)x^k\ dx \\
&= \int_0^\infty W_F(x,y)B_G(n,x)dx = \int_0^\infty dx\ W_F(x,y)\int_0^\infty z^n W_G(z,x)dz \\
&= \int_0^\infty z^n \left(\int_0^\infty W_F(x,y)W_G(z,x)dx\right) dz
\end{aligned} \tag{34}$$

and this implies that

$$B_{F(G)}(n,y) = \int_0^\infty x^n W_{F(G)}(x,y)dx \tag{35}$$

where $W_{F(G)}(x,y)$ is a positive function given by

$$W_{F(G)}(x,y) = \int_0^\infty W_F(z,y)W_G(x,z)dz. \tag{36}$$

We remark that the arguments of the weight functions in Eq.(36) need not satisfy any particular symmetry properties.

More generally, for p-fold substitution $F_1(F_2(\ldots(F_p)\ldots))$ one obtains

$$W_{F_1(F_2(\ldots(F_p)\ldots))}(x,y) = \int_0^\infty dz_1\ W_{F_1}(z_1,y)\int_0^\infty dz_2\ W_{F_2}(z_2,z_1)\ldots$$
$$\ldots\int_0^\infty dz_p\ W_{F_{p-1}}(z_p,z_{p-1})W_{F_p}(x,z_p). \qquad (37)$$

Eq.(37) reveals a typical structure appearing in the iterated-kernel method of solving integral equations [24],[25].

In other words; for the Sheffer-type polynomials the property of being a solution of the Stieltjes moment problem is *reproduced* by the mechanism of compositional substitution, under the evident condition that the integrals in Eqs.(36) and (37) exist. In the following section we provide a number of examples of substitutions $F(G(x))$ for which an explicit evaluation of $W_{F(G)}(x,y)$ and $B_{F(G)}(n,y)$ can be carried through.

4. Compositional Dobiński-type relations

A rather large reservoir of solutions of the Stieltjes moment problem is contained in the formulas (13) and (14). For any $r=1,2,\ldots$ $B_{r,1}(n,y)$ is the moment of a positive function $W_r(x,y)$, which can be written down explicitly, for instance by extending to $y\neq 1$ the results given in [4],[5], [6]. The examples are:

$$W_1(x,y) = e^{-y}\sum_{k=1}^{\infty}\frac{y^k\delta(x-k)}{k!} \qquad (38)$$

$$W_2(x,y) = ye^{-(x+y)}\frac{I_1(2\sqrt{xy})}{\sqrt{xy}} \qquad (39)$$

$$W_3(x,y) = \frac{1}{12\sqrt{\pi}x}e^{-\frac{x}{2}-y}y\left(6\sqrt{2x\pi}+3xy\sqrt{\pi}\ {}_0F_2(\tfrac{3}{2},2;\tfrac{xy^2}{8})\right. \qquad (40)$$

$$\left.+\sqrt{2}x^{3/2}y^2\ {}_1F_3(1;\tfrac{3}{2},2,\tfrac{5}{2};\tfrac{xy^2}{8})\right), \qquad (41)$$

where $\delta(z)$ is the Dirac delta function, $I_\nu(z)$ is the modified Bessel function of first kind and ${}_0F_2$ and ${}_1F_3$ are hypergeometric functions. Eqs.(39) and (41) were obtained using the inverse Mellin transform. See [26] for its exposition and [27] for examples of applications.

Note, that whereas $W_1(x,y)$ is a discrete distribution in the form of a Dirac comb concentrated on positive integers, the functions $W_r(x,y)$ for $r>1$ are continuous distributions [6]. Observe also that they are not normalized, in the sense of their zero moments: $\int_0^\infty W_1(x,1)dx = 1$ whereas $\int_0^\infty W_r(x,1)dx \neq 1$, $r>1$.

In this section we demonstrate that the reproducing character of the compositional moment problem, see Eq.(36), implies the reproducing character of the Dobiński-type relations. In the following paragraph, with given $F(x)$ and $G(x)$ of Eqs.(17) and (18) we will carry out explicit substitutions $F(G(x))$ and analyze the weight functions $W_{F(G)}(x)$ obtained from Eq.(36) and the resulting Dobiński-type relations.

