Enumeration & generation of Young tableaux with walls: the density method

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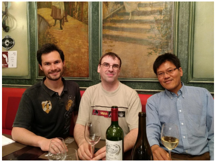
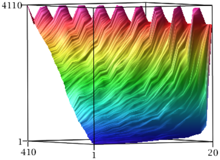
3	10	5	6	12	16	13	14
1	2	4	7	8	9	11	15





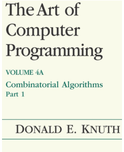
Based on articles with Michael Wallner (TU Wien) & Philippe Marchal (LAGA, Univ. Paris-Nord):

[Gascom 2018], [Annals of Proba 2020], [FPSAC 2021]

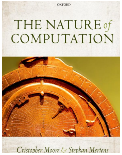
Uniform random generation of combinatorial structures

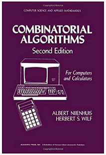
Many approaches:

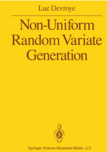
- ad hoc methods (& general principles: bijections, rejection, (un)ranking, etc.)
- Markov chain Monte Carlo algorithms, e.g. coupling from the past [Propp Wilson 1998]
- recursive method [Nijenhuis Wilf 1975, Flajolet Zimmermann Van Cutsem 1994, Denise Dutour Zimmermann 1998]
  $\rightsquigarrow$ packages in MuPAD/SageMath/Maple
- generating trees [West 1990, Dulucq Gire Guibert 1996, Barcucci Del Lungo Pergola Pinzani 1998, Banderier Bousquet-Mélou Denise Flajolet Gardy Gouyou-Beauchamps 1998...]
- Boltzmann method [Duchon Flajolet Louchard Schaeffer 2002, Fusy Pivoteau Salvy Soria Bodini Ponty Dovgal Bendkowsky Dien Papin Bacher Sportiello Stufler...]: the cherry on the cake of Flajolet's symbolic method!
- density method $\rightsquigarrow$ this talk!

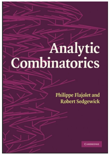


The Art of Bijective Combinatorics video-book









Part I:

Enumerative and bijective results

Young tableaux with local decreases

7	18	19	12	21	20	17
2	6	8	9	10	14	16
1	3	4	5	11	13	15

We consider Young tableaux in which some pairs of (horizontally or vertically) consecutive cells are allowed to have decreasing labels. Places where a decrease is allowed (but not compulsory) are drawn by a red edge, which we call a "wall".

Nice formulas for some specific tableaux of shape $n \times 2$:

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Nice formulas for some specific tableaux of shape $n \times 2$:

- no walls: $\frac{1}{n+1}\binom{2n}{n}$

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10	13
3	11
8	7
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9	1

Nice formulas for some specific tableaux of shape $n \times 2$:

- no walls: $\frac{1}{n+1}\binom{2n}{n}$
- walls everywhere:

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- no walls: $\frac{1}{n+1}\binom{2n}{n}$
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- horizontal walls everywhere:

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- horizontal walls everywhere in 2^{nd} col.:

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- vertical walls everywhere:

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- vertical walls everywhere: $\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$
- all k vertical walls: $\frac{1}{n+1}\binom{n+1}{k}\binom{2n}{n}$ (We give 2 proofs)

Proof #1: bijection with paths

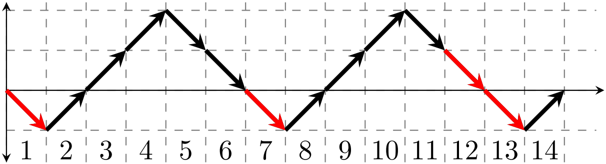
Theorem

The number of $n \times 2$ Young tableaux $\mathcal{Y}$ with k vertical walls is equal to

$$v_{n,k} = \frac{1}{n+1}\binom{n+1}{k}\binom{2n}{n}.$$

14	13
10	12
9	11
8	7
4	6
3	5
2	1

Proof: (part 1) Bijection with Dyck bridges on $\mathbb{Z}$: steps ± 1, length $2n$, k marked down steps:



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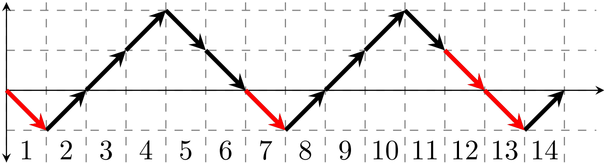
- The kth step is an up step iff the entry k appears in the first column of $\mathcal{Y}$; otherwise it is a down step.
- Down steps coming from a row with a wall are colored.



