ONE-PARAMETER GROUPS AND COMBINATORIAL PHYSICS

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In this communication, we focus on operators of the type

$$\Omega = \sum_{\alpha+\beta=e+1} c_{\alpha,\beta}(a^+)^\alpha a(a^+)^\beta,$$

the integration of the one-parameter groups $e^{\lambda\Omega}$ and their combinatorial byproducts. In particular we show how these groups can be represented as groups of substitutions with prefunctions.

1. Introduction

This text is the continuation of a series of works overs the combinatorial and analytic aspects of normal forms of Boson strings and combination of these [1,2,3,4,5,11,13,14,15,18,19,20].
Let $w \in \{a, a^+\}^*$ be a word in the letters $\{a, a^+\}$ (i.e. a boson string), and define (as in Blasiak, Penson and Solomon[1,2,3,4]) by r, s and e, respectively $|w|_{a^+}$ (the number of creations), $|w|_a$ (the number of annihilations) and $r - s$ (the excess), then the normal form of w^n reads

$$\mathcal{N}(w^n) = (a^+)^{ne}\left(\sum_{k=0}^{\infty} S_w(n,k)(a^+)^k a^k\right) \tag{1}$$

when e is positive (i.e. there is more creations than annihilations).
In the opposite case (i.e. there is more annihilations than creations) the normal form of w^n is

$$\mathcal{N}(w^n) = \left(\sum_{k=0}^{\infty} S_w(n,k)(a^+)^k a^k\right)(a)^{n|e|} \tag{2}$$

in each case, the coefficients S_w are well defined by the corresponding equation (1 and 2).
Now, for any boson string u one has

$$\mathcal{N}(u) = (a^+)^{|u|_{a^+}} a^{|u|_a} + \sum_{|v|<|u|} \lambda_v v. \tag{3}$$

It has been observed [12,20] that the numbers λ_v are indeed rook numbers.

Let us give, as examples, the upper-left corner of these (doubly infinite) matrices.

For $w = a^+a$, one gets the usual matrix of Stirling numbers of the second kind.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 7 & 6 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 15 & 25 & 10 & 1 & 0 & \cdots \\ 0 & 1 & 31 & 90 & 65 & 15 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{4}$$

For $w = a^+aa^+$, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 4 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 6 & 18 & 9 & 1 & 0 & 0 & 0 & \cdots \\ 24 & 96 & 72 & 16 & 1 & 0 & 0 & \cdots \\ 120 & 600 & 600 & 200 & 25 & 1 & 0 & \cdots \\ 720 & 4320 & 5400 & 2400 & 450 & 36 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{5}$$

For $w = a^+aaa^+a^+$, one gets

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 12 & 60 & 54 & 14 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 144 & 1296 & 2232 & 1296 & 306 & 30 & 1 & 0 & 0 & \cdots \\ 2880 & 40320 & 109440 & 105120 & 45000 & 9504 & 1016 & 52 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{6}$$

Remark 1.1. In each case, the matrix S_w has a staircase form and the "step" depends of the number of a's in the word w. More precisely, due to equation (3) one can prove that the ones ending each row are have (n, nr) as addresses (where $r = |w|_a$). Thus all the matrices are row finite and unitriangular iff $r = 1$, which case will be of special interest in the following. Moreover, the first column is $(1, 0, 0 \cdots, 0, \cdots, 0, \cdots)$ iff w ends with an a (this means that $\mathcal{N}(w^n)$ is free of constant for all $n > 0$).

In this communication, we focus on bonson strings and more generally (homogeneous) boson operators involving only one "a". We will see that

this case is strongly connected with the one-parameter groups of substitutions and their conjugates.
The structure of the paper is the following.
In section 2 we define the framework for our matrices of transformation (spaces, topology and decomposition), then we concentrate on the Riordan subgroup (i.e. transformations which are substitutions with prefunctions) and adaapt the classical theory (Sheffer condition) to our needs. In section 3 we analyse unipotent transformations (Lie group structure and combinatorial examples). But the divisibility property of the group of unipotent transformations tells us that every transformation is embedded in a one-paramater group. This will be analysed in section 4 from the formal and analytic points of view. Section 5 is devoted to some concluding remarks and further interesting possibilities.