4.1. $F(x) = G(x) = e^x - 1$

In the following the subscript $B(B)$ stands for "substitute Bell into Bell". We investigate the polynomials $B_{B(B)}(n, y)$ resulting from

$$e^{y\left(e^{e^x-1}-1\right)} = \sum_{n=0}^{\infty} B_{B(B)}(n, y)\frac{x^n}{n!} \tag{42}$$

which correspond to the ordinary Stirling transform [23] of the Bell polynomials $B_{1,1}(n, y)$

$$B_{B(B)}(n, y) = \sum_{k=1}^{n} S(n, k)B_{1,1}(k, y) \tag{43}$$

where $S(n, k)$ are the conventional Stirling numbers of the second kind. The polynomial $B_{1,1}(n, y)$ is the n-th moment of the Dirac comb [3],

$$W_B(x, y) = e^{-y}\sum_{k=1}^{\infty}\frac{y^k\delta(x-k)}{k!}, \tag{44}$$

and the weight function resulting from the substitution $F(F(x))$ is through Eq.(36) equal to

$$\begin{aligned} W_{B(B)}(x, y) &= \int_0^{\infty} W_B(z, y)W_B(x, z)dz = \\ &= \int_0^{\infty}\left(e^{-y}\sum_{k=1}^{\infty}\frac{y^k\delta(z-k)}{k!}\right)\left(e^{-z}\sum_{p=1}^{\infty}\frac{z^p\delta(x-p)}{p!}\right)dz \\ &= e^{-y}\sum_{p=1}^{\infty}\frac{\delta(x-p)}{p!}\left(\sum_{k=1}^{\infty}\frac{k^p}{k!}(ye^{-1})^k\right) \\ &= e^{y(e^{-1}-1)}\sum_{p=1}^{\infty}\frac{\delta(x-p)}{p!}\left(\sum_{r=1}^{p} S(p, r)(ye^{-1})^r\right), \end{aligned} \tag{45}$$

where the last equality results from the original Dobiński formula Eq.(13). This result shows that

$$B_{B(B)}(n) = B_{B(B)}(n, 1) = e^{\left(e^{-1}-1\right)}\sum_{k=1}^{\infty}\frac{k^n}{k!}\left(\sum_{r=1}^{p} S(k, r)e^{-r}\right), \tag{46}$$

with the initial terms $B_{B(B)}(n) = 1, 1, 3, 12, 60, 358, 2471, 19302, \ldots$, for $n = 0, 1, \ldots$. $B_{B(B)}(n)$ counts the number of partitions of a set of n distinguishable elements, in which every part is again partitioned [19].

Multiple substitutions of Bell egf's into themselves result in hierarchical, chain-like formulas for corresponding partition numbers, i.e. for $F(F(F(x)))$ one obtains for $n = 0, 1, \ldots$

$$\begin{aligned} B_{B(B(B))}(n) = \mathrm{e}^{\left(e^{(e^{-1}-1)}-1\right)}\sum_{k=1}^{\infty}\frac{k^n}{k!}\,\cdot \\ \cdot\left(\sum_{p=1}^{k} S(k, p)e^{-p}\left(\sum_{r=1}^{p} S(p, r)e^{r(e^{-1}-1)}\right)\right). \end{aligned} \tag{47}$$

For example, $B_{B(B(B))}(n) = 1, 1, 4, 22, 154, 1304, 12915, 146115, \ldots,$ for $n = 0, 1 \ldots,$ which counts the number of "triple" partitions of an n-set.

We conclude that the substitution $F(F(x))$ results in a formula for $B_{B(B)}(n)$ which conserves the original Dobiński-type structure of $B_B(n)$ as in Eq.(3); and also gives a Dirac comb type of weight function with modified weights concentrated on positive integers. These results also hold good for higher order substitutions.

