Maximally Dense Sphere Packings

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Outline

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Thue's theorem
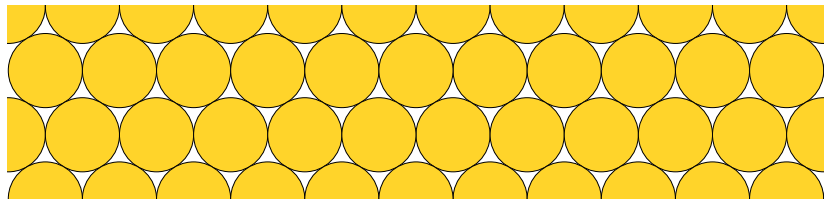
A packing of spheres in $\mathbb{R}^n$ is a union of interior disjoint balls.
Its density is defined by

$$\delta := \limsup_{k \to \infty} \frac{\text{area of } B_n(0, k) \text{ covered by the spheres}}{\text{volume of } B_n(0, k)}.$$

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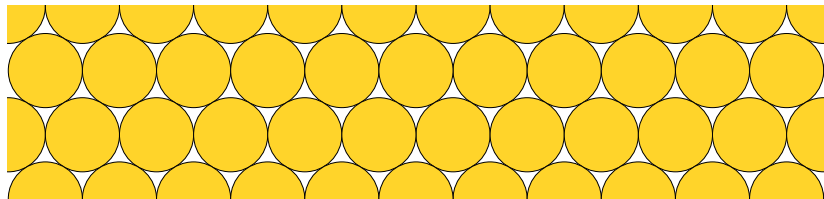
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Theorem (Thue 1910, Tóth 1943)

Any packing of equal circles in $\mathbb{R}^2$ has density at most $\frac{\pi}{2\sqrt{3}}$.

Proof (following H.-C. Chang & L.-C. Wang, 2010)

Delaunay triangulation: circumscribed circles have empty interior.

Saturated packing: no further circle can be added.

Lemma
In a Delaunay triangulation of the centers of a saturated packing of unit circles, the density within each triangle is at most $\frac{\pi}{2\sqrt{3}}$.

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▶ Let ABC be a triangle with largest angle $\widehat{A} \leadsto \widehat{A} \geq \frac{\pi}{3}$.

▶ If $\widehat{A} \geq \frac{2\pi}{3}$, then its smallest angle, say $\widehat{B}$, satisfies $\widehat{B} \leq \frac{\pi}{6}$. This yields a circumradius $R = \frac{\overline{AC}}{2\sin\widehat{B}} \geq 2$: a new circle can thus be added at the circumcenter. Contradiction $\leadsto \widehat{A} < \frac{2\pi}{3}$.

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▶ The area is $\frac{1}{2} \sin \widehat{A} \cdot \overline{AB} \cdot \overline{AC} \geq \sqrt{3}$, covered by half a unit disc.
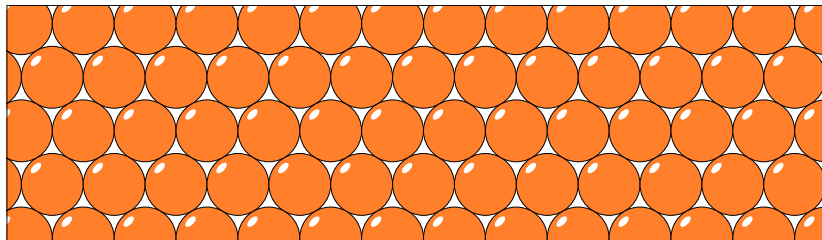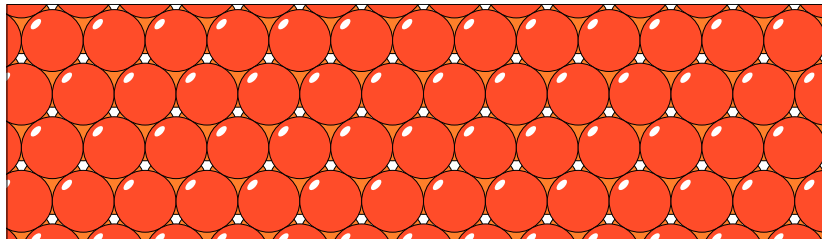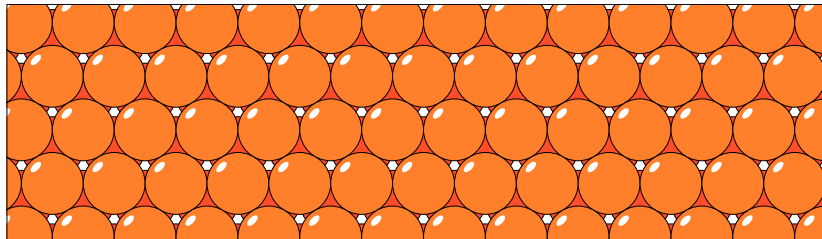
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The Kepler conjecture