2. The algebra $\mathcal{L}(\mathbf{C}^{\mathbf{N}})$ of sequence transformations

Let $\mathbf{C}^{\mathbf{N}}$ be the vector space of all complex sequences, endowed with the Frechet product topology. It is easy to check that the algebra $\mathcal{L}(\mathbf{C}^{\mathbf{N}})$ of all continuous operators $\mathbf{C}^{\mathbf{N}} \to \mathbf{C}^{\mathbf{N}}$ is the space of *row-finite* matrices with complex coefficients. Such a matrix M is indexed by $\mathbf{N} \times \mathbf{N}$ and has the property that, for every fixed row index n, the sequence the sequence $(M(n,k))_{k\geq 0}$ has finite support. For a sequence $A = (a_n)_{n\geq 0}$, the transformed sequence $B = MA$ is given by $B = (b_n)_{n\geq 0}$ with

$$b_n = \sum_{k\geq 0} M(n,k) a_k \tag{7}$$

Remark that the combinatorial coefficients S_w defined above are indeed row-finite matrices.

To a sequence $(a_n)_{n\in\mathbf{N}}$ can be associated (univariate) series. It's generating series, formal or not, with a sequence of prescribed (non-zero) denominators $(d_n)_{n\in\mathbf{N}}$ is

$$\sum_{n\geq 0} a_n \frac{z^n}{d_n}. \tag{8}$$

For example, with $d_n = 1$, we get the ordinary generating functions (OGF), with $d_n = n!$, we get the exponential generating functions (EGF) and with $d_n = (n!)^2$, the doubly exponential generating functions (DEGF) and so on. Thus, once the denominators have been choosen, to every (linear continuous) transformation of generating functions, one can associate it's matrix.

The algebra $\mathcal{L}(\mathbf{C}^{\mathbf{N}})$ possesses many interesting subalgebras and groups as the algebra of lower triangular transformations $\mathcal{T}(\mathbf{N},\mathbf{C})$, the group of inversible elements of the latter $\mathcal{T}_{inv}(\mathbf{N},\mathbf{C})$ (which is the set of infinite lower triangular matrices with non-zero elements on the diagonal), the subgroup of unipotent transformations $\mathcal{UT}(\mathbf{N},\mathbf{C})$ (i.e. the set of infinite lower triangular matrices with elements on the diagonal alla equal to 1) and it's Lie algebra $\mathcal{NT}(\mathbf{N},\mathbf{C})$, the algebra of locally nilpotent transformations (with zeroes on the diagonal). One has the inclusions (with $\mathcal{D}_{inv}(\mathbf{N},\mathbf{C})$, the set of invertible diagonal matrices)

$$\begin{gathered}\mathcal{UT}(\mathbf{N},\mathbf{C}) \subset \mathcal{T}_{inv}(\mathbf{N},\mathbf{C}) \subset \mathcal{T}(\mathbf{N},\mathbf{C}) \subset \mathcal{L}(\mathbf{C}^{\mathbf{N}})\\ \mathcal{D}_{inv}(\mathbf{N},\mathbf{C}) \subset \mathcal{T}_{inv}(\mathbf{N},\mathbf{C}) \text{ and } \mathcal{NT}(\mathbf{N},\mathbf{C}) \subset \mathcal{L}(\mathbf{C}^{\mathbf{N}}).\end{gathered} \tag{9}$$

We can remark that $\mathcal{T}_{inv}(\mathbf{N},\mathbf{C}) = \mathcal{D}_{inv}(\mathbf{N},\mathbf{C}) \bowtie \mathcal{UT}(\mathbf{N},\mathbf{C})$ because $\mathcal{UT}$ is normalized by $\mathcal{D}_{inv}$ and $\mathcal{T}_{inv} = \mathcal{D}_{inv}.\mathcal{UT}$ (every invertible transformation is the product of it's diagonal by a unipotent transformation).