4.2. $F(x) = G(x) = \frac{x}{1-x}$

This case corresponds to $B_{2,1}(n, y)$ which from Eq.(14) is

$$B_{2,1}(n, y) = \frac{1}{e^y}\sum_{k=1}^{\infty}\frac{\Gamma(n+k)}{k!\Gamma(k)}y^k = n!\sum_{k=1}^{n}\frac{1}{k!}\binom{n-1}{k-1}y^k, \tag{48}$$

and can be also written as

$$B_{2,1}(n, y) = (n-1)!\; yL^{(1)}_{n-1}(-y) \tag{49}$$

by using the standard form of the generating function of generalized Laguerre polynomials $L^{(\lambda)}_n(x)$. With the notational convention introduced above we rewrite Eq.(49) as (here L stands for Laguerre)

$$B_L(n, y) = \sum_{k=1}^{n} S_L(n, k)y^k \tag{50}$$

where $S_L(n, k)$ are the unsigned Lah numbers, see Eq.(16). For $y = 1$, the integers $B_L(n, 1) \equiv B_L(n)$ count binary ordered forests of n nodes [13] (the initial terms are $B_L(n) = 1, 3, 13, 73, 501, 4051\ldots,\ n = 1, 2, \ldots$). For other combinatorial interpretations see [28].

The polynomial $B_L(n, y)$ is the n-th moment of [6] (see Eq.(40)):

$$W_L(x, y) = ye^{-(x+y)}\frac{I_1(2\sqrt{xy})}{\sqrt{xy}} \tag{51}$$

By $F(F(x))$-type composition the function $\exp\left(\frac{yx}{1-2x}\right)$ generates $B_{L(L)}(n, y)$ through

$$e^{y\frac{x}{1-2x}} = \sum_{n=0}^{\infty} B_{L(L)}(n, y)\frac{x^n}{n!} \tag{52}$$

where $L(L)$ stands for "substitute Laguerre into Laguerre", which are the n-th moments of

$$\begin{aligned} W_{L(L)}(x, y) &= \int_0^{\infty} W_L(z, y)W_B(x, z)dz = \\ &= \int_0^{\infty}\left(ye^{-(z+y)}\frac{I_1(2\sqrt{zy})}{\sqrt{zy}}\right)\cdot\left(ze^{-(x+z)}\frac{I_1(2\sqrt{xz})}{\sqrt{xz}}\right)dz. \end{aligned} \tag{53}$$

By virtue of the entry 2.15.20.8 of [29], this yields a continuous distribution

$$W_{L(L)}(x, y) = ye^{-\frac{x+y}{2}}\frac{I_1(\sqrt{xy})}{2\sqrt{xy}} = \frac{1}{2}W_L(\tfrac{x}{2}, \tfrac{y}{2}), \tag{54}$$

thus preserving the original structure encountered in Eq.(51). In addition, simple use of the generating function of the generalized Laguerre polynomials yields

$$B_{L(L)}(n,y)=\int_0^\infty x^n W_{L(L)}(x,y)dx=2^{n-1}(n-1)!\ yL_{n-1}^{(1)}(-\tfrac{y}{2}) \tag{55}$$

whose initial terms for $y=1$ are $B_{L(L)}(n)=1,5,37,361,4361,62701\ldots$, $n=1,2,\ldots$. The p-fold substitution, $p=1,2,\ldots$, gives in this case the compact expression:

$$B_{L(L(\ldots(L)\ldots))}(n)=p^{n-1}(n-1)!L_{n-1}^{(1)}(-\tfrac{1}{p}),\qquad n=1,2,\ldots \tag{56}$$

4.3. $F(x)=e^x-1,\ G(x)=\frac{x}{1-x}$

Here we substitute Laguerre (continuous distribution) into Bell (discrete distribution) and vice versa.

The calculations are analogous to those in 4.1 and 4.2 with repeated use of integrals listed in [29]. We only quote the final results:

$$B_{B(L)}(n,y)=\int_0^\infty x^n W_{B(L)}(x,y)dx, \tag{57}$$

where

$$W_{B(L)}(x,y)=\frac{e^{-(x+y)}}{\sqrt{x}}\sum_{k=1}^{\infty}\frac{y^k}{k!}\sqrt{k}e^{-k}I_1(2\sqrt{kx}) \tag{58}$$

which is a continuous distribution. The polynomials $B_{B(L)}(n,y)$ are generated by

$$e^{y(e^{\frac{x}{1-x}}-1)}=\sum_{n=0}^{\infty}B_{B(L)}(n,y)\frac{x^n}{n!}. \tag{59}$$

The initial terms of $B_{B(L)}(n)$ are $1,4,23,171,1552,16583\ldots$ for $n=1,2,\ldots$. These integers count structures called *sets of sets of lists*, where *list* means an ordered subset [28]. A closed-form Dobiński-type formula for $B_{B(L)}(n)$ can be obtained by calculating the moments of $W_{B(L)}(x,1)$. A longer but straightforward calculation gives

$$B_{B(L)}(n)=e^{-1}\sum_{k=1}^{\infty}\frac{(n-1)!L_{n-1}^{(1)}(-k)}{(k-1)!}\ . \tag{60}$$

Higher order substitutions yield formulas of similar type.

