

# Towards An Institution for Graph Transformation

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# institutions vs. DPO rewr.

- [ Institutions: abstract model theory

- sentences, models (and satisfiability)

- [ DPO rewriting: abstract rewriting formalism

- rules, rule applications (and replacement)

- lacking both sentences and models  
(hence, lacking an entailment system)

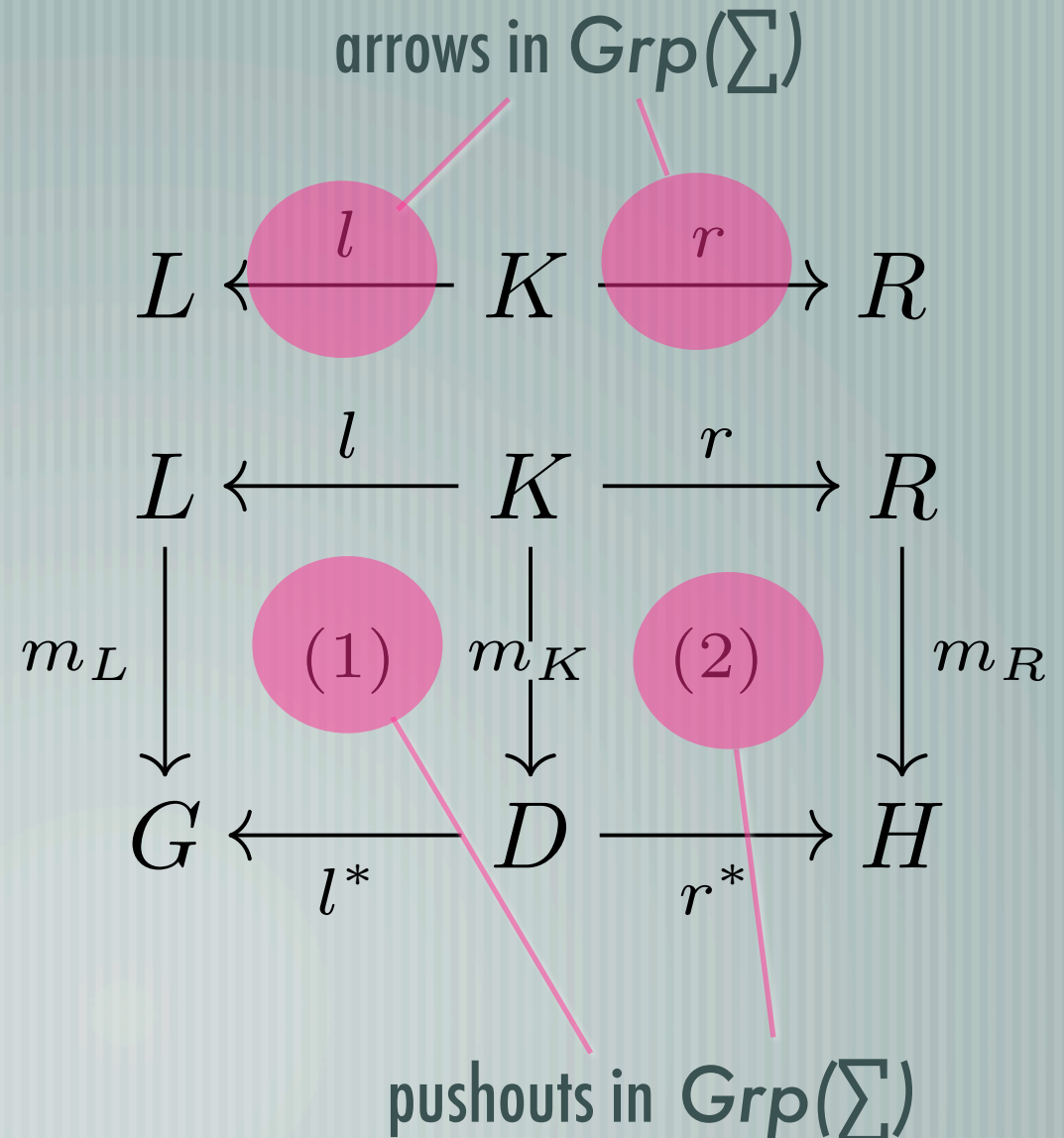