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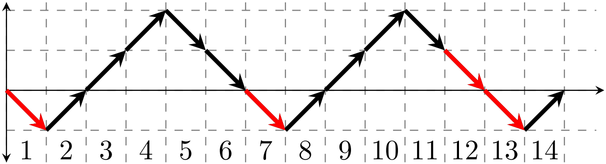
$\Rightarrow$ $v_{n,k}$ counts the number of paths with exactly k colored steps.



Proof #1: bijection with paths and Chung–Feller

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10	12
9	11
8	7
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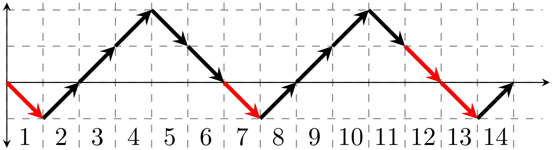
3	5
2	1



Proof (part 2):

- A down step below 0 is always colored because a wall has to be involved, a down step above 0 has a choice.
- By [Chung–Feller 49] the number of Dyck bridges of length $2n$ with i down steps below 0 is independent of i and equal to $\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$.

$$\Rightarrow \qquad v_{n,k} = \sum_{i=0}^{k}\binom{n-i}{k-i}\mathrm{Cat}(n) = \binom{n+1}{k}\mathrm{Cat}(n). \qquad \square$$

Proof #2: bijection to leaf-marked binary trees

Theorem

The GF for a fixed size n and an arbitrary number of walls is

$$v_n(u) := \sum_{k=0}^{n} v_{n,k} u^k = \mathrm{Cat}(n) \left((1+u)^{n+1} - u^{n+1}\right).$$

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11	13
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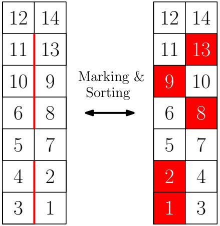

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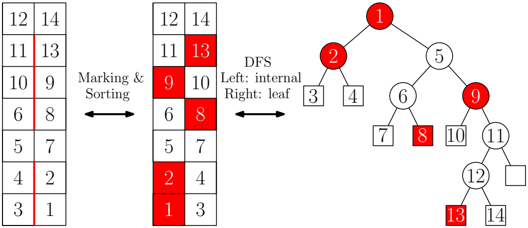

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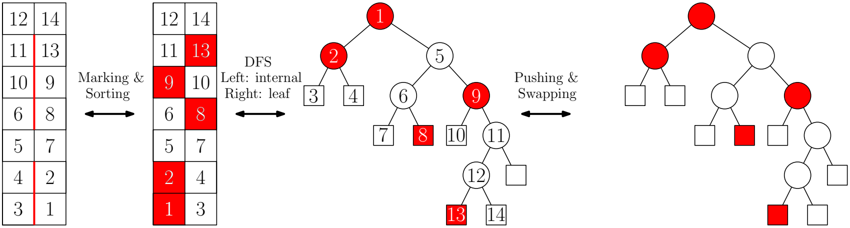

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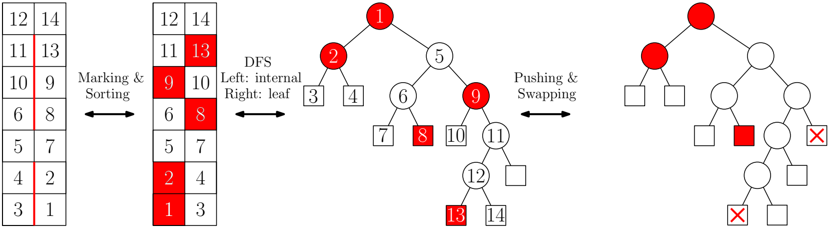