For the opposite substitution ("Bell into Laguerre" denoted $L(B)$) generated by

$$e^{y\frac{e^x-1}{2-e^x}}=\sum_{n=0}^{\infty}B_{L(B)}(n,y)\frac{x^n}{n!}. \tag{61}$$

we obtain

$$B_{L(B)}(n,y)=\int_0^\infty x^n W_{L(B)}(x,y)dx, \tag{62}$$

where

$$W_{L(B)}(x,y)=\frac{y}{2}e^{-\frac{y}{2}}\sum_{k=1}^{\infty}\frac{\delta(x-k)}{2^k\cdot k}L_{k-1}^{(1)}(-\tfrac{y}{2}) \tag{63}$$

which is a discrete (Dirac comb) distribution, with moments

$$B_{L(B)}(n) = B_{L(B)}(n,1) = \frac{1}{2}e^{-\frac{1}{2}}\sum_{k=1}^{\infty}\frac{k^{n-1}}{2^k}L_{k-1}^{(1)}(-\tfrac{1}{2}) \tag{64}$$

and initial terms $B_{B(L)}(n) = 1, 4, 23, 173, 1602, 17575\ldots$, for $n = 1, 2, \ldots$.

4.4. Bell numbers vs. "ordered" Bell numbers

As the last example we shall consider a slightly more general substitution problem in which only the "internal" egf $G(x)$ is of Sheffer-type. In other words, the egf of one of the sequences is not an exponential. A case in point is given by the so called "ordered" Bell numbers [3],[9] $B_O(n)$ defined through

$$B_O(n) = \sum_{k=1}^{n} S(n,k)\ k!\ . \tag{65}$$

Their extension to polynomials $B_O(n,y) = \sum_{k=1}^{n} S(n,k)\ k!\ y^k$ is generated by [21]

$$\frac{1}{1-y(e^x-1)} = \sum_{n=0}^{\infty} B_O(n,y)\frac{x^n}{n!}. \tag{66}$$

Thus the $B_O(n,y)$ are not of Sheffer-type.

We now perform the substitution "Bell into ordered Bell", denoted by the subscript $O(B)$. Although Eq.(30) is no longer valid, we can still define the numbers $B_{O(B)}(n)$ through Eq.(31):

$$B_{O(B)}(n) = \sum_{k=1}^{n} S(n,k)B_O(k), \tag{67}$$

or equivalently by

$$\frac{1}{2-e^{e^x-1}} = \sum_{n=0}^{\infty} B_{O(B)}(n)\frac{x^n}{n!}. \tag{68}$$

Recalling the Dobiński-type expression for $B_O(n)$ [3], [9]

$$B_O(n) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{k^n}{2^k}\ , \tag{69}$$

the formula Eq.(36), now for $y = 1$ only, carries over and after straightforward calculation we obtain the Dobiński-type formula for $B_{O(B)}(n)$:

$$B_{O(B)}(n) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{k^n}{k!}Li_{-k}(\tfrac{1}{2e}) \tag{70}$$

where $Li_m(y)$ is the polylogarithm of order m of y. The initial terms are $B_{O(B)}(n) = 1, 4, 23, 175, 1662, 18937, \ldots$, $n = 1, 2, \ldots$.

Similarly, from the substitution "double Bell into ordered Bell" (denoted by $O(B(B))$ below) we obtain

$$B_{O(B(B))}(n) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{k^n}{k!}\left(\sum_{r=1}^{\infty}\frac{e^{-r}r^k}{r!}Li_{-r}(\tfrac{1}{2e})\right) \tag{71}$$

where

$$\frac{1}{2 - e^{e^{e^x-1}-1}} = \sum_{n=0}^{\infty} B_{O(B(B))}(n) \frac{x^n}{n!} \,, \qquad \textit{etc.} \tag{72}$$

Clearly, Eqs.(70) and (71) again give rise to Dirac comb weight functions.

