Some new "Taylor BKAR" formulas

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Introduction

Tensorial quantum field theories are a promising way to study random geometries with weights corresponding to some discretization of Einstein-Hilbert action, the ultimate goal being to quantize gravity.

Regularity structures (the Hairer theory) is based on a suitable generalisation of the Taylor formula for distributions.

It is highly regarded in the domain of quantum field theory and renormalisation.

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So we thought that it may be astuce to offer a suitable generalisation of the BKAR formula into a Taylor-BKAR formula.

The BKAR Forest Formula

Let f be a smooth function of $n(n-1)/2$ line variables $x_\ell \in [0,1]$, $\ell = (i,j)$, $1 \le i < j \le n$. The forest formula states:

$$\mathrm{f}(1,...,1) = \sum_{\mathfrak{F}} \Big\{ \prod_{\ell \in \mathfrak{F}} \Big[\int_0^1 dw_\ell \Big] \Big\} \Big\{ \prod_{\ell \in \mathfrak{F}} \frac{\partial}{\partial x_\ell} \mathrm{f} \Big\} \big[X^{\mathfrak{F}}(w_{\mathfrak{F}}) \big], \text{ where}$$

- the sum over $\mathcal{F}$ is over all forests over n vertices,
- the "weakening parameter" $X_{ij}^{\mathfrak{F}}(w_{\mathfrak{F}})$ is 0 if i and j dont belong to the same connected component of $\mathfrak{F}$; otherwise it is the minimum of the $w_{\ell'}$ for ℓ' running over the unique path from i to j in $\mathfrak{F}$.
- Furthermore the real symmetric matrix $X_{ij}^{\mathfrak{F}}(w_{\mathfrak{F}})$ (completed by 1 on the diagonal $i = j$) is positive.

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The BKAR Forest Formula

The set PS_n of positive n by n symmetric matrices with 1 on the diagonal and off-diagonal entries between 0 and 1 is convex.

We order the parameters w: $w_{|\mathfrak{F}|} \leq w_{|\mathfrak{F}|-1} \leq \cdots \leq w_1 \leq w_0 = 1$.
These orderings are also called Hepp sectors in quantum field theory.

$X^{\mathfrak{F}}(w_{\mathfrak{F}}) = \sum_{k=1}^{|\mathfrak{F}|} (w_{k-1} - w_k)\, \Pi_k$, where Π_k is a block matrix.

This convex combination depends on the ordering of the $w's$.

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Classical Constructive Theory

Cluster expansion = Taylor-Lagrange expansion of the functional integral:

$$F = 1 + H,\ \ H = -\lambda \int_0^1 dt \int_{-\infty}^{+\infty} x^4 e^{-\lambda t x^4 - x^2/2} \frac{dx}{\sqrt{2\pi}}$$

Mayer expansion: define $H_i = -\lambda \int_0^1 dt \int_{-\infty}^{+\infty} x_i^4 e^{-\lambda t x_i^4 - x_i^2/2} \frac{dx_i}{\sqrt{2\pi}} = H\ \forall i$, $\varepsilon_{ij} = 0\ \forall i,j$ and write

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Defining $\eta_{ij} = -1$, $\varepsilon_{ij} = 1 + \eta_{ij} = 1 + x_{ij}\eta_{ij}|_{x_{ij}=1}$ and apply the forest formula.

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The logarithm of the forest formula is simply a tree BKAR formula. Then defining $G = \log F$,

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where the sum over trees!

- The convergence is easy because each H_i contains a different "copy" $\int dx_i$ of functional integration, and $|1 + \eta_\ell x_\ell^{\mathcal{F}}(\{w\})| \leq 1$.
- Borel summability now easily follows from the Borel summability of H.
- It generalizes well to case of lattice statistical mechanics ($d > 0$).

However the link with Feynman graphs is somewhat lost, and furthermore it may be not optimal for curved or random space-time geometries.

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Fermionic Rewriting of the Mayer Expansion

We have

$$a = \int d\bar{\chi} d\chi e^{-\bar{\chi} a \chi} = \int d\mu(\bar{\chi}, \chi) e^{-\bar{\chi}(a-1)\chi}$$

hence

$$\prod_{\ell \in K_n} (1 + \eta_\ell) = \int d\mu(\bar{\chi}_\ell, \chi_\ell) e^{-\sum_\ell \bar{\chi}_\ell \eta_\ell \chi_\ell}$$

and we can apply the forest formula to the interaction term $e^{-\sum_\ell \bar{\chi}_\ell \eta_\ell \chi_\ell}$.

