$$h(x)=\sum_{n=0}^{\infty}\frac{(6n)!\,n!}{(3n)!\,(2n)!\,(2n)!}\left(\frac{x}{108}\right)^{n}\in\mathbb{Z}\left[\left[\frac{x}{108}\right]\right]\qquad 108=\frac{6^6\cdot 1^1}{3^3\,2^2\cdot 2^2}$$

no known combinatorial interpretation

Coefficient $\in\mathbb{Z} \Leftarrow \forall$ prime p $\operatorname{ord}_p(\text{Coeff.})\geq 0.$

$$h(x)\in\overline{\mathbb{Q}(x)}\qquad h(x)=\frac{1}{\sqrt{1-x}}\left(\left(1-2x+2\sqrt{x(x-1)}\right)^{+\frac{1}{3}}+\left(1-2x+2\sqrt{x(x-1)}\right)^{-\frac{1}{3}}\right)$$

imaginary for $x\in(0,1)$ but the sum is real.

Karol Penson: moments of a probability density

$z\in(0,1)$:
$$W(z)=\frac{1}{2\pi}\frac{1}{\sqrt{1-z}}\left(\frac{\left(1+\sqrt{1-z}\right)^{+\frac{2}{3}}}{z^{5/6}}+\frac{\left(1+\sqrt{1-z}\right)^{-\frac{2}{3}}}{z^{1/6}}\right)$$

$$\boxed{\int_0^1\frac{W(z)}{1-xz}\,dz=h(z)}$$

$Y(z):=z(1-z)\,W(z)^2$

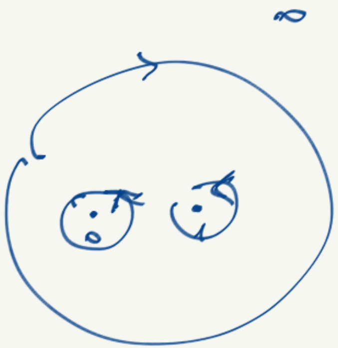
$\rightsquigarrow\ Y\cdot(Y-3)^2z^2=4(z-2)^2$

assume all α_j, β_i are distinct mod $\mathbb{Z}$

Hypergeometric equation:

$\alpha_1 \ldots \alpha_N, \beta_1 \ldots \beta_N \in \mathbb{C}$ (later $\alpha_1 \ldots \alpha_N, \beta_1, \ldots \beta_N \in \mathbb{Q}$)

$T_0 T_1 T_\infty = \mathrm{id}$



$$\left[\prod_{i=1}^{N}\left(z\frac{d}{dz} - \beta_i\right) - \prod_{j=1}^{N}\left(z\frac{d}{dz} - \alpha_j\right)\cdot z\right]\psi = 0$$

Singularities: $z = 0, 1, \infty$

Monodromy

0: eigenvalues $b_i = \exp(2\pi\sqrt{-1}\,\beta_i) \in \mathbb{C}^\times$

∞: $a_j = \exp(-2\pi\sqrt{-1}\,\alpha_j) \in \mathbb{C}^\times$

1: quasi-reflection $\begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots 1 & \\ 0 & & & & c \end{pmatrix}$ $\quad c = \exp\left(2\pi\sqrt{-1}\left(\sum\alpha_j - \sum\beta_i\right)\right)$

Basis of solutions at $z=0$, $1 \le i_0 \le N$:

$$\psi_{i_0}(z) := z^{\beta_{i_0}} \cdot \sum_{n \geq 0} \frac{\prod_j \Gamma(1+n+\beta_{i_0}-\alpha_j)}{\prod_{i \neq i_0} \Gamma(1+n+\beta_{i_0}-\beta_i) \cdot \Gamma(1+n)} z^n$$

$$= \frac{\prod_j \Gamma(1+\beta_{i_0}-\alpha_j)}{\prod_{i \neq i_0} \Gamma(1+\beta_{i_0}-\beta_i)} \cdot z^{\beta_{i_0}} \cdot {}_N\Gamma_{N-1}\left(\beta_{i_0}-\alpha_1+1, \dots, \beta_{i_0}-\alpha_N ; \beta_{i_0}-\beta_1+1, \dots, \widehat{i_0}, \dots, \beta_{i_0}-\beta_N+1\right)(z)$$