In $\mathbb{R}^3$, close-packings of equal spheres achieve the density $\frac{\pi}{3\sqrt{2}}$:

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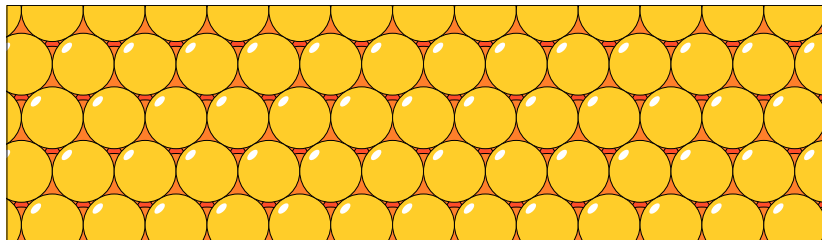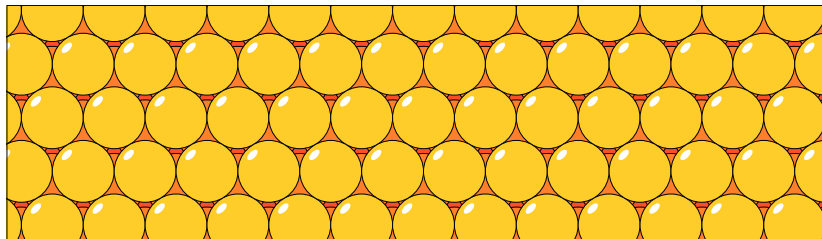In $\mathbb{R}^3$, close-packings of equal spheres achieve the density $\frac{\pi}{3\sqrt{2}}$:

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Theorem (Hales-Ferguson, 1992–2014)

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Frustration

One could try to mimic the proof of Thue's theorem:

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The global optimum (densest packing) is not locally optimal.

This lemma only yields an upper bound on the optimal density:

$$\sqrt{2}\arccos\tfrac{23}{27} \approx 0.7796 > \tfrac{\pi}{3\sqrt{2}} \approx 0.7405.$$

More frustration

Already in $\mathbb{R}^2$, this situation is the norm for unequal circles:

Theorem (Florian, 1960)

In a Delaunay triangulation of the centers of a saturated packing of unequal circles, the densest possible triangle connects the centers of two smallest circles and a largest one, all mutually tangent.
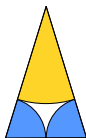
There is no decomposition made of such triangles only (exercise). Again, this only yields an upper bound on the optimal density.

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Even better:
Can we express it in the elementary language of the real numbers (quantifiers, logical connectives, $+$, $-$, $\times$, variables, constants)?
$\rightsquigarrow$ This would, at least, be decidable (Tarski, 1931)

Localization (following J. C. Lagarias, 2002)

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Each local density inequality yields an upper bound on the density.

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We have to find a local density inequality which

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For unequal circles (see suite), I used an enhanced mix of the localizations proposed by Heppes in 2003 and Kennedy in 2004.

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Theorem (Kennedy, 2006)

There are exactly 9 sizes allowing a triangulated binary packing.
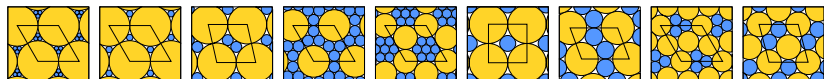
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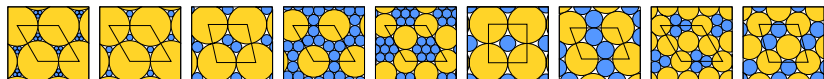
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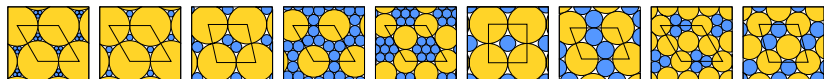
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Theorem (Fernique-Hashemi-Sizova, 2019)

There are exactly 164 sizes allowing a triangulated ternary packing.