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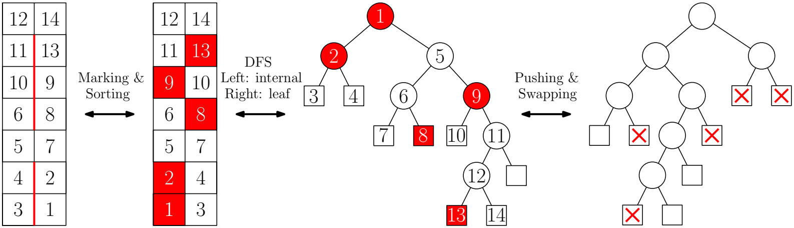

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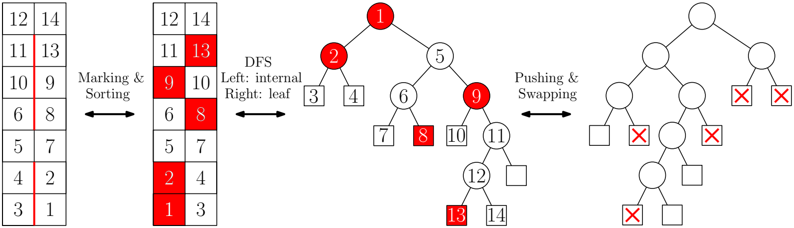


Open problem: combinatorial explanation of $\mathrm{Cat}(k-1) \mid v_{n,k}$?

Closed form of $v_{n,k}$ also proves $\sum_{n\geq 0} v_{n,k} z^n = \mathrm{Cat}(k-1)\frac{z^{k-1}}{(1-4z)^{(2k-1)/2}}$.

Long walls with small holes: hook-length type formulas

Holes of size 1 on the border

13	14	16	17	19	20	21	25	27
11	2	10	12	15	18	6	23	26
4	1	8	5	7	9	3	22	24

λ_1 λ_2 λ_3 λ_4

Theorem

The number of $n \times m$ Young tableaux of size mn with k walls from column 1 to $m-1$ at distance $0 < d_i := \sum_{j=1}^{i} \lambda_i < n$, $i = 1, \ldots, k$ with $h_i < h_{i+1}$ is equal to

$$\frac{(m-1)!}{(mn+m-1)_{m-1}} \left(\prod_{i=1}^{k+1} \prod_{j=1}^{m-2} \binom{\lambda_i + j}{j}^{-1} \right) \left(\prod_{i=1}^{k+1} \binom{md_i + m - 1}{\lambda_i, \ldots, \lambda_i} \right),$$

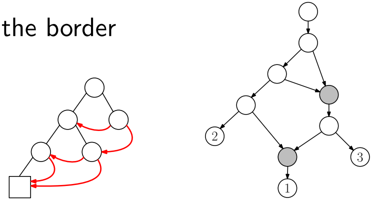
where the multinomial coefficients contain $m-1$ λ_i's.

$\rightsquigarrow$ hook-length formulas to a give a combinatorial explanation of the integrality of $\frac{(6n)!n!}{(3n)!(2n)!^2}$ or $\frac{(30n)!n!}{(15n)!(10n)!(6n)}$? (alternative to the Landau criterion, 1900)

Larger holes lead to unusual asymptotics

The "simplest case" of holes of size 2 on the border

6	10	14	15	17	18
3	5	9	12	13	16
2	1	7	4	11	8



BAADBACFCBEDECDFEF

Theorem

The number f_n of such Young tableaux of size $n \times 3$ satisfies

$$f_n = \Theta\left(n!\ 12^n e^{a_1 (3n)^{1/3}} n^{-2/3}\right),$$

where $a_1 \approx -2.338$ is the largest root of the Airy function of the first kind.