# shortly, the DP0 approach

$$\text{Grp}(\Sigma) = \text{Grp} \downarrow \Sigma$$

a rule

a derivation step

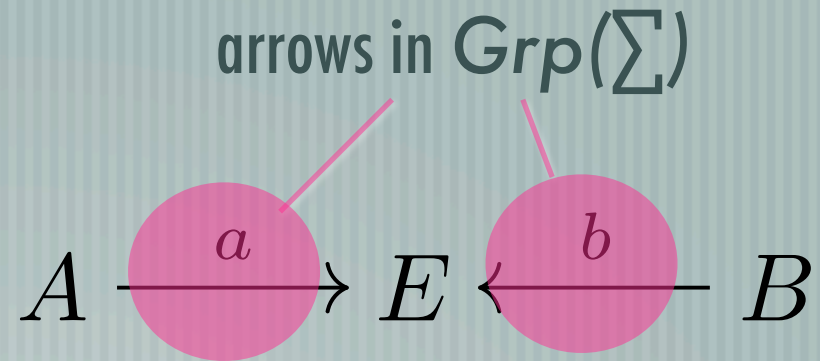
set of theoretical tools  
(concurrency, mostly)  
[holding for adhesive cats]



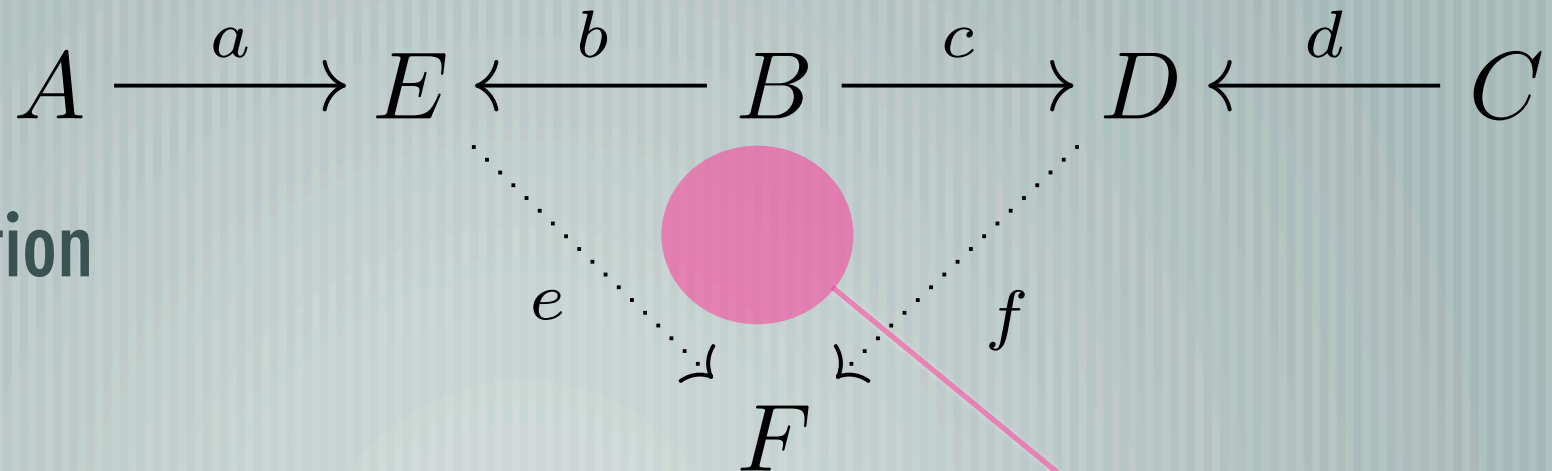
# the CoSpan (bi-)category

[holding for any cocomplete cat]

an arrow



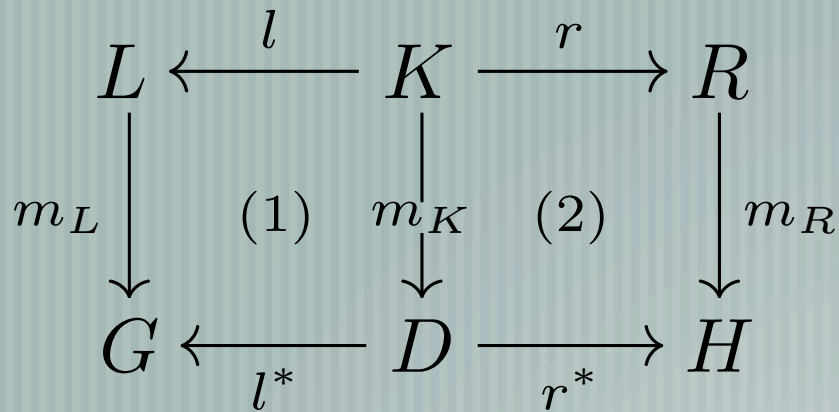
composition



powerful categorical tool  
(cats of relations)