Unique (up to scalar) solution at $z = 1-\varepsilon$ $\quad 0<\varepsilon \ll 1$

purely ramified branch :

$$W(z) = \sum_{i_0=1}^{N} \frac{\prod_j \sin(\pi(\alpha_j - \beta_{i_0}))}{\prod_{i \neq i_0} \sin(\pi(\beta_i - \beta_{i_0}))} \psi_{i_0}(z)$$

Claim:

$$\int_0^1 W(z) \cdot z^s \frac{dz}{z} = \frac{\prod_i \Gamma(\beta_i + s)}{\prod_j \Gamma(\alpha_j + s)} \cdot \pi$$

Mellin transform $MW(s)$

Equation $\left(\prod_1^N \left(\frac{z\partial}{\partial t} - \beta_i\right) - \prod \left(\frac{z\partial}{\partial t} - \alpha_j\right) \cdot z\right) \psi = 0$

$$\frac{M\psi(s+1)}{M\psi(s)} = \frac{\prod (s+\beta_i)}{\prod (s+\alpha_j)}$$

$\Leftrightarrow \quad \prod(-s-\beta_i)\, M\psi(s) = \prod(-s-\alpha_j)\, M\psi(s+1)$

$W(s)$ is Meijer's G-function: (1936).

in general 4 groups of parameters

$$\text{Mellin transform} = \frac{\prod \Gamma(s+\dots)\ \prod \Gamma(-s+\dots)}{\prod \Gamma(s+\dots)\ \prod \Gamma(-s+\dots)}$$

F. Beukers, G. Heckman "Monodromy for hypergeometric functions ${}_nF_{n-1}$"
Inv. Math., 95, 325-354 (1989).

Full list of algebraic ones:

Thm. for $\alpha_1 \dots \alpha_N, \beta_1 \dots \beta_N \in \mathbb{Q}$, all $\alpha_i \bmod \mathbb{Z}$, $\beta_i \bmod \mathbb{Z}$ are distinct in $\mathbb{Q}/\mathbb{Z}$

All are equivalent:

next page

1) monodromy of hypergeometric equation is finite

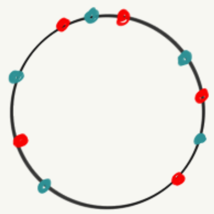
$\Downarrow$ 2) all hypergeom. series are algebraic

$\Downarrow$ 3) one of hypergeom. series is algebraic

$\Downarrow$ 4) $\exists\, C$ $\forall$ prime $p > C$ ord_p(all coeff. of one hyperg. solutions) ≥ 0.

$\Downarrow$ 5) $\exists\, C$ $\forall$ prime $p > C$ $\exp(2\pi i\, p\alpha_i)$, $\exp(2\pi i\, \beta_i)$ are interlacing

call it "universally interlacing"

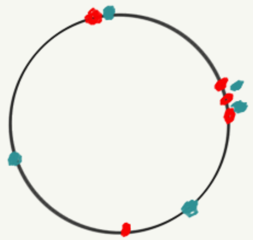


Main Lemma:

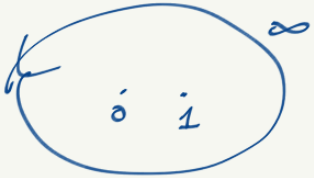
$$\left\{ (T_0, T_\infty^{-1}) \;\middle|\; \begin{array}{l} T_0, T_\infty^{-1} \in U(N) \\ \mathrm{rk}\,(T_0 - T_\infty^{-1}) \le 1 \\ \Leftrightarrow \mathrm{rk}\,(T_0 T_\infty - \mathrm{Id}_N) \le 1 \end{array} \right\} \Big/ \begin{array}{c} \text{conjugation} \\ T_i \to S T_i S^{-1} \\ S \in U(N) \end{array}$$

$\overset{1:1}{\Longleftrightarrow}$ two non-strictly interlacing N-element multisets $\in U(1)$

$\mathrm{Spec}\, T_0, \ \mathrm{Spec}\, T_\infty^{-1}$



$G =$ Finite subgroup generated by $T_0 =$ 

, $T_\infty^{-1} =$ 

$\supset$ Normal subgroup G' generated by $T_1^{-1} = T_\infty T_0$ 

and its conjugates.