- Bijections to phylogenetic networks, special words with n distinct letters, and related to compacted trees (special DAGs) [Fuchs–Yu–Zhang 21]
- General method to prove stretched exponentials in bivariate recurrences [Elvey Price–Fang–Wallner 21]. Here:

$$y_{n,k} = y_{n,k-1} + (2n + k - 1) y_{n-1,k} \qquad \text{and} \qquad f_n = y_{n,n}.$$

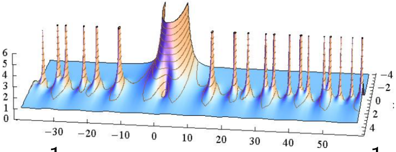
Part II:

The density method

- * far origins in poset theory (volume of polytopes, log-concavity) [Stanley 1981]
- * avatars in number theory [Zagier, Beukers Kolk Calabi 1993, Elkies 2003]
- * applied to square Young tableaux [Barishnikov 2001]
  and variants of alternating permutations [Baryshnikov Romik 2010, Stanley 2010]
- * generalized to further posets & random generation [Banderier Marchal Wallner 2016–2021]

Values of the zeta function and geometry



$$\zeta(s) = \sum_{k \geq 1} \frac{1}{k^s} \quad \to \quad S(s) := \zeta(s)(1 - \frac{1}{2^s}) = \sum_{k \geq 0} \frac{1}{(2k+1)^s}$$

$$S(2) = \sum_{k \geq 0} \frac{1}{(2k+1)^2} = \sum_{k \geq 0} \int_0^1 \int_0^1 (xy)^{2k} = \int_0^1 \int_0^1 \frac{dxdy}{1-(xy)^2}$$

Change of variable $x = \frac{\sin u}{\cos v}$ and $y = \frac{\sin v}{\cos u}$.
The integration domain becomes the triangle $T = \{u > 0, v > 0, u + v < \pi/2\}$.

$$S(2) = \int_T dudv = \pi^2/8$$

n even: $\quad S(n) = \text{vol(polytope of dimension } n) = (\pi/2)^n/n!\, A(n)$

$$A(n) = \#\text{ alternating permutations of length } n.$$

$\rightsquigarrow$ Kontsevich–Zagier periods / "dimension" of a number / Hilbert 3rd problem (congruence of polytopes)

Uniform random generation and enumeration

6	15	16
1	13	14
8	10	18
3	9	12
4	7	17
2	5	11

This example is "without loss of generality"
(i.e., our method works also
for non-periodic shapes).

How to generate/enumerate such tableaux? Brute-force is hopeless!

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How to generate/enumerate such tableaux? Brute-force is hopeless!
Solution = use our density method!

The density method will give thousands of coefficients in a few seconds.
The number of tableaux of size $2n \times 3$ is $f_n = (6n+1)! \int_0^1 p_n(z)dz$, with

$$p_{n+1}(z) = \int_0^z \frac{1}{24}(z-1)(x-z)(3x^3 - 7x^2z - xz^2 - z^3 - 2x^2 + 4xz + 4z^2)p_n(x)\,dx.$$

$\{f_n\}_{n\geq 0}=\{$1, 12, 8550, 39235950, 629738299350, 26095645151941500, 2323497950101372223250, 392833430654718548673344250, 115375222087417545717234273063750, 55038140590519890608190921051205837500, $\ldots\}$.

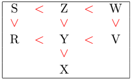
From tableaux to tuples of real numbers, and polytopes

6	15	16
1	13	14
8	10	18
3	9	12
4	7	17
2	5	11

7	16	17
2	14	15
9	11	19
4	10	13
5	8	18
3	6	12
	1	

.74	.96	.97
.25	.94	.95
.85	.91	.99
.42	.90	.93
.54	.82	.98
.35	.57	.92
	.06	

S	Z	W
R	Y	V
	X	



Left: $2n \times 3$ Young tableau with walls.