pushout in  $\mathbf{Grp}(\Sigma)$

# connecting with the DPO



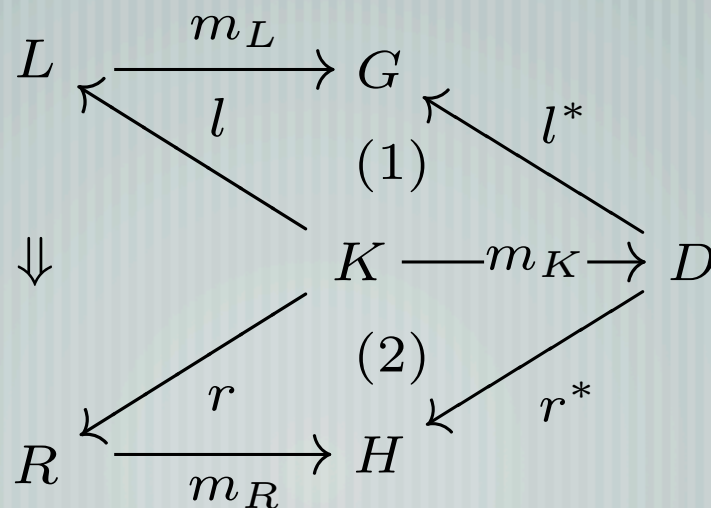
a rule

a derivation step

operational vs.  
induction-based

a "cell"

$\emptyset$



$\emptyset$

"whiskering"

# DPO vs. CoSpans

- [ (A factorized sub-category of) Cospans over graphs (typed over  $\Sigma$ ) form the free compact-closed category built from  $\Sigma$  (with operators as basic arrows)
- [ The DPO approach is operational: search for the match, build the PO complement...
- [ The free construction (concretely, via cospans) is algebraic: inductive closure of a set of basic rules

# first step: inductive sentences

[  $DGSTh(\Sigma)$  is the self-dual, free symmetric (strict) monoidal category equipped with symmetric monoidal transformations

$$\nabla_a : a \longrightarrow a \otimes a \qquad !_a : a \longrightarrow e$$

[ (intuitively representing pairing tuple  $\langle x, x \rangle$  and empty tuple)

[ plus two additional laws (relating transfs. and their dual)

# main correspondence result

I) (Isomorphic classes of) Cospans over graphs (typed on  $\Sigma$ ) and sets of nodes as objects  $\ast$ 1-1 correspond to  $\ast$  arrows in  $DGSTh(\Sigma)$  (indeed, a categorical equivalence)

— You abstract the identity of nodes not in the interface

— ...but this way graphs get a “standard” notion of sentence

II) The preorder on arrows obtained by replacing each DPO rule with an order on graphs  $\ast$ 1-1 corresponds to  $\ast$  DPO rewrites