We will examine now an important class of transformations of $\mathcal{T}$ as well as their diagonals: the substitutions with prefunctions.

2.1. *Substitutions with prefunctions*

Let $(d_n)_{n\geq 0}$ bet a fixed set of denominators. We consider, for a generating function f, the transformation

$$\Phi_{g,\phi}[f](x) = g(x)f(\phi(x)) \tag{10}$$

the matrix of this transformation $M_{g,\phi}$ is given by the transforms of the monomials $\frac{x^k}{d_k}$ hence

$$\sum_{n\geq 0} M_{g,\phi}(n,k)\frac{x^n}{d_n} = \Phi_{g,\phi}\left[\frac{x^k}{d_k}\right] = g(x)\frac{\phi(x)^k}{d_k} \tag{11}$$

If $g,\phi \neq 0$ (otherwise the transformation is trivial), we can write

$$g(x) = a_l\frac{x^l}{d_l} + \sum_{r>l} a_r\frac{x^r}{d_r},\ \ \phi(x) = \alpha_m\frac{x^m}{d_m} + \sum_{s>m}\alpha_s\frac{x^s}{d_s} \tag{12}$$

with $a_l, \alpha_m \neq 0$ and then, by (11,12)

$$\Phi_{g,\phi}\left[\frac{x^k}{d_k}\right] = a_l(\alpha_m)^k\frac{x^{l+mk}}{d_l d_m^k d_k} + \sum_{t>l+mk} b_t\frac{x^t}{d_t} \tag{13}$$

one has, then

$$M_{g,\phi} \text{ is row} - \text{finite} \Longleftrightarrow \phi \text{ has no constant term} \tag{14}$$

and, in this case, it is always lower triangular.

The converse is true in the following sense. Let $T \in \mathcal{L}(\mathbf{C}^{\mathbf{N}})$ be a matrix with non-zero two first columns and suppose that the first index n such that $T(n,k) \neq 0$ is less for $k = 0$ than $k = 1$ (which is, from (11) the case when $T = M_{g,\phi}$). Set

$$g(x) := d_0 \sum_{n\geq 0} T(n,0)\frac{x^n}{d_n};\ \phi(x) := \frac{d_1}{g(x)} \sum_{n\geq 0} T(n,1)\frac{x^n}{d_n} \tag{15}$$

then $T = M_{g,\phi}$ iff

$$\sum_{n\geq 0} T(n,k)\frac{x^n}{d_n} = g(x)\frac{\phi(x)^k}{d_k}. \tag{16}$$

Remark 2.1. Eq. (11) is called Sheffer condition (see [16,19,21,23]) and, for EGF ($d_n = n!$) it amounts to state

$$\sum_{n,k\geq 0} T(n,k)\frac{x^n}{n!}y^k = g(x)e^{y\phi(x)} \tag{17}$$

From now on, we will suppose that ϕ has no constant term ($\alpha_0 = 0$). Moreover $M_{g,\phi} \in \mathcal{T}_{inv}$ if and only if f $a_0, \alpha_1 \neq 0$ and then the diagonal term with address (n,n) is $\frac{a_0}{d_0}\left(\frac{\alpha_1}{d_1}\right)^n$. We get

$$M_{g,\phi} \in \mathcal{UT} \Longleftrightarrow \frac{a_0}{d_0} = \frac{\alpha_1}{d_1} = 1 \tag{18}$$

In particular for the EGF and the OGF, we have the condition that

$$g(x) = 1 + \text{higher terms } \textit{and } \phi(x) = x + \text{higher terms} \tag{19}$$

Note 2.1. In classical combinatorics (and then for OGF and EGF), the matrices $M_{g,\phi}(n,k)$ are known under the name of *Riordan matrices* (see [16,17] for example).