Centre: A related tableau with one more cell (removing this cell + relabel: bijection with left tableau).

Our algorithm generates real numbers between 0 and 1, with same relative order.

All possible values = a polytope $\mathcal{P} \in [0,1]^{6n+1}$.

Right: The "building block" of 7 cells. Each polyomino is made of the overlapping of n such building blocks.

Density method: key ideas

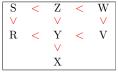
• geometric point of view:
Associate with a poset of size N its *order polytope* $\mathcal{P}$ (it is a subset of $[0,1]^N$).
Generate a random element of $\mathcal{P}$ slice by slice using conditional densities.
In the present example, $N = 6n+1$ and the slices are the building blocks of size 6 (except for the first one).

• sequence of densities:
sequence of polynomials $p_n(x)$, defined by the following recurrence (which in fact encodes the full structure of the problem, building block after building block):
$p_0 = 1$ and by induction,

$$p_{n+1}(z) = \int_{0<x<z} \int_{x<y<z} \int_{0<r<y} \int_{r<s<z} \int_{z<w<1} \int_{y<v<w} p_n(v)\, dv\, dw\, ds\, dr\, dy\, dx.$$

The density method algorithm

.74	.96	.97
.25	.94	.95
.85	.91	.99
.42	.90	.93
.54	.82	.98
.35	.57	.92
	.06	



1 Initialization: Precompute the polynomials $p_0(z), \dots, p_n(z)$.
Label the building blocks from $k = n-1$ to $k = 0$ (top to bottom).
Start at the top, i.e. $k := n-1$.
Put into the top cell Z a random number z with density $p_n(z)/\int_0^1 p_n(t)\,dt$.

2 Filling: Now that Z is known, put into the cells X, Y, R, S, V, W random numbers x, y, r, s, v, w with conditional density

$$g_{k,z}(x,y,r,s,v,w) := \frac{1}{p_{k+1}(z)} p_k(x) \mathbf{1}_{\mathcal{P}_k},$$

where $\mathbf{1}_{\mathcal{P}_k}$ is the indicator function of the k-th building block (with value z in cell Z):

$$\mathbf{1}_{\mathcal{P}_k} := \mathbf{1}_{\{0 \le x \le y \le z, 0 \le r \le y, r \le s \le z, z \le w \le 1, y \le v \le w\}}.$$

3 Iteration: Consider X as the Z of the next building block. Set $k := k-1$ and go to step 2 (until $k = 0$).

A very efficient algorithm!

Theorem

The density method algorithm is a uniform random generation algorithm with quadratic time complexity (including precomputations).

Proof

Our algorithm yields a tuple $\mathbf{x} := (x_i, y_i, r_i, s_i, v_i, w_i)_{0<i\leq n}$ with density

$$\frac{p_n(x_n)}{\int_0^1 p_n(t)dt} \prod_{i=1}^{n} g_{n-i,x_{n-i+1}}(x_{n-i}, y_{n-i}, r_{n-i}, s_{n-i}, v_{n-i}, w_{n-i})$$

$$= \frac{p_n(x_n)}{\int_0^1 p_n(t)dt} \prod_{k=0}^{n-1} \frac{p_k(x_k)\mathbf{1}_{\mathcal{P}_{x_k}}}{p_{k+1}(x_{k+1})} \underset{\uparrow}{=} \frac{p_0(x_0)\mathbf{1}_{\mathcal{P}}}{\int_0^1 p_n(t)\,dt} = \frac{\mathbf{1}_{\mathcal{P}}}{\int_0^1 p_n(t)\,dt} \qquad (\text{as } p_0 = 1),$$

telescopic product!

where $\mathbf{1}_{\mathcal{P}_{x_k}}$ is as in the algorithm above the indicator function of the k-th block.
$\int$ (any density) $= 1$ implies:

$$\mathrm{vol}(\mathcal{P}) = \int_{[0,1]^{6n+1}} \mathbf{1}_{\mathcal{P}}\,d\mathbf{x} = \int_0^1 p_n(t)\,dt.$$