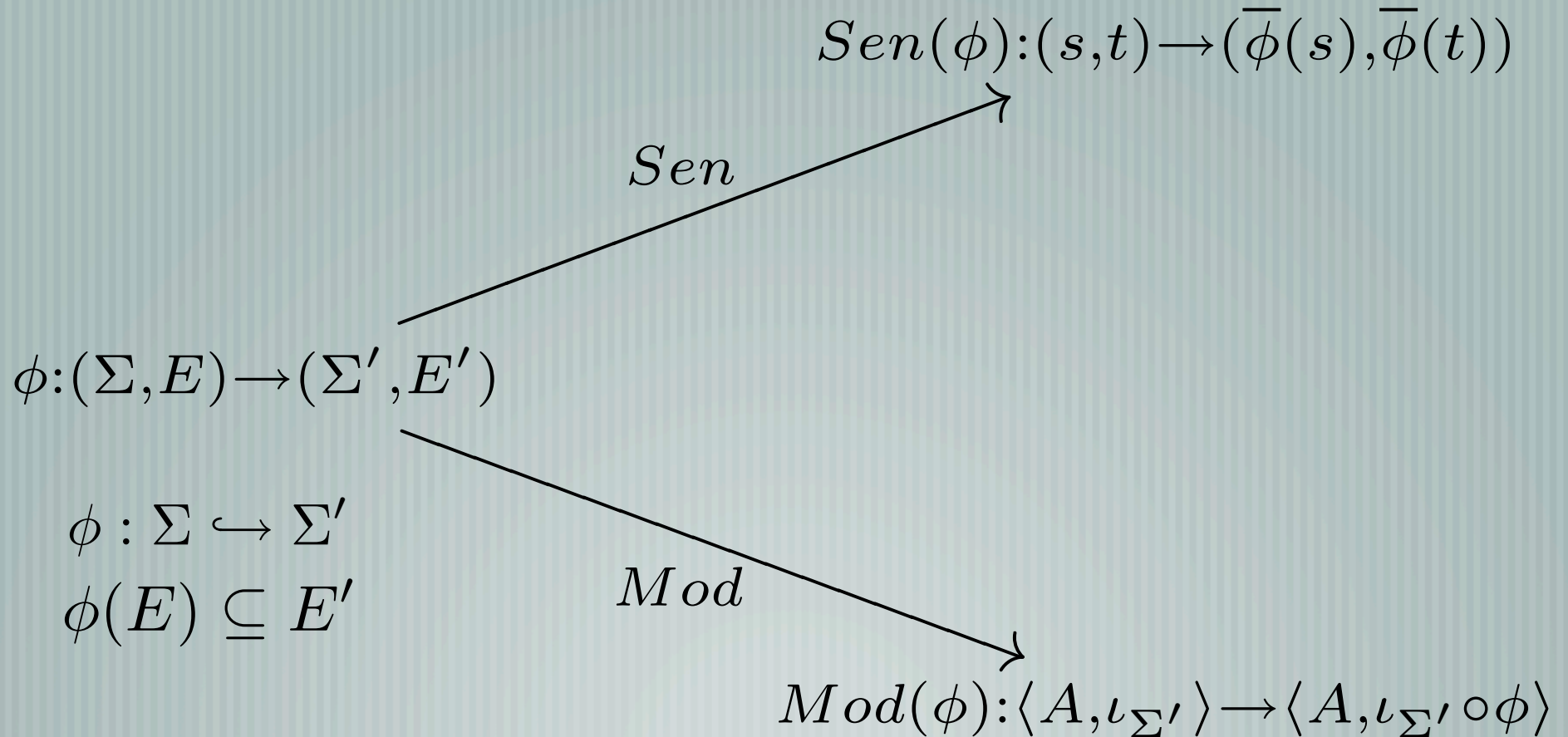
— This way DPO rewriting gets an entailment system



