Subobject Transformation Systems and Elementary Net Systems

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Joint work with

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Outline

- Motivations
- Subobject Transformation Systems
- Elementary Net Systems as STSs
- Relations among productions in STSs
- From derivation trees of a GTS to an STS
- Analysis of dependencies using relations
- Future perspectives

Motivations

In rule-based computational formalisms, a fundamental ingredient of the theory is the analysis of computations:

- equivalences among computations
- partial order or branching structures (processes, unfoldings)

- **Term Rewriting Systems**: permutation equivalence
- **Petri Nets**: processes, unfolding
- **Graph Transformation Systems**: shift equivalence, processes, unfolding
- **Transformation Systems over Adhesive Categories**: ...

Motivations (cont'd)

The analysis of computations is based on the **analysis of relations among rule occurrences**.
Examples:

- **conflict**, **causal dependence** between transitions of Petri Nets
- **parallel/sequential independence**, **conflict**, **asymmetric conflict** among productions of GTS
- **co-causality**, **disabling**, **co-disabling** in TS over adhesive categories

Such relations are meaningful on the computation space of a system, sometimes represented as a system satisfying safety and acyclicity constraints (occurrence system).

Motivations (cont'd)

Natural questions arise:

- is **conflict** the negation of **parallel independence**?
- how are related **conflict** and **asymmetric conflict**?
- which relations can be defined in terms of the others? which ones are primitive?

A systematic study of such relations is missing...

We introduce Subobject Transformation Systems as a formal framework for the analysis of the relations among production occurrences of a **DPO system**.

Double-pushout rewriting in C

- A rule is a span of mono $q = L \stackrel{\alpha}{\leftarrow} K \stackrel{\beta}{\rightarrow} R$
- A match is an arrow $m : L \rightarrow G$
- Direct derivation $A \stackrel{\langle m,q \rangle}{\Longrightarrow} B$ if the following double-pushout diagram can be constructed:

$$\begin{array}{ccccc} L & \stackrel{\alpha}{\longleftarrow} & K & \stackrel{\beta}{\longrightarrow} & R \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle c} \\ A & \underset{\gamma}{\longleftarrow} & D & \underset{\delta}{\longrightarrow} & B \end{array}$$

Double-pushout rewriting in C

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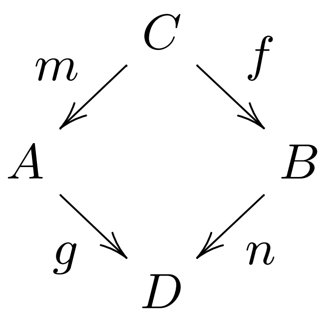
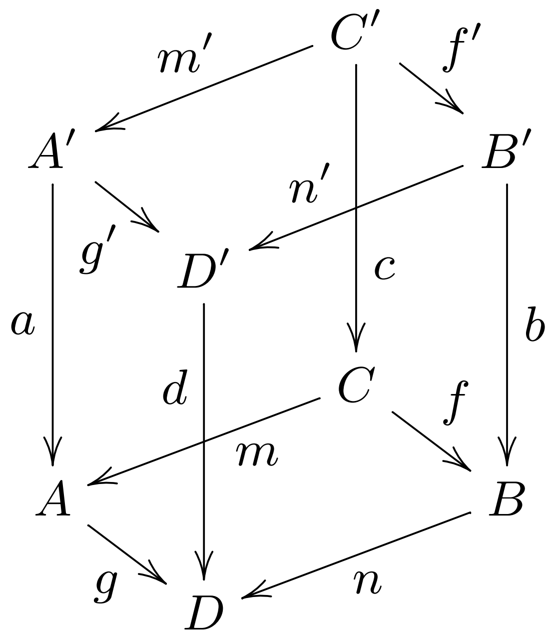
- Theory of DPO originally developed for $\mathbf{C} = \mathbf{Graph}$
- Recently generalized to *adhesive categories*

Adhesive categories

An adhesive category:

- has pullbacks, has pushouts along monos
- pushouts along monos are Van Kampen squares





DPO theory in quasi-adhesive cats

- Parallel and Sequential Independence
- Parallel Productions and Derivations
- Local Church-Rosser and Parallelism Theorem
- Shift Equivalence and Canonical Derivations
- Concurrency Theorem
- Embedding and extensions
- Critical pair lemma
- ...

The category of subobjects

Given category $\mathbf{C}$ and $T \in \mathbf{C}$, $\mathbf{Sub}(T)$ is the full subcategory of $\mathbf{C}/T$ with monos as objects.