A very efficient algorithm!

end of proof (uniformity & complexity):

Now if we choose a random uniform element in $[0,1]^{6n+1}$, the probability that it belongs to $\mathcal{P}$ is

$$\int_{[0,1]^{6n+1}} \mathbf{1}_{\mathcal{P}}\, d\mathbf{x}.$$

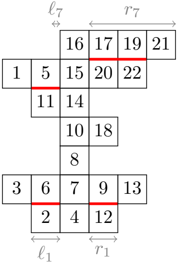
This is also the probability that a random uniform filling of our building block is correct (i.e., respects the order constraints).

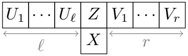
This implies that $f_n = (6n+1)! \int_0^1 p_n(t)dt = |\mathcal{P}|! \operatorname{vol}(\mathcal{P})$.

Each step = computation and evaluation of the associated polynomial $p_n(z)$ (of degree proportional to n) $\Rightarrow$ quadratic time complexity and quadratic space. $\square$

Jenga tableaux and the density method

Jenga! = Construct! in Swahili.





Given a shape encoded by $(\ell_i, r_i)_{i\in\mathbb{N}}$, what is the number of tableaux with n lines?

$$f_n = \left(\sum_{i=1}^{n}(\ell_i + r_i + 1)\right)! \int_0^1 p_n(x)\,dx.$$

$$p_n(z) = \int_{z<v_1<1} \cdots \int_{v_{r-1}<v_r<1} \int_{0<u_\ell<z} \cdots \int_{0<u_1<u_2} \int_{0<x<z} p_{n-1}(x)\,dx\,du_1 \ldots du_\ell\,dv_r \ldots dv_1$$

$$= \frac{z^{\ell_n}(1-z)^{r_n}}{\ell_n!\,r_n!} \int_0^z p_{n-1}(x)\,dx \qquad \text{with} \qquad p_1(z)\frac{z^{\ell_1}(1-z)^{r_1}}{\ell_1!\,r_1!}.$$

Special Jenga shapes

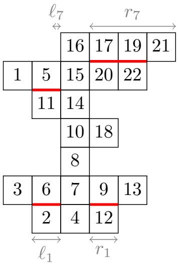
Theorem (1-periodic Jenga tableaux are D-finite)

The bivariate GF $P(t,z) = \sum_{n\geq 1} p_n(z) t^n$ *is D-finite in* t *and* z.

Proposition

For $r_i = 0$, $\forall i \geq 1$, the number f_n of Jenga tableaux is

$$f_n = \frac{\left(\sum_{i=1}^{n}(\ell_i + 1)\right)!}{\prod_{i=1}^{n} \ell_i! \left(\sum_{j=1}^{i}(\ell_j + 1)\right)}.$$



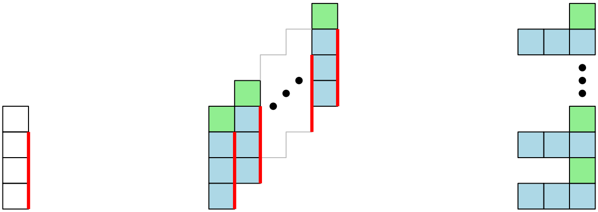
Proposition

For Jenga tableaux with period p, $L := \sum_{i=1}^{p} \ell_i$, and $(r_i)_{i=0}^{p} = (0,\ldots,0)$, one has:

$$f_{kp+m} = f_m \left(\frac{(L+p)^L}{\prod_{i=1}^{p} \ell_i!} \right)^k \prod_{\substack{j=1 \\ j\neq \ell_1+\cdots+\ell_i+i}}^{L+p} \frac{\Gamma\left(k + \frac{j+m}{L+p}\right)}{\Gamma\left(\frac{j+m}{L+p}\right)}.$$

Accordingly, the GF of such tableaux is the sum of p hypergeometric functions.