3. Unipotent transformations

3.1. *Lie group structure*

We first remark that $n \times n$ truncations (i.e. the fact of taking the $[0..n] \times [0..n]$ submatrix of a matrix) are algebra morphisms

$$\tau_n : \mathcal{T}(\mathbf{N},\mathbf{C}) \to \mathcal{M}([0..n] \times [0..n], \mathbf{C}) \tag{20}$$

we can endow $\mathcal{T}(\mathbf{N},\mathbf{C})$ with the Frechet topology given by these morphisms. We will not develop this point in details here, but this topology is metrisable and given by the following convergence criterium :

a sequence (M_k) of matrices in $\mathcal{T}(\mathbf{N},\mathbf{C})$ converges iff
for all fixed $n \in \mathbf{N}$
the sequence of truncated matrices $(\tau_n(M_k))$ converges.

This topology is compatible with the structure of $\mathbf{C}$-algebra of $\mathcal{T}(\mathbf{N},\mathbf{C})$.

The two maps $\exp : \mathcal{NT}(\mathbf{N},\mathbf{C}) \to \mathcal{UT}(\mathbf{N},\mathbf{C})$ and $\log : \mathcal{UT}(\mathbf{N},\mathbf{C}) \to \mathcal{NT}(\mathbf{N},\mathbf{C})$ are continous and mutually inverse.

3.2. *Examples*

3.2.1. *Provided by the exponential formula*

The "classical exponential formula" [7,?,21] says us that, for a class [a] of finite labelled graphs $\mathcal{C}$, denoting $\mathcal{C}^c$, the subclass of connected graphs in $\mathcal{C}$, the exponential generating series of $\mathcal{C}$,we have

$$EGF(\mathcal{C}) = e^{EGF(\mathcal{C}^c)}. \tag{21}$$

The following examples gives us a taste of why combinatorial matrices of the type:

$T(n,k) =$ *Number of grahs of $\mathcal{C}$ on n vertices having k connected components*

give us substitution transformations.

Example 3.1. Stirling numbers.
We here use the class of graphs of equivalence relations. Then using the statistics $x^{(\text{number of points})}y^{(\text{number of connected components})}$ we get

$$\sum_{n,k\geq 0} S(n,k)\frac{x^n}{n!}y^k =$$

$$\sum_{\text{all equivalence graphs }\Gamma} \frac{x^{(\text{number of points of }\Gamma)}}{(\text{number of points of }\Gamma)!} y^{(\text{number of connected components of }\Gamma)} =$$

$$\exp\left(\sum_{\Gamma\ \mathit{connected}} \frac{x^{(\text{number of vertices of }\Gamma)}}{(\text{number of points of }\Gamma)!} y^{(\text{number of connected components of }\Gamma)}\right) =$$

[a] Closed under relabelling (of the vertices), disjoint union, and taking connected components.

$$\exp\left(\sum_{n\geq 1} y\frac{x^n}{n!}\right) = e^{y(e^x-1)} \tag{22}$$

we will see that the transformation associated with the matrix $S(n,k)$ is
$f \to f(e^x - 1)$.

Example 3.2. Idempotent numbers.
We consider the graphs of endofunctions[b]. Then, using the statistics $x^{(\text{number of points of the set})}y^{(\text{number of connected components of the graph})}$ and denoting $I(n,k)$ the number of endofunctions of a given set with n elements having k connected components, we get

$$\sum_{n,k\geq 0} I(n,k)\frac{x^n}{n!}y^k =$$
$$\sum_{\text{all graphs of endofunctions }\Gamma} \frac{x^{(\text{number of vertices of }\Gamma)}}{(\text{number of vertices of }\Gamma)!} y^{(\text{number of connected components of }\Gamma)} =$$
$$\exp\left(\sum_{\Gamma\ \mathit{connected}} \frac{x^{(\text{number of vertices of }\Gamma)}}{(\text{number of vertices of }\Gamma)!} y^{(\text{number of connected components of }\Gamma)}\right) =$$
$$\exp\left(\sum_{n\geq 1} y\frac{nx^n}{n!}\right) = e^{yxe^x} \tag{23}$$