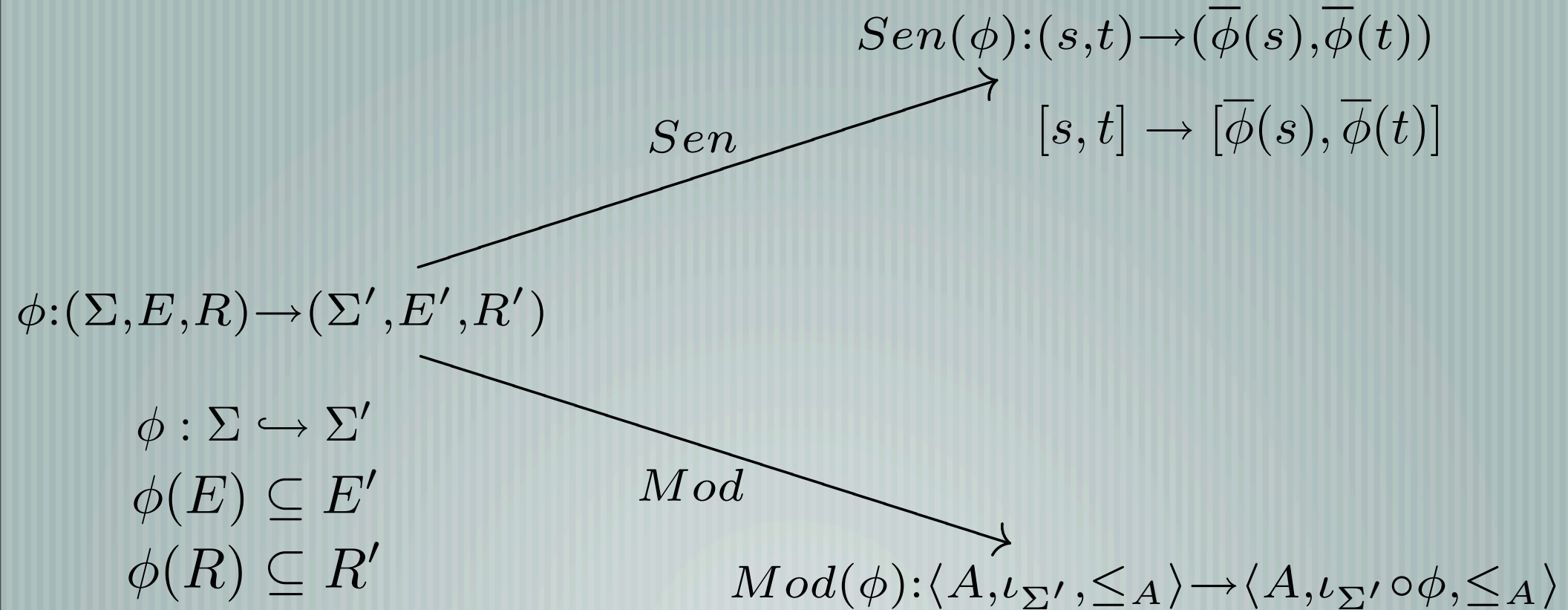
# very shortly, institutions

- [ A category  $Sign$  of specifications
- [ a functor  $Sen: Sign \rightarrow Set$  for sentences
- [ a functor  $Mod: Sign \rightarrow Cat^{op}$  for models
- [ a (satisfiability) relation  $\models_\Sigma$  on  $|Mod(\Sigma)| \times Sen(\Sigma)$
- [ a coherence axiom  $\forall \phi: \Sigma \rightarrow \Sigma', e \in Sen(\Sigma), M' \in Mod(\Sigma')$   
$$M' \models_{\Sigma'} Sen(\phi)(e) \Leftrightarrow Mod(\phi)(M') \models_\Sigma e$$

# institution for algebraic specs.



# institution for rewriting specs.



# the easy way out...

- [ exploit the categorical laws...
  - sentences as pairs of arrows in  $DGSTh(\Sigma)$  (same homs.)
  - models as dgs-monoidal categories
    - obvious satisfiability
    - reductions via order enrichment
- [ unsatisfactory: looking for a “concrete” model characterisation, in terms of “classical” algebraic models (algebras for specs.)

# a functorial detour

- [ The algebraic theory  $Th(\Sigma)$  is concretely defined as
  - lists of vars as objects, (tuples of) typed terms as arrows
  - term substitution as composition
- [ (the theory is also the free cartesian category over  $\Sigma$ )
- [ Algebras over  $\Sigma$  and axioms in  $E$  as functors
$$M \in [Th(\Sigma) \rightarrow Set]_E^\times$$
- [ product and axioms preserving (homs as natural transfs.)

# a functorial detour, II

— [ PreAlgebras as rule-preserving functors

$$M \in [Th(\Sigma) \rightarrow Pre]_{E,R}^{\times}$$

$$s \rightarrow t \in R \Leftrightarrow \forall X. M(s) \leq M(t)$$

— (still homomorphisms as natural transformations)

— [ How to generalize? Note that functors

$$M \in [Th(\Sigma) \rightarrow Rel]_E^{\times}$$

— [ still define algebras!!

# alternative take on $Th(\Sigma)$

— [  $Th(\Sigma)$  is the free symmetric (strict) monoidal category equipped with symmetric monoidal natural transformations

$$\nabla_a : a \longrightarrow a \otimes a \qquad !_a : a \longrightarrow e$$

— [ (intuitively representing pairing tuple  $\langle x, x \rangle$  and empty tuple)

# explicit definition of a theory

$$\begin{array}{ccc}
 a & \xrightarrow{\nabla_a} & a \otimes a \\
 \downarrow s & & \downarrow s \otimes s \\
 b & \xrightarrow{\nabla_b} & b \otimes b
 \end{array}$$

$$\begin{array}{ccc}
 a & \xrightarrow{!_a} & e \\
 \downarrow s & & \downarrow e \\
 b & \xrightarrow{!_b} & e
 \end{array}$$

$$\begin{array}{ccc}
 a & \xrightarrow{\nabla_a} & a \otimes a \\
 \downarrow \nabla_a & & \downarrow a \otimes \nabla_a \\
 a \otimes a & \xrightarrow{\nabla_a \otimes a} & a \otimes a \otimes a
 \end{array}$$

$$\begin{array}{ccc}
 a & \xrightarrow{\nabla_a} & a \otimes a \\
 \searrow \nabla_a & & \downarrow \gamma_{a,a} \\
 & & a \otimes a
 \end{array}$$