In some cases (not all), the density is maximized by such a packing.

Playing with stoichiometry
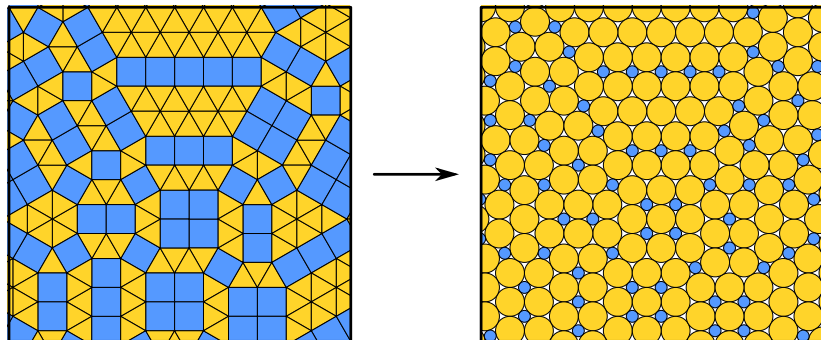
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The densest packings with circles of size 1 and $\sqrt{2} - 1$ are

▶ *recodings of* square-triangle tilings *for a large circles excess;*

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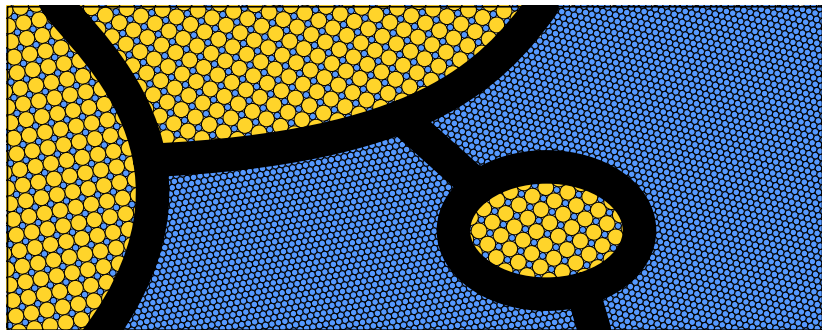
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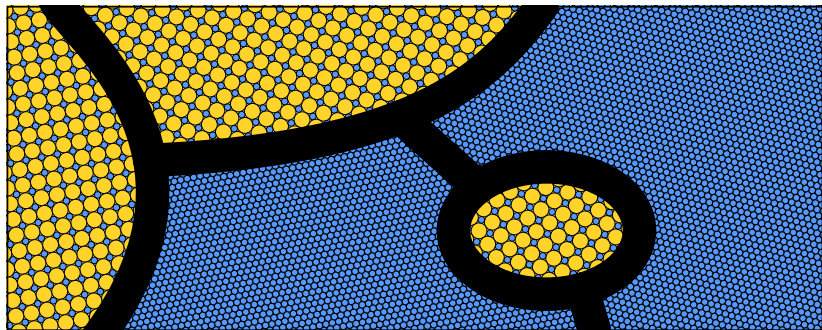
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The maximum density as a function of the stoichiometry follows.

Global picture for two sizes of circle
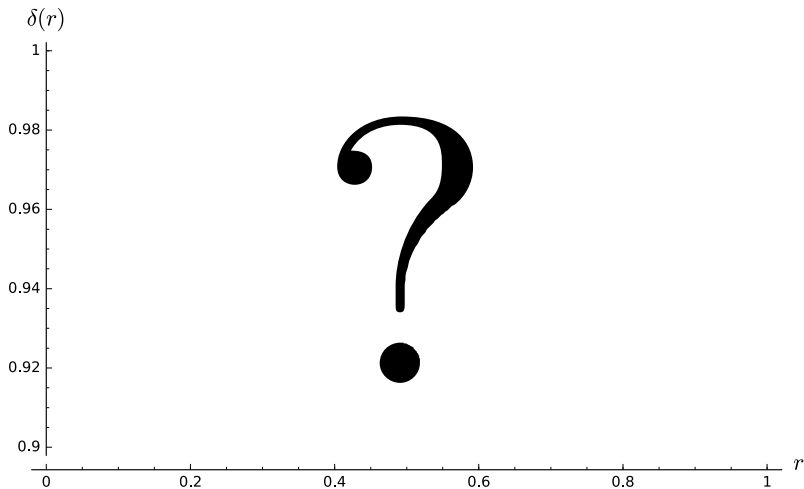
Maximal density $\delta(r)$ of packings of circles of size 1 and r?