for these numbers, we get the (doubly) infinite matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 3 & 6 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 4 & 24 & 12 & 1 & 0 & 0 & \cdots \\ 0 & 5 & 80 & 90 & 20 & 1 & 0 & \cdots \\ 0 & 6 & 240 & 540 & 240 & 30 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{24}$$

we will see that the transformation associated with this matrix is $f \to f(xe^x)$

[b]Functions from a finite set into itself.

3.2.2. *Provided by normal ordering powers of boson strings*

To get unipotent matrices, one has to consider boson strings with only one annihilation operator. In the introduction, we have given examples with a^+a, a^+aa^+ (the matrix of the third string, with two derivations, $a^+aaa^+a^+$ is not unipotent). Such a string then reads $w = (a^+)^{r-p}a(a^+)^p$ and we will see in a moment that

- if $p = 0$, $S_w(n,k)$ is the matrix of a unipotent substitution
- if $p > 0$, $S_w(n,k)$ is the matrix of a unipotent substitution with prefunction

To cope with the matrices coming from the normal ordering of powers of boson strings we have to do a small excursion to analysis and formal groups.

4. One-parameter subgroups of $UT(\mathbf{N},\mathbf{C})$

4.1. *Exponential of elements of $NT(\mathbf{N},\mathbf{C})$*

Let $M = I + N \in UT(\mathbf{N},\mathbf{C})$ ($I = I_{\mathbf{N}}$ is the indentity matrix). One has

$$M^t = \sum_{k \geq 0} \binom{t}{k} N^k \tag{25}$$

where $\binom{t}{k}$ is the generalized binomial coefficient defined by

$$\binom{t}{k} = \frac{t(t-1)\cdots(t-k+1)}{k!} \tag{26}$$

one can see that, for $k \leq n$, due to the local nilpotency of N, the matrix coefficient $M^t(n,k)$ is well defined and, in fact, a polynomial of degree $n-k$ in t (for $k > n$, this coefficient is 0). We have the additive property $M^{t_1+t_2} = M^{t_1}M^{t_2}$ and the correspondence $t \to M^t$ is continuous. Conversely, let $t \to M_t$ is a continous local one-parameter group in $UT(\mathbf{N},\mathbf{C})$ that is, for some real $\epsilon > 0$

$$|t_1| \text{ and } |t_2| < \epsilon \Longrightarrow M_{t_1}M_{t_2} = M_{t_1+t_2} \tag{27}$$

then there exists a unique matrix $H \in NT(\mathbf{N},\mathbf{C})$ such that $M_t = exp(tH)$ (one can prove it using the projections τ_n and the theorem about continous one-parameter groups in Lie groups, see [10], for example).

In case $M_t = M^t$ is defined by formula (25) we have
$H = log(I+N) = \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} N^k$.

The mapping $t \to M^t$ will be called a *one parameter group* of $UT(\mathbf{N}, \mathbf{C})$.

Proposition 4.1. *Let M be the matrix of a substitution with prefunction, so is M^t for all $t \in \mathbf{C}$.*

The proof will be detailed in a forthcoming paper and uses the fact that "to be the matrix of a substitution with prefunction" is a property of polynomial type. But, using composition, it is straightforward that M^t is the matrix of a substitution with prefunction for all $t \in \mathbf{N}$. Thus, using a "Zariski-type" argument, we get the fact that the property is true for all $t \in \mathbf{C}$.