$G' \subset U(N)$ is a finite subgroup generated by quasi-reflections. ($\Leftrightarrow$ algebra of invariants is $\approx$ polynomial algebra)

Generalization of a Coxeter group

Classified by Shephard & Todd - 1954

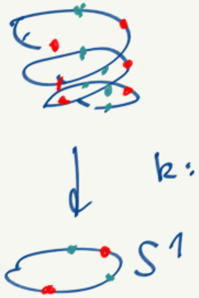
G/G' is quotient of $\pi_1(\mathbb{CP}^1 - \{0, \infty\}) = \mathbb{Z}$ hence is a cyclic group.

if $\{\alpha\}_{\bmod \mathbb{Z}}$ $\{\beta\}_{\bmod \mathbb{Z}}$ is a universally interlacing pair of N-element subsets.

$\Rightarrow$ 1) $\forall\ \gamma \in \mathbb{Q}$ $\quad \{\alpha_j+\gamma\}_{\bmod \mathbb{Z}}$, $\{\beta_i+\gamma\}_{\bmod \mathbb{Z}}$ also universally interlacing pair

2) $\forall\ k \geq 2$ $\quad \left\{\frac{\alpha_j}{k}+\frac{l}{k}\right\}_{\bmod \mathbb{Z}}$ $\begin{matrix} l=0,\dots,k-1 \\ j=1\dots N \end{matrix}$, $\left\{\frac{\beta_i}{k}+\frac{l}{k}\right\}_{\bmod \mathbb{Z}}$ $\begin{matrix} l=0\dots k-1 \\ j=1\dots N \end{matrix}$

is again an universally interlacing pair of $N \cdot k$-element subsets



3) $\{\alpha\},\{\beta\}$ univ. interlacing $\Longleftrightarrow$ $\{\alpha'=\beta\}$, $\{\beta'=\alpha\}$ is univ. interlacing

4) $\hat{\mathbb{Z}}^\times = \mathrm{Gal}(\mathbb{Q}(\sqrt[\infty]{1})/\mathbb{Q})$ - action. $\left(\sim (\alpha_\cdot)(\beta_\cdot) \Rightarrow (\alpha p \bmod \mathbb{Z}), (\beta p \bmod \mathbb{Z}) \quad p \gg 1\right)$.

Classification: 1. ∞ family, pair of coprime integers

(up to rotations, coverings, swaps, $\mathbb{Z}^\times$-action)

$N = a+b-1$ $\quad (a,b)=1, \quad a,b \geq 1$

(Series A_N)

$$\{\alpha\} = \left\{ \frac{1}{a+b}, \frac{2}{a+b}, \dots, \frac{a+b-1}{a+b} \right\}$$

$$\{\beta\} = \{0\} \cup \left\{ \frac{1}{a}, \frac{2}{a}, \dots, \frac{a-1}{a} \right\} \cup \left\{ \frac{1}{b}, \frac{2}{b}, \dots, \frac{b-1}{b} \right\}$$

Hypergeom. series

$$\sum_{n\geq 0} \frac{((a+b)n)!}{(an)!\,(bn)!} \left(z^n \cdot \frac{a^a b^b}{(a+b)^{a+b}} \right)^n$$

$\boxed{\beta_{i_0} = 0}$

trivial case
$N=1$
∞ . family
$\cong \mathbb{Z}^{\times}$ $(0), \left(\frac{1}{q}\right)$, $q \geq 2$

($\mathbb{Z}^{\times}$ orbit
$\rightsquigarrow$ $(0), \left(\frac{p}{q}\right)$)
$0 < \frac{p}{q} < 1$