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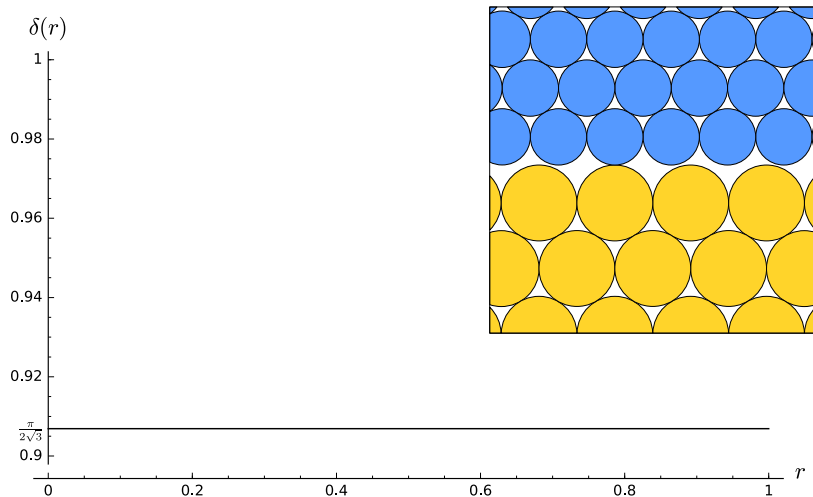
Trivial lower bound: the density of the hexagonal compact packing.

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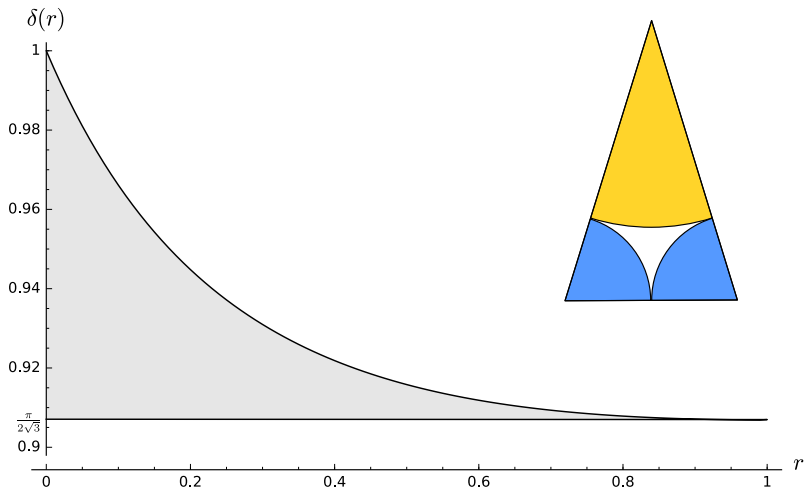
The already mentionned Florian's upper bound.

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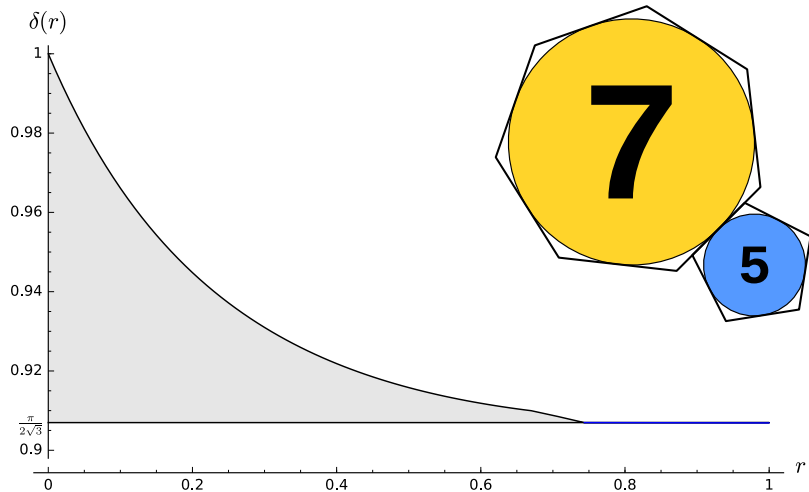
Blind's upper bound (1969) yields a sharp bound over $[0.74\ldots, 1]$.