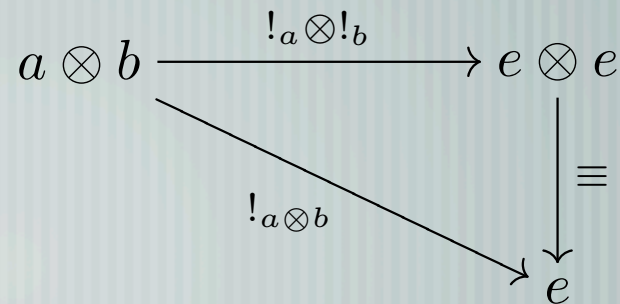
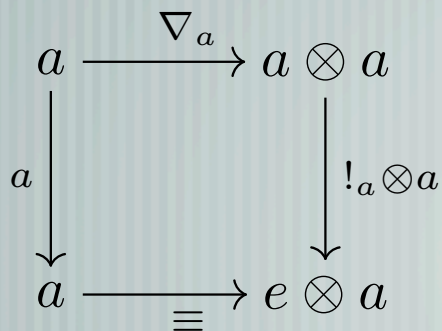
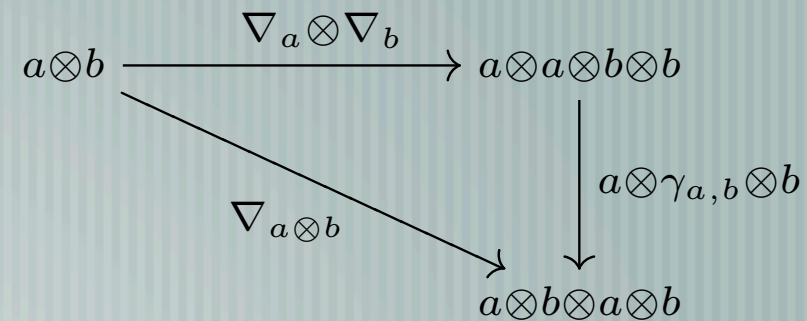
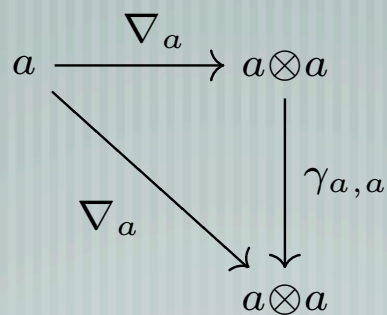
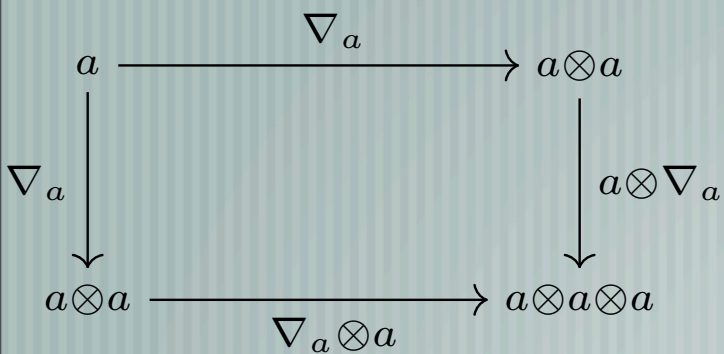
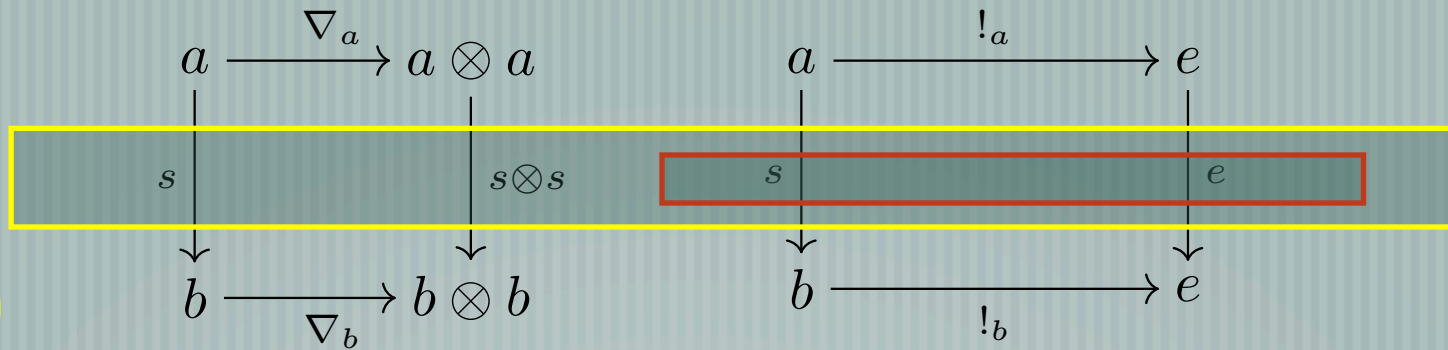
$$\begin{array}{ccc}
 a \otimes b & \xrightarrow{\nabla_a \otimes \nabla_b} & a \otimes a \otimes b \otimes b \\
 \searrow \nabla_{a \otimes b} & & \downarrow a \otimes \gamma_{a,b} \otimes b \\
 & & a \otimes b \otimes a \otimes b
 \end{array}$$