4.2. *Link with local Lie groups : Straightening vector fields on the line*

Let us treat first the case of $p = 0$. The string $(a^+)^r a$ corrresponds, in the Bargmann-Fock representation, to the vector field $x^r \frac{d}{dx}$ defined on the whole line.
Now, we can try (at least locally) to straighten this vector field by a diffeomorphism u to get the constant vector field (this procedure has been introduced by G. Goldin in the context of algebras of currents [8]). As the one-parameter group generated by a constant field is the shift, the one-parameter (local) group of transformations should read, on a suited domain

$$U_\lambda[f](x) = f\left(u^{-1}\left(u(x) + \lambda\right)\right) \tag{28}$$

Now, we know from section (4.1) that, if two one-parameter groups have the same tangent vector at the origin, then they coincide *(tangent paradigm)*. Direct computation gives this tangent vector :

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} f\left(u^{-1}\left(u(x) + \lambda\right)\right) = \frac{1}{u'(x)} f'(x) \tag{29}$$

so the local one-parameter group U_λ has $\frac{1}{u'(x)} \frac{d}{dx}$ as tangent vector field.
Here, we have to solve $\frac{1}{u'(x)} = x^r$ in order to get the diffeomorphism u.
In the case $r \neq 1$, we have (with $\mathcal{D} =]0, +\infty[$ as domain

$$u(x) = \frac{x^{1-r}}{1-r} = y;\ u^{-1}(y) = ((1-r)y)^{\frac{1}{1-r}} \tag{30}$$

and

$$e^{\lambda x^r \frac{d}{dx}}[f](x) = f\left(\frac{x}{(1-\lambda(r-1)x^{r-1})^{\frac{1}{r-1}}}\right) \tag{31}$$

The substitution factor $s_\lambda(x) = \frac{x}{(1-\lambda(r-1)x^{r-1})^{\frac{1}{r-1}}}$ has been already obtained by other means in [1]. The computation is similar for the case when $r = 1$ and, for this case, we get

$$e^{\lambda x \frac{d}{dx}}[f](x) = f\left(e^{\lambda}x\right) \tag{32}$$

with $s_\lambda(x) = e^{\lambda}x$ as substitution factor.
The first examples are summarized in the following table

$r =$	$s_\lambda(x) =$	Name
0	$x + \lambda$	Shift
1	$e^{\lambda}x$	Dilation
2	$\frac{x}{1-\lambda x}$	Homography
3	$\frac{x}{\sqrt{1-2\lambda x^2}}$	-

Comment 4.1. If one uses classical analysis (i.e. convergent Taylor series), one must be careful about the domain where the substitutions are defined and the one-parameter groups are defined only locally.
On each of these examples, one can check by hand that, for suitable (and small) values of λ, μ, one has $s_\lambda(s_\mu(x)) = s_{\lambda+\mu}(x)$ (one-parameter group property).
It is possible to get rid of the discussion over the domains by considering λ, μ as new variables and applying the "substitution principle" saying that it is possible to substitute a series without constant term in a series (in the algebra $\mathbf{C}[[x, \lambda, \mu]]$).

Using the same method, one can start wiith more complicated operators. Examples and substitution factors are given below

Operator	Substitution Factor	Description
$\left(1+(a^+)^2\right)a$	$s_\lambda(x) = \dfrac{xcos(\lambda)+sin(\lambda)}{cos(\lambda)-xsin(\lambda)}$	One-parameter group of homographies
$\dfrac{\sqrt{1+(a^+)^2}}{a^+}$	$s_\lambda(x) = \sqrt{x^2+2\lambda\sqrt{1+x^2}+\lambda^2}$	Composition of quadratic direct and inverse functions

4.3. *Case $p > 0$: another conjugacy trick and a shocking formula*

Now, seing vector fields as infinitesimal generators of one-parameter groups, leads to conjugacy as, if U_λ is a one-parameter group of transformation, so is $VU_\lambda V^{-1}$ (V being a continous invertible operator). The case $(a^+)^{r-p}a(a^+)^p;\ p>0$ belongs to this setting as $(a^+)^{-p}\left((a^+)^r a\right)(a^+)^p$. More generally, supposing all the terms defined, with

$$\Omega = u_1(x)\frac{d}{dx}u_2(x) = \frac{1}{u_2(x)}\left(u_1(x)u_2(x)\frac{d}{dx}\right)u_2(x)$$

one has

$$e^{\lambda\Omega} = \frac{1}{u_2(x)}\left(e^{\lambda u_1(x)u_2(x)\frac{d}{dx}}\right)u_2(x) \tag{33}$$