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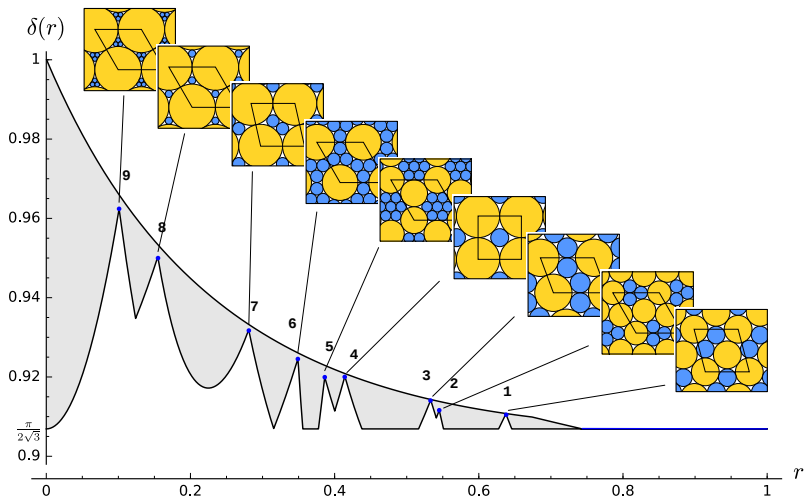
Flip & flow on the 9 triangulated binary packing ⇝ lower bound.

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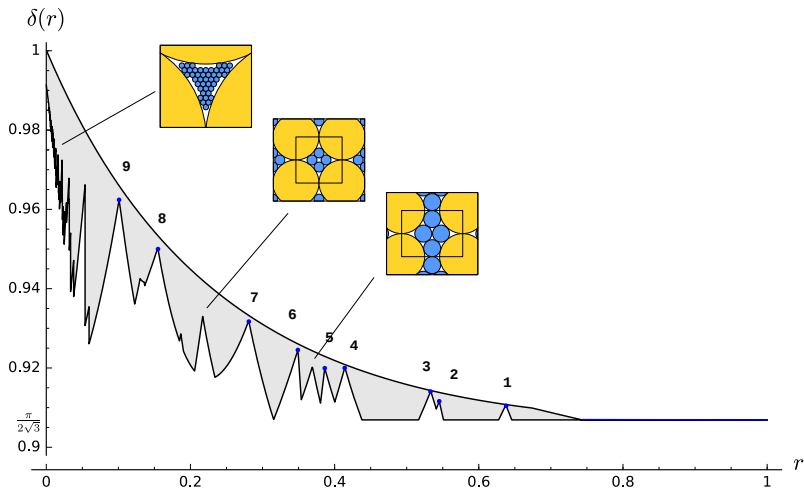
Flip & flow on any suitable packing improves the lower bound.

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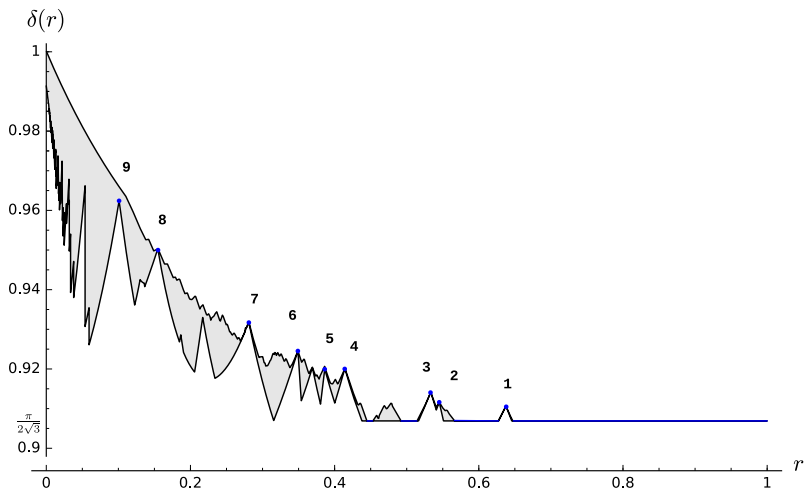
Massive computations for localization improve the upper bound.