$$\begin{array}{ccc}
 a & \xrightarrow{\nabla_a} & a \otimes a \\
 \downarrow a & & \downarrow !_a \otimes a \\
 a & \xrightarrow{\equiv} & e \otimes a
 \end{array}$$

$$\begin{array}{ccc}
 a \otimes b & \xrightarrow{!_a \otimes !_b} & e \otimes e \\
 \searrow !_a \otimes b & & \downarrow \equiv \\
 & & e
 \end{array}$$



# two alternative takes



# another alternative take

— [  $DGSTh(\Sigma)$  as self-dual  $GSTh(\Sigma)$  satisfying

$$\begin{array}{ccc}
 a \otimes a & \xrightarrow{\nabla_a^{op}} & a \\
 \nabla_a \otimes a \downarrow & & \downarrow \nabla_a \\
 a \otimes a \otimes a & \xrightarrow{a \otimes \nabla_a^{op}} & a \otimes a
 \end{array}$$

$$\begin{array}{ccc}
 a & \xrightarrow{\nabla_a} & a \otimes a \\
 a \searrow & & \downarrow \nabla_a^{op} \\
 & & a
 \end{array}$$

# some characterization results

- [ arrows in  $DGSTh(\Sigma)$  are (isomorphic classes of) cospans of graphs (typed over  $\Sigma$ )
- [ arrows in  $GSTh(\Sigma)$  are (isomorphic classes of) cospans of term graphs (typed over  $\Sigma$ )
- [ arrows in  $GTh(\Sigma)$  are conditioned terms  $s \mid D$  (over  $\Sigma$ )
  - $s$  a term (the functional)
  - $D$  a sub-term closed set of terms (the domain restriction)

# functorial characterizations

Partial algebras with  $\perp$ -preserving operators, tight homomorphisms and conditioned Kleene (in)equations

Multialgebras with tight point-to-set operators, tight point-to-point homomorphisms and “term graph” (in)equations

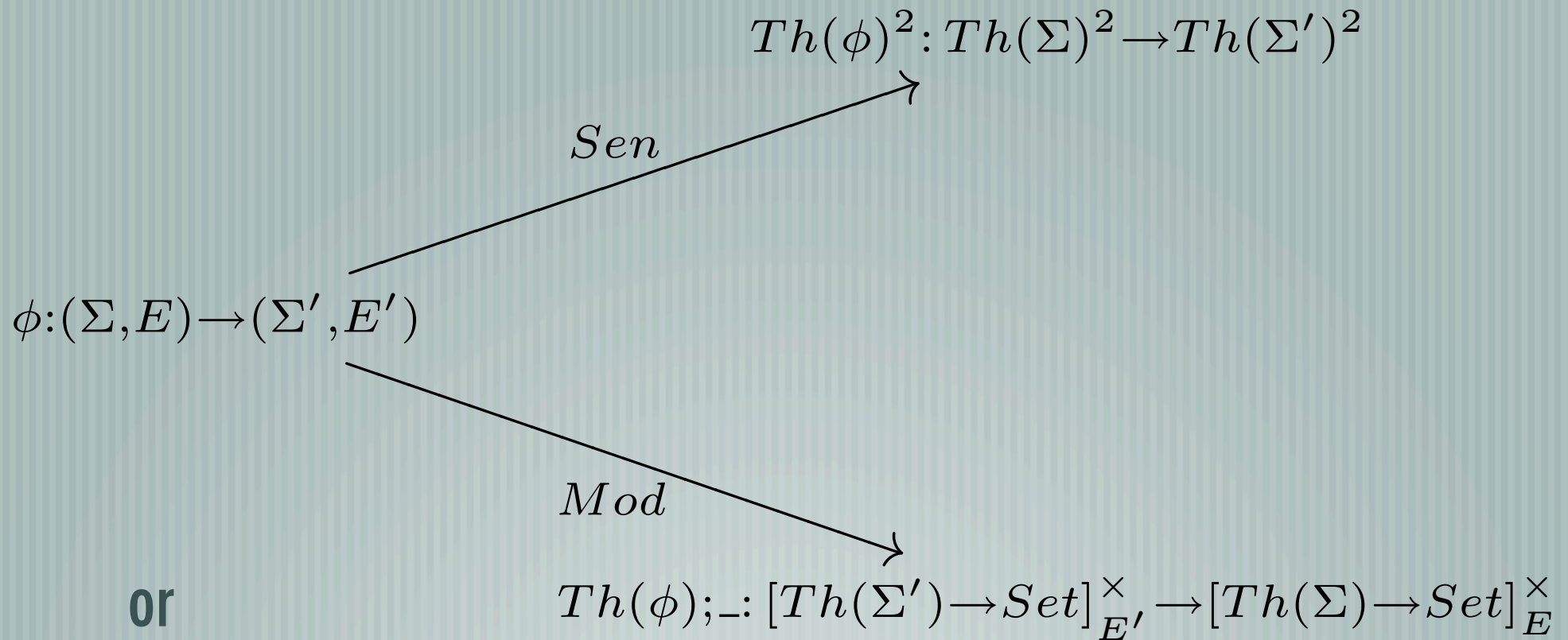
Multialgebras with tight point-to-set operators, tight point-to-point homomorphisms and “graph” (in)equations