5. Discussion and conclusions

The main result of this work can be viewed from different perspectives. It is primarily a method for the generation of new solutions of moment problems. As such it is of potential importance for the construction of new generalized coherent states. Refs. [5] and [4] should be considered as first steps in this direction. The iterative method based on Eqs.(36) and (37) appears to be straightforward under the condition of the existence of the relevant integrals. This will definitely extend and enrich the families of currently known solutions of the moment problem.

A closer look at the examples above based on Eq.(36) leads to the conclusion that if $e^{G(x)}$ generates the moments of a discrete distribution then the moments generated by $e^{F(G(x))}$ are those of a discrete distribution. Similarly, when $e^{G(x)}$ gives a continuous distribution, the composition $e^{F(G(x))}$ gives rise to a continuous distribution.

We are dealing here with Sheffer-type polynomials which are also solutions of the moment problem; it should be borne in mind that these are quite strong restrictions. It is easy to construct Sheffer-type polynomials which are *not* solutions of the moment problem. For example, the polynomials $p_n(y)$, which are related to Bessel polynomials [22], are generated by

$$e^{y(\sqrt{1+2x}-1)} = 1 + \sum_{n=1}^{\infty} p_n(y) \frac{x^n}{n!} \tag{73}$$

and can take on negative values for $y = 1$; they are therefore not acceptable solutions of the moment problem. On the other hand, for $s > 1$, the polynomials $B_{r,s}(n, y)$ defined by Eq.(6) are solutions of the moment problem [6] but are *not* of Sheffer-type [1],[2].

Referring to various Dirac comb-type distributions obtained by compositions (see Eqs.(44), (46), (47), (64), (69), (70) and (71)) we observe that the substitution $B(n) \to B(\alpha n^2 + \beta n + \gamma)$, ($\alpha, \beta, \gamma$ - integers, $\alpha > 0$) gives sequences $\tilde{B}(n) = B(\alpha n^2 + \beta n + \gamma)$ which are the n-th moments of continuous measures; they are infinite, weighted sums of log-normal distributions [3].

The reproducing nature of Dobiński-type relations under composition also follows from the scheme presented here. It has already provided a number of new closed-form expressions for combinatorial numbers, Eqs.(55),(56),(60),(64),(70) and (71), together with the associated weight functions. It seems that this method can be also applied to various generalizations of combinatorial numbers, e.g. q-deformations [30] and to more involved substitution schemes such as those considered in [31].

Acknowledgments

We thank D. Barsky, M. M. Mendez and C. Quesne for important discussions.