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The Complete picture for two sizes of circle
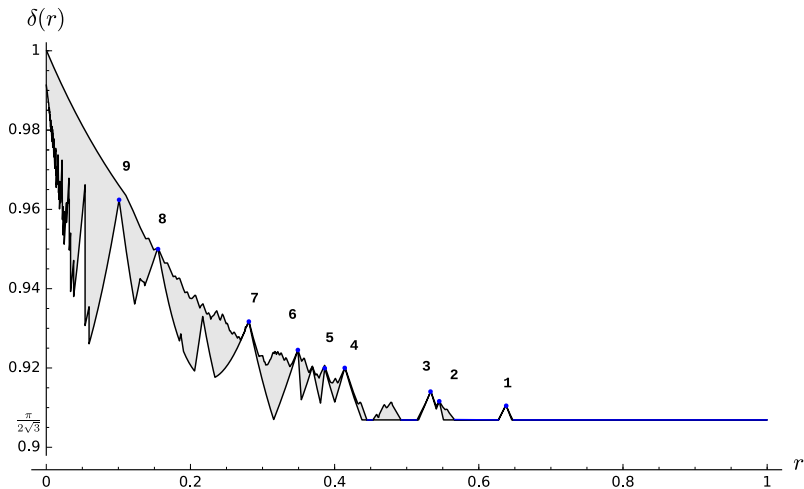
The localization is challenged by nontriangulated optimal packings and circles that fit well locally (around a circle), but not globally.

The most uniform packing

What is the largest r such that circles with sizes in $[r, 1]$ can be packed more densely than all equal circles?
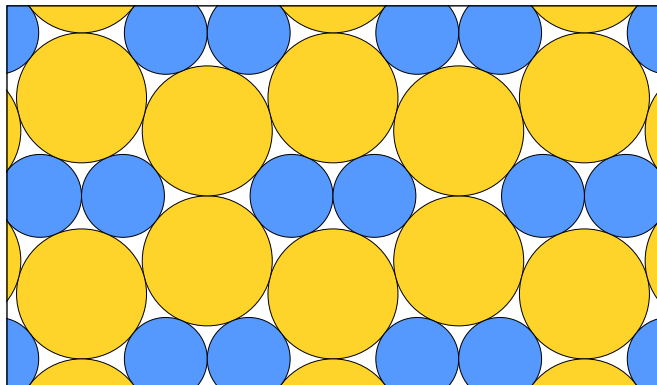
$r \geq 0.6375\ldots$ (Tóth, 1964) $r \leq 0.7430\ldots$ (Blind, 1969).

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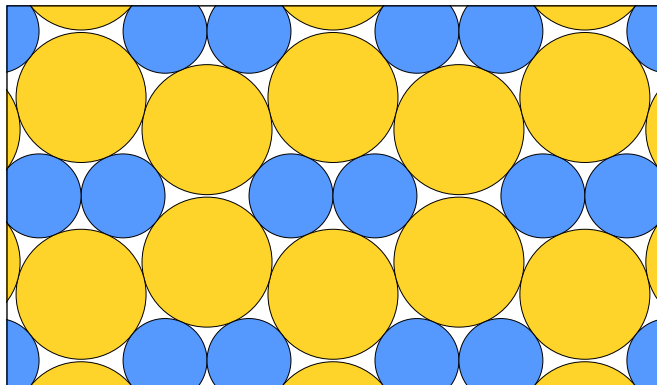
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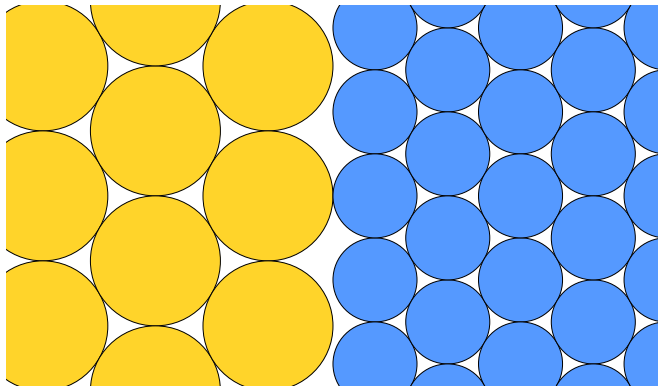
$r < 0.6464$ if there are only two sizes of circle (Fernique, 2020).