$$[GTh(\Sigma) \rightarrow Set_{\perp}]_E^{\times}$$

$$[DGSTh(\Sigma) \rightarrow 2^{Set}]_E^{\times}$$

$$[GSTh(\Sigma) \rightarrow 2^{Set}]_E^{\times}$$

# back to institutions



$$[GTh(\Sigma) \rightarrow Set_\perp]_E^\times$$

**or**

$$[(D)GSTh(\Sigma) \rightarrow 2^{Set}]_E^\times$$

# on entailment systems

— [ Claim: complete entailment system for partial algebras

$$\frac{s \mid D_s \equiv t \mid D_t}{s \mid D_s \cup D \equiv t \mid D_t \cup D} \quad \frac{u_i \mid D_u \quad (s \mid D_s, t \mid D_t) \in E}{s[\overline{u}/\overline{x}] \mid D_s[\overline{u}/\overline{x}] \cup D_u \equiv t[\overline{u}/\overline{x}] \mid D_t[\overline{u}/\overline{x}] \cup D_u}$$

— [ Conjecture: complete entailment system for multi-algebras

$GSTh(\Sigma)$  plus

$$\begin{array}{ccc} a & \xrightarrow{\nabla_a} & a \otimes a \\ s \downarrow & & \downarrow s \otimes s \\ b & \xleftarrow{!_a \otimes a} & b \otimes b \end{array}$$

# back to insts. on preorders

$$\begin{array}{ccc}
 & 2Th(\phi)^2: 2Th(\Sigma)^2 \rightarrow 2Th(\Sigma')^2 & \\
 \nearrow^{Sen} & & \\
 \phi: (\Sigma, E, R) \rightarrow (\Sigma', E', R') & & \\
 \searrow_{Mod} & & \\
 & Th(\phi); -: [Th(\Sigma') \rightarrow Set]_{E', R'}^\times \rightarrow [Th(\Sigma) \rightarrow Set]_{E, R}^\times &
 \end{array}$$

or

$$\begin{array}{ccc}
 [GTh(\Sigma) \rightarrow Pre_{\perp}]_{E, R}^\times & \text{Smyth power-domain} & X \leq Y \Leftrightarrow \forall y \in Y. \exists x \in X. x \leq y \\
 \text{NOT! the bottom} & & \\
 \text{of the preorder} & \text{or} & [(D)GSTh(\Sigma) \rightarrow 2^{Pre}]_{E, R}^\times
 \end{array}$$

# preliminary conclusions

- [ uniform presentation of institutions for the DPO rewriting formalism over various graph-like structures
- [ sound and complete “abstract” entailment systems
- [ sound (possibly complete) “concrete” entailment systems



# to be addressed...

- [ completeness for the entailment system
  - (rewriting) interpretation for up-to garbage law
- [ tackling hyper-graphs and hyper-signature
  - (singular vs plural) interpretation for hyper-operators
- [ considering cospans of adhesive categories
  - free construction for suitable algebraic varieties