References

[1] Blasiak P, Penson K A and Solomon A I 2003 The general boson normal ordering problem *Phys. Lett.* A **309** 198
[2] Blasiak P, Penson K A and Solomon A I 2003 The boson normal ordering problem and generalized Bell numbers *Ann. Comb.* **7** 127
[3] Blasiak P, Penson K A and Solomon A 2003 Dobiński-type relations and the log-normal distribution, *J. Phys. A: Math. Gen.* **36** L273
[4] Blasiak P, Penson K A and Solomon A I 2003 Combinatorial coherent states via normal ordering of bosons *Lett. Math. Phys.*, in press, *Preprint* arXiv:quant-ph/0311033
[5] Penson K A and Solomon A I 2002 Coherent states from combinatorial sequences *Proc. 2nd Internat. Symp. on Quantum Theory and Symmetries (Cracow, Poland) July 2001* eds E Kapuścik and A Horzela (Singapore: World Scientific) p 527, *Preprint* arXiv:quant-ph/0111151
[6] Penson K A and Solomon A I 2003 Coherent state measures and the extended Dobiński relations In: *Symmetry and Structural Properties of Condensed Matter: Proc. 7th Int. School of Theoretical Physics (Myczkowce, Poland) September 2002* eds Lulek T, Lulek B and Wal A (Singapore: World Scientific) p 64 *Preprint* arXiv:quant-ph/0211061
[7] Katriel J 1974 Combinatorial aspects of boson algebra *Lett. Nuovo Cim.* **10** 565
[8] Katriel J and Duchamp G 1995 Ordering relations for q-boson operators, continued fractions techniques and the q-CBH enigma *J. Phys. A: Math. Gen.* **28** 7209
[9] Wilf H S 1994 *Generatingfunctionology* (New York: Academic Press)
[10] Lang W 2000 On generalizations of the Stirling number triangles *J. Int. Seqs.* Article 00.2.4, available electronically at: http://www.research.att.com/~njas/sequences/JIS/
[11] Pitman J 1997 Some probabilistic aspects of set partitions *Amer. Math. Monthly* **104** 201
[12] Constantine G M and Savits T H 1994 A stochastic process interpretation of partition identities *SIAM J. Discrete Math.* **7** 194; Constantine G M 1999 Identities over set partitions *Discrete Math.* **204** 155
[13] Blasiak P, Duchamp G, Horzela A, Penson K A and Solomon A I 2004 Normal ordering of bosons - combinatorial interpretation, in preparation
[14] Mendez M M, Penson K A, Blasiak P and Solomon A I 2004 A combinatorial approach to generalized Bell and Stirling numbers, in preparation
[15] Klauder J R and Skagerstam B-S 1985 *Coherent States; Applications in Physics and Mathematical Physics* (Singapore: World Scientific)
[16] Luque J-G and Thibon J-Y 2003 Hankel hyperdeterminants and Selberg integrals *J. Phys. A: Math. Gen.* **36** 5267
[17] Shapiro L W, Getu S, Woan W J and Woodson L 1991 The Riordan group *Discrete Appl. Math.* **34** 229
[18] Zhao X and Wang T 2003 Some identities related to reciprocal functions *Discrete Math.* **265** 323
[19] Stanley R P 1999 *Enumerative Combinatorics* vol 2 (Cambridge: University Press)
[20] Aldrovandi R 2001 *Special Matrices of Mathematical Physics* (Singapore: World Scientific)
[21] Flajolet P and Sedgewick R 2003 *Analytic Combinatorics - Symbolic Combinatorics*, *Preprint* http://algo.inria.fr/flajolet/Publications/books.html
[22] Roman S 1984 *The Umbral Calculus* (New York: Academic Press)
[23] Bernstein M and Sloane N J A 1995 Some canonical sequences of integers *Linear Algebra Appl.* **226-228** 57
[24] Morse P M and Feshbach H 1953 *Methods of Theoretical Physics* (New York: McGraw Hill)
[25] Krasnov M, Kissélev A and Makarenko G 1976 *Equations Intégrales* (Moscow: Editions Mir)

[26] Sixdeniers J-M, Penson K A and Solomon A I 1999 Mittag-Leffler coherent states *J. Phys. A: Math. Gen.* **32** 7543; Klauder J R, Penson K A and Sixdeniers J-M 2001 Constructing coherent states through solutions of Stieltjes and Hausdorff moment problems *Phys. Rev. A* **64** 013817

[27] Quesne C 2001 Generalized coherent states associated with the C_λ-extended oscillator *Ann. Phys. (N.Y.)* **293** 147; Quesne C 2002 New q-deformed coherent states with an explicitly known resolution of unity *J. Phys. A: Math. Gen.* **35** 9213; Popov D 2002 Photon-added Barut-Girardello coherent states of the pseudoharmonic oscillator *J. Phys. A: Math. Gen.* **35** 7205; Quesne C, Penson K A and Tkachuk V M 2003 Math-type q-deformed coherent states for $q > 1$ *Phys. Lett.* A **313** 29

[28] Sloane N J A 2003 *Encyclopedia of Integer Sequences*, available electronically at http://www.research.att.com/~njas/sequences

[29] Prudnikov A P, Brychkov Y A and Marichev O I 1986 *Integrals and Series: Special Functions* vol 2 (Amsterdam: Gordon and Breach)

[30] Schork M 2003 On the combinatorics of normal ordering bosonic operators and deformations of it *J. Phys. A: Math. Gen.* **36** 4651

[31] Sloane N J A and Wieder T 2003 The number of hierarchical orderings *Preprint* arXiv:math.CO/0307064