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Loop Vertex Expansion

Intermediate field representation

$$
\begin{aligned}
F &= \int_{-\infty}^{+\infty} e^{-\lambda x^4 - x^2/2} \frac{dx}{\sqrt{2\pi}} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-i\sqrt{2\lambda}\sigma x^2 - x^2/2 - \sigma^2/2} \frac{dx}{\sqrt{2\pi}}\frac{d\sigma}{\sqrt{2\pi}} \\
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$$

Let us apply the forest formula, but using **"replicas"** of the intermediate field:

$$V^n(\sigma) \to \prod_{i=1}^{n} V_i(\sigma_i), \quad d\mu(\sigma) \to d\mu_C(\{\sigma_i\}),$$

$C_{ij} = \mathbb{1}_n = x_{ij}|_{x_{ij}=1}$, where $\mathbb{1}_n$ is the $n \times n$ matrix with entries one everywhere.

Loop Vertex Expansion

Intermediate field representation

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Loop Vertex Expansion II

$$F = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{F}} \Big\{ \prod_{\ell \in \mathcal{F}} \Big[\int_0^1 dw_\ell \Big] \Big\} \int \Big\{ \prod_{\ell \in \mathcal{F}} \frac{\partial}{\partial \sigma_{i(\ell)}} \frac{\partial}{\partial \sigma_{j(\ell)}} \prod_{i=1}^{n} V(\sigma_i) \Big\} d\mu_{C^{\mathcal{F}}}$$

where $C^{\mathcal{F}}_{ij} = x^{\mathcal{F}}_{\ell}(\{w\})$ if $i < j$, $C^{\mathcal{F}}_{ii} = 1$.

$$G = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{T}} \Big\{ \prod_{\ell \in \mathcal{T}} \Big[\int_0^1 dw_\ell \Big] \Big\} \int \Big\{ \prod_{\ell \in \mathcal{T}} \frac{\partial}{\partial \sigma_{i(\ell)}} \frac{\partial}{\partial \sigma_{j(\ell)}} \prod_{i=1}^{n} V(\sigma_i) \Big\} d\mu_{C^{\mathcal{T}}}$$

where the second sum runs over trees!

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Advantages

One can picture the result as a sum over trees on loops, or "cacti". Since

$$\frac{\partial^k}{\partial\sigma^k}\log[1+i2\sqrt{2\lambda}\sigma] = -(k-1)!(-i2\sqrt{2\lambda})^k[1+i2\sqrt{2\lambda}\sigma]^{-k},$$

- Convergence is easy because $|[1+i2\sqrt{2\lambda}\sigma]^{-k}| \leq 1$
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- Link with Feynman graphs can be recovered

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A few notations

Scope : to be able to perform a further expansion

Must generalise both the BKAR Forest Formula and Taylor expansion

One still denotes $x_{ij} \in [0, 1]$ the line variables, and consider the expansion of a function of $\frac{n(n-1)}{2}$ variable. For compactness, we now adopt matricial notations, the line variables being in one to one correspondence with symmetric matrices with only ones on their diagonal, so that the point $(1, \dots, 1)$ now reads $\mathbb{1}_n$, the $n \times n$ matrix with entries one everywhere.

Let's finally denote $\mathbf{T}^p_{A \to B}(\mathrm{f})$ the order p Taylor expansion of one such matrix-valued function f between A and B :

$$\mathbf{T}^p_{A_{ij} \to B_{ij}}(\mathrm{f}) = \sum_{i=0}^{p} \frac{1}{i!} \sum_{(j_1 < k_1)\dots(j_i < k_i)} \prod_{l=1}^{i} (B-A)_{j_l, k_l} \partial^i_{(j_1,k_1)\dots(j_i,k_i)} \mathrm{f}(A_{ij})$$

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The formula

The Taylor-BKAR formula reads:

$$\mathrm{f}(\mathbb{1}_n) = \sum_{\substack{\mathfrak{F} \text{ forest} \\ 0 \leq |\mathfrak{F}| \leq n-1}} \int dw_{\mathfrak{F}} \prod_{i=1}^{|\mathfrak{F}|} \left(\frac{(w_{e_{i-1}} - w_{e_i})^p}{p!} \right) \sum_{\substack{\{e^i_j\}_{1 \leq i \leq |\mathfrak{F}|, 1 \leq j \leq p} \\ \forall i,j \ \ \{e^i_j \cup \{e_1, \dots, e_{i-1}\}\} \text{ forest}}} \times$$

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and the edges of the forest are ordered according to the values of the parameters w_ℓ so that $w_{e_{|\mathfrak{F}|}} \leq \dots \leq w_{e_1} \leq w_{e_0} = 1$

These orderings (also called Hepp sectors) crucially depend on the sector of integration.