Skew tableaux with walls



Building block Skew shape Jenga shape

Proposition

The number of above skew tableaux with n columns of height h is

$$f_n = \left(\frac{h^{h-2}}{(h-2)!}\right)^n \prod_{j=1}^{h-2} \frac{\Gamma\left(n+\frac{j}{h}\right)}{\Gamma\left(\frac{j}{h}\right)}.$$

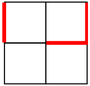
Proof: Use a bijection between this class and periodic Jenga tableaux of period $p=2$, $\ell_1 = h-2$, $\ell_2 = 0$, and $r_i = 0$. □

A classification of 2×2 periodic shapes

A periodic shape is the concatenation of n copies of a building block $\mathcal{B}$ of ¡cells:

$$\mathcal{Y} = \mathcal{B}^n.$$

A tableau $\mathcal{Y}$ with periodic walls is a periodic shape filled with all integers from $\{1, \dots, |\mathcal{B}|n\}$ respecting the induced order constraints.

$\mathcal{B} =$ 

A filling of $\mathcal{B}^4$:

3	10	5	6	12	16	13	14
1	2	4	7	8	9	11	15

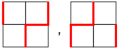
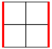
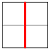
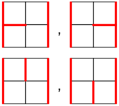
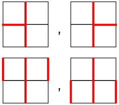
There are a priori $2^6 = 64$ shapes, but some are in bijection (e.g., turn by 180 degrees and reverse labels). It turns out that it leads to 32 different sequences.

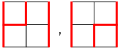
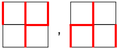
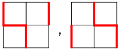
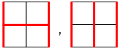
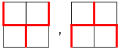
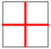
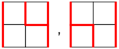
We now characterize all 2×2 shapes according to the nature of the counting sequence/generating function, which is either

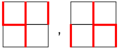
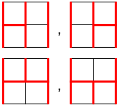
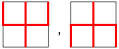
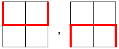
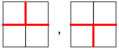
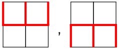
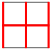
- "simple" hypergeometric
- hypergeometric,
- algebraic,
- D-algebraic and beyond.

“Simple” hypergeometric cases

= cases where walls split tableaux into independent regions ⇒ product formulas

Class	Shape	Formula
P1		$4\frac{(4n)!}{24^n}$
P2		$\frac{(4n)!}{12^n}$
P3		$3\frac{(4n)!}{12^n}$
P4		$\frac{(4n)!}{8^n}$
P5		$4\frac{(4n)!}{8^n}$

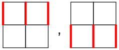
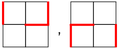
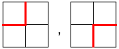
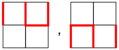
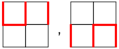
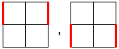
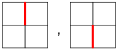
Class	Shape	Formula
P6		$\frac{(4n)!}{6^n}$
P7		$3\frac{(4n)!}{6^n}$
P8		$\frac{8}{5}(4n)!\,\frac{5^n}{24^n}$
P9		$\frac{(4n)!}{4^n}$
P10		$2\frac{(4n)!}{4^n}$
P11		$4\frac{(4n)!}{4^n}$
P12		$\frac{(4n)!}{3^n}$

Class	Shape	Formula
P13		$\frac{3}{2}\frac{(4n)!}{3^n}$
P14		$\frac{(4n)!}{2^n}$
P15		$2\frac{(4n)!}{2^n}$
P16		$\frac{(4n)!}{(2n)!\,2^n}$
P17		$2\frac{(4n)!}{(2n)!\,2^n}$
P18		$\frac{(4n)!}{(2n)!}$
P19		$(4n)!$

Proofs: Choose and distribute elements according to constraints.