This shocking formula (33) may be understood as an operator equality. Now, the tangent paradigm (see section 4.2) tels us that, if we adjust this tangent vector to coincide with $x^{r-p}\frac{d}{dx}x^p$ (recall that the original problem was the integration of the operator $\Omega = (a^+)^{r-p}a(a^+)^p;\ p>0$), then we get the right one-parameter group. Using this "conjugacy trick" we get

$$e^{\lambda\Omega}[f](x) = \left(\frac{s_\lambda(x)}{x}\right)f(s_\lambda(x)) \text{ with } s_\lambda(x) = \frac{x}{(1-\lambda(r-1)x^{r-1})^{\frac{1}{r-1}}} \tag{34}$$

Remark 4.1. (i) It can be checked that, if $s_\lambda(x)$ is a substitution factor (i.e. at least locally $s_\lambda(s_\mu(x)) = s_{\lambda+\mu}(x)$) such that $s_\lambda(0)=0$ for evary λ (which is the case in most of our examples) then the transformations defined by $U_\lambda[f](x) = \left(\frac{s_\lambda(x)}{x}\right)f(s_\lambda(x))$ form a one-parameter (possibly local) group.

(ii) It is also possible to use the "ad" operator (Lie adjoint) instead of "Ad" (conjugacy) to obtain integration formulas (see Dattoli [6]).

4.4. *Characteristic series ↔ one parameter group correspondence*

In fact, what precedes allows us to extend integration process to linear combination of boson strings in the following sense. The algebra $W_{1,\infty}$ generated by $a^+, (a^+)^{-1}, a$ is graded by

$$\text{weight}(a^+) = 1,\ \text{weight}\left((a^+)^{-1}\right) = \text{weight}(a) = -1 \tag{35}$$

and every homogeneous operator of this algebra which is of the form

$$\Omega = \sum_{|w|_a=1,\ \text{weight}(w)=e} \alpha_w w \tag{36}$$

(there is only one derivative in each monomial) can be integrated in the preceding manner. So one would like to reconstruct the characteristic series

$$\sum_{n,k} S_\Omega(n,k)\frac{x^n}{n!}y^k \tag{37}$$

from the knowledge of the one-parameter subgroup $e^{\lambda\Omega}$.
This is the aim of the following paragraph.

For every homogeneous operator as above with $e \geq 0$, one defines the coefficients $S_\Omega(n,k)$ as in the introduction of this text by

$$\mathcal{N}(\Omega^n) = (a^+)^{ne}\sum_{k=0}^{\infty} S_\Omega(n,k)(a^+)^k a^k \tag{38}$$

One has the following proposition

Proposition 4.2. *With the preceding denotations, the following conditions are equivalent:*

$$\sum_{n,k\geq 0} S_\Omega(n,k)\frac{x^n}{n!}y^k = g(x)e^{y\phi(x)} \tag{39}$$

$$U_\lambda[f](x) = g(\lambda x^e) f\left(x\left(1+\phi(\lambda x^e)\right)\right) \tag{40}$$

Which solves the problem.

5. Conclusion and remaining problems

We have considered a class of elements of $W_{1,\infty}$ (see section 4.4 for a definition) which describe some rational vector fields on the line. For these operators, we have established a correspondence

One-parameter group (=integration of the field) $\leftrightarrow$
Characteristic series (=coefficients of the normal ordering)

We have then seen families of combinatorial matrices giving rise through the exponential formula, to substitutions.
Remains to study the vector fields associated with these combinatorial matrices. Also one would desire to adapt this machinery to other algebras ($q$uons, several Bosons).

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