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Comments

The Taylor-BKAR formula does reduce to the classical BKAR forest formula when $p = 0$ (immediate to check)

It also does reduce to the Taylor-Lagrange expansion with integral reminder for $n = 2$ (also immediate)

The function f and its derivatives are always evaluated at $X^{\mathfrak{F}}(w_{\mathfrak{F}})$ which is positive, hence the formula preserves the positivity of the interaction, and can therefore be trusted to perform Mayer expansion, and both horizontal and vertical cluster expansion

This new formula is plague by a severe drawback : it no longer factors on connected components:

$$f(x_{12} = 1) = \underbrace{\sum_{k=0}^{p} \frac{\partial_{12}^k f(0)}{k!}}_{A(\mathfrak{F}^{\emptyset}=\{\{v_1\},\{v_2\}\})\neq A(\{v_1\})A(\{v_2\})} + \int_0^1 \frac{(1-t)^p}{p!} \partial_{12}^k f(t) dt$$

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$$\mathrm{f}(x_{12} = 1) = \underbrace{\sum_{k=0}^{p} \frac{\partial_{12}^k \mathrm{f}(0)}{k!}}_{A(\mathfrak{F}^{\emptyset}=\{\{v_1\},\{v_2\}\})\neq A(\{v_1\})A(\{v_2\})} + \int_0^1 \frac{(1-t)^p}{p!} \partial_{12}^k \mathrm{f}(t) dt$$

Sketch of the proof I

To interpolate, one introduces an auxiliary matrix defined by:

$$W_{ij}^{\mathfrak{F}}(w_{\mathfrak{F}}, t) = \begin{cases} 0 & \text{si } X_{ij}^{\mathfrak{F}} \neq 0 \\ t & \text{sinon} \end{cases} \tag{4.1}$$

such that

$$\frac{d}{dt}\mathfrak{f}(W_{ij}^{\mathfrak{F}}(w_{\mathfrak{F}}, t)) = \sum_{\substack{e \\ e \cup \mathfrak{F} \text{ forest}}} \partial_e \mathfrak{f}(W_{ij}^{\mathfrak{F}}(w_{\mathfrak{F}}, t))$$

which implies, by Taylor expansion of order p, that:

$$\mathfrak{f}(W^{\mathfrak{F}}(u_{\mathfrak{F}}, t)) = \sum_{k=0}^{p} \frac{t^k}{k!} \sum_{\substack{e_1, \dots, e_k \\ \forall i,\ \mathfrak{F} \cup e_i \text{ forest}}} \partial^k_{e_1, \dots, e_k} \mathfrak{f}(W^{\mathfrak{F}}(u_{\mathfrak{F}}, 0))$$

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Sketch of the proof II

To demonstrate the Taylor-BKAR formula, one applies interatively the previous formula to $f(\mathbb{1}_n) = f(W^{\emptyset}(u_{\emptyset}, 1))$, first noticing that $W^{\emptyset}(u_{\emptyset}, 0) = X^{\emptyset} = \mathrm{Id}_n$ and $W^{\emptyset}(u_{\emptyset}, 1) = Y^{\emptyset} = \mathbb{1}_n$. This firstly yields to:

$$\mathrm{f}(\mathbb{1}_n) = \mathbf{T}^p_{X^{\emptyset}(\emptyset) \longrightarrow Y^{\emptyset}(\emptyset)} \mathrm{f} + \sum_e \int_0^1 dw_e \, \frac{(1-u_e)^p}{p!} \sum_{e_1^1, \ldots, e_p^1} \partial^p_{e_1^1, \ldots, e_p^1} \partial_e \mathrm{f}(W^{\emptyset}(u_{\emptyset}, u_e))$$

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Then, we iterate the Taylor expansions, still using the following interpolation rule:

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Sketch of the proof III

n-1 successive Taylor expansions lead to:

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Grouping together the $|\mathfrak{F}|!$ contributions to the forest $\mathfrak{F}$, the previous formula boils down to the Taylor-BKAR formula. □

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Conclusion and outlooks

We presented here a new formula mixing Taylor expansion and BKAR forest formula.

It could help to construct some non-local quantum field theories of matricial and tensorial type in the just renormalisable case : for instance the Grosse-Wulkenhaar theory, or the T^4_5 theory, that are respectively asymptotically safe and free.

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