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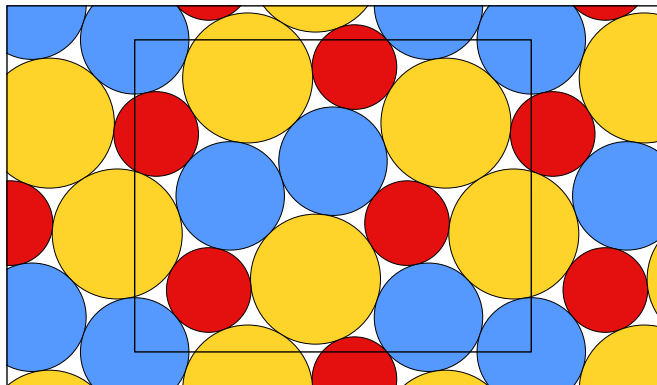
$r \geq 0.6510\ldots$ (Fernique-Hashemi-Sizova, 2018).

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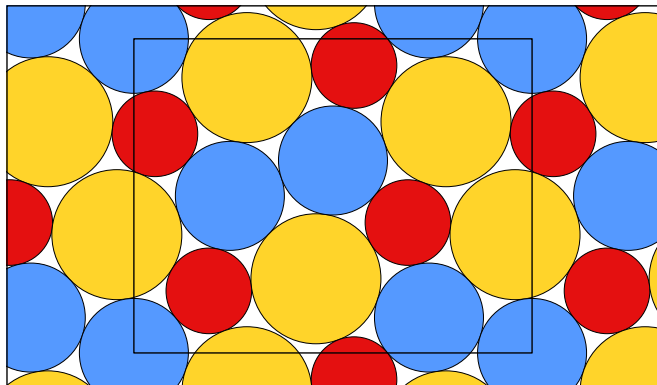
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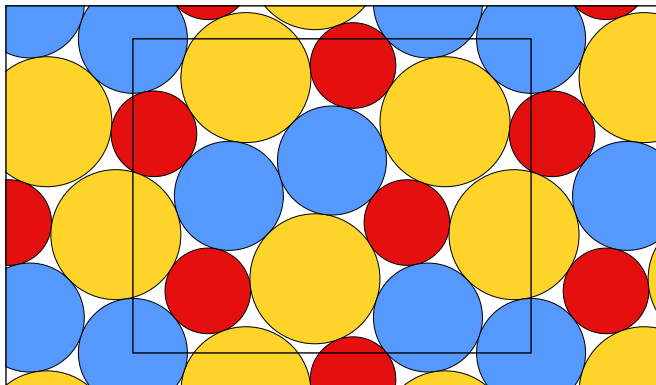
$r \geq 0.6585\ldots$ (Connelly–Pierre, 2019). Optimal? (Connelly–Zhang)

An aperiodic densest packing

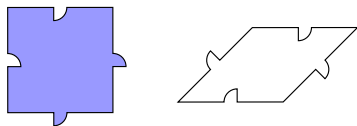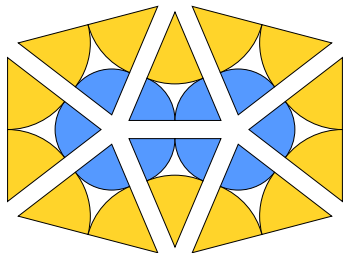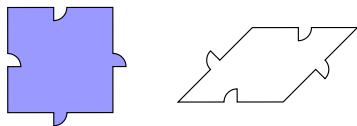Does exist circle sizes such that the densest packings are aperiodic?
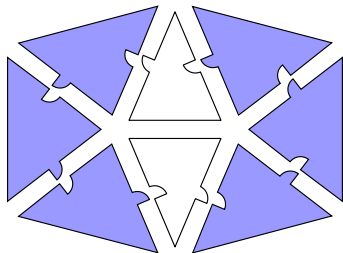
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Theorem (Bédaride-Fernique, 2012)

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Some references

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📄 T. Kennedy, *A Densest compact planar packing with two sizes of discs*, preprint, 2004 (link).

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