$N=2$ H.A. Schwarz, (1873) not listed in Beukers-Heckman.

two infinite families : $\left(0, \frac{1}{2}\right), \left(\frac{1}{q}, \frac{q-1}{q}\right)$ & $\left(0, \frac{1}{2}\right), \left(\frac{1}{q}, \frac{1}{2}+\frac{1}{q}\right)$ (dihedral case)
$q \geq 3$

23 exceptional examples
e.g
$\left(0, \frac{1}{3}\right) \left(\frac{1}{60}, \frac{49}{60}\right)$

• Exceptional examples for $N \geq 3$
$N = 3, 4, 5, 6, 7, 8$ 77 of them.

the last one:

$$\left\{\frac{1}{24}, \frac{5}{24}, \frac{7}{24}, \frac{11}{24}, \frac{13}{24}, \frac{17}{24}, \frac{19}{24}, \frac{23}{24}\right\}, \quad \left\{0, \frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{1}{2}, \frac{5}{9}, \frac{7}{9}, \frac{8}{9}\right\}$$

group E_8.

How to write an explicit algebraic equation for $W(z)$?

New observation: Riemann surface of $W(z)$

$$\{ (z, W^{K}(z)) \in \mathbb{C}^2 \}$$

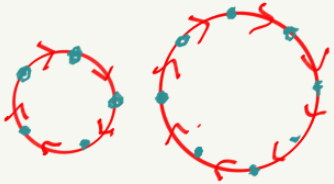
is of a very low genus, often $=0$

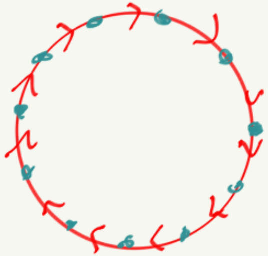
K: order of ramification of $W(z)$ at $z=1$

Example: $\sum_{n\geq 0} \frac{((a+b)n)!}{(an)!\,(bn)!} z^n$ series $\qquad N = a+b-1$

$$\pi_1(\mathbb{CP}^1 - \{0,1,\infty\}) \to S_{a+b} \to GL(a+b-1, \mathbb{Q})$$

$T_0 = (a)(b)$ $\qquad T_1 = (2)(1^{a+b-2})$ $\qquad T_\infty = (a+b)$



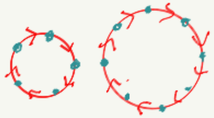


Action on $\{$ conjugacy class of $T_1 \}$ = orbit of the vector

$\pm$

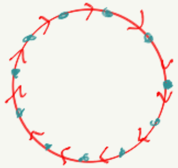
Claim Riemann surface even of the action on $(a+b)(a+b-1)$ ordered pairs has genus $=0$

Calculation: $\chi = \#\{\text{ordered pairs}\} \cdot \chi(\mathbb{P}^1 - \{0,1,\infty\}) = -(a+b)(a+b-1)$



$+$ # of orbits of T_0 : $(a-1) + (b-1) + 2$

$+$ # of orbits of T_1 $(a-1)(a-2) + a-1 + (b-1)(b-2) + (b-1)$
$+1$



$+$ # of orbits of T_∞ $(a+b-1)$

$\Rightarrow \chi = 2 \quad g=0$

$N = 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 8$

$0,1,2 \quad 0,1,2,3,4,5 \quad 1,2 \quad 1,3,4,5 \quad 1,2 \quad 1,2,3,4,5$

almost all $g=0$

4 examples of $g=0$

<u>Challenge</u>: in $g=0$ cases: find a parametrization of $(z, W(z)^k) \overset{?}{\simeq} \mathbb{Q}P^1$

$\rightsquigarrow \quad W(z)^k = R_2^{-1} \circ R_1 \qquad R_1, R_2$ rational functions in <u>one</u> variable

$g \geq 1$: Are curves hyperelliptic?

Dessins d'enfants : (preimages of $\underset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{1}{\bullet}$):