Hypergeometric cases

$=$ cases with uniquely determined minimum or maximum

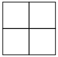
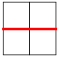
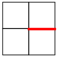
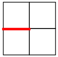
Class	Shape	Sequence	OEIS
H1		$\prod_{i=1}^{n}(4i-1)(4i-3)$	A101485
H2		$\prod_{i=1}^{n}(2i-1)(4i-1)$	A159605
H3		$2^{n+1}n!\prod_{i=1}^{n}(4i-3)$	2^{n+1}·A084943
H4		$\binom{4n}{n}\prod_{i=1}^{n}(3i-1)$	$\binom{4n}{n}$·A008544
H5		$\binom{4n}{n}\prod_{i=1}^{n}(3i-2)$	$\binom{4n}{n}$·A007559
H6		$2^{n}n!\prod_{i=1}^{n}(4i-3)$	$n!$ ·A084948
H7		$\prod_{i=1}^{n}(2i-1)(4i-1)$	A159605

Proofs:

- Models H1–H5: variants of Jenga tableaux with $r_i = 0$ for all i
- Models H6–H7: recursively decompose with respect to the location of the unique minimum or maximum.

Algebraic cases

= cases with no vertical walls

Class	Shape	Sequence	OEIS
A1		$\mathrm{Cat}(2n) = \frac{1}{2n+1}\binom{4n}{2n}$	A048990
A2		$\binom{4n}{2n}$	A001448
A3	 , 	$2^{2n+1}\,\mathrm{Cat}(n) - \mathrm{Cat}(2n+1)$	A079489

Proofs:

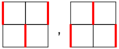
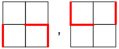
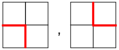
- Models A1 and A2: Use bijections to Dyck bridges.
- Model A3: Decomposing at the first wall that cannot be removed gives

$$f_n = \mathrm{Cat}(2n) + \sum_{i=1}^{n} \mathrm{Cat}(2i-1) f_{n-i},$$

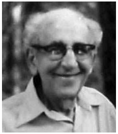
which we then solve with generating functions.

D-algebraic cases?

$\approx$ cases with a zig-zag-like pattern

Class	Shape	GF	OEIS	Example
Z1		D-algebraic, and not D-finite: $\frac{\cos(t/\sqrt{2})^2 + \cosh(t/\sqrt{2})^2}{2\cos(t/\sqrt{2})\cosh(t/\sqrt{2})}$	related to A211212	12 16 6 15 13 14 7 10 / 8 3 5 9 11 2 4 1
Z2		open problem!	—	3 5 8 9 11 13 14 15 / 2 10 4 7 1 16 6 12
Z3		open problem!	—	2 4 5 8 11 12 14 15 / 13 3 16 7 9 6 10 1

Proof for Z1: A permutation $(a_1, \ldots, a_n)$ is an alternating permutation of type $(k_1, \ldots, k_m)$ if
$a_1 < \cdots < a_{k_1} > a_{k_1+1} < \cdots < a_{k_1+k_2} > a_{k_1+k_2+1} < \cdots < a_n$.
Then, $k_i = 1$ gives classical alternating permutations;
while $k_1 = 3$, $k_2 = \cdots = k_n = 4$, and $k_{n+1} = 1$ gives Z1.
A generalization of [Carlitz 73] then leads to



Leonard Carlitz
(1907-1999)
771 articles!

$$F(t) = \frac{E_{4,3}(t)E_{4,1}(t)}{E_{4,0}(t)} + E_{4,0}(t) \quad \text{where} \quad E_{k,r}(t) = \sum_{n\geq 0}(-1)^n \frac{t^{nk+r}}{(nk+r)!}. \qquad \square$$

Conclusion

- 3 ways to enumerate and generate Young tableaux with walls: hook-length type formulas, bijections, density method.
- Approach different from [Greene Nijenhuis Wilf 84]. They used the existence of a simple product formula (hook-length formula).
- Brute-force generation → exponential cost. Generation via our density method → $O(n^2)$ cost.
- A field to explore: examine more families of posets (e.g., permutations, Young tableaux, increasing trees, urn models in [Banderier Marchal Wallner 20]).
- Asymptotics? D-finite? D-algebraic? Links with other objects?

3	5	8	9	11	13	14	15
2	10	4	7	1	16	6	12

$$\Theta\left(n!\, C^n e^{a_1 n^\sigma} n^\alpha\right)\ ?$$