Objects: $a : A \rightarrowtail T$, denoted simply as A

Arrows: $f : (a : A \rightarrowtail T) \to (b : B \rightarrowtail T)$ such that $b \circ f = a$, denoted as $A \subseteq B$, because it is a **preorder**

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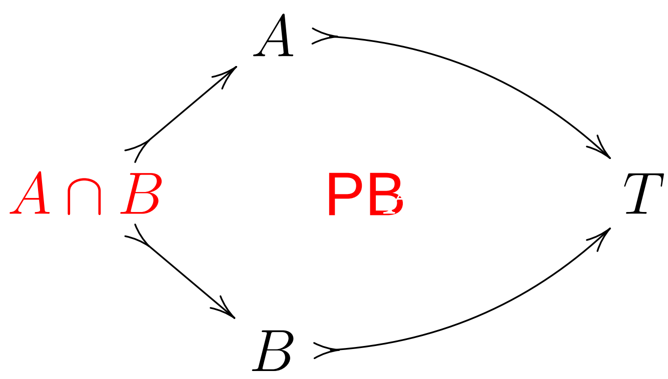
- If $\mathbf{C}$ has pullbacks, $\mathbf{Sub}(T)$ has **products** (**intersections**)



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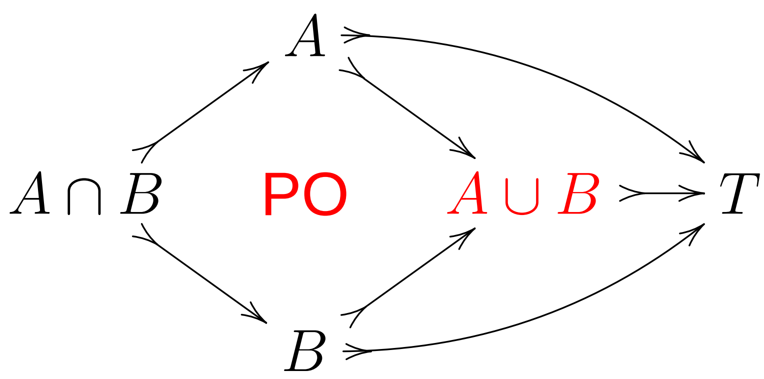
- If $\mathbf{C}$ is adhesive, $\mathbf{Sub}(T)$ has **coproducts** (**unions**), and it is **distributive**



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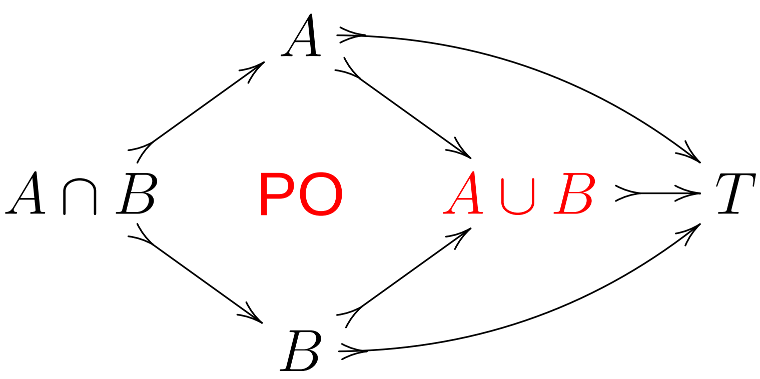


Note: $\mathbf{Sub}(T)$ is not adhesive!

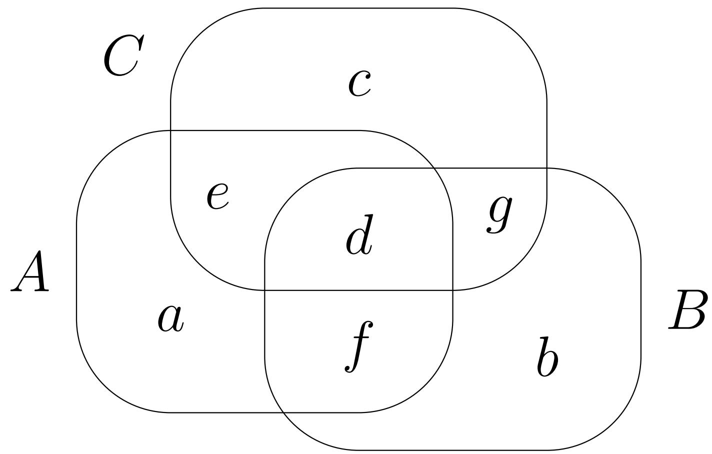
Representing subobjects with Venn diags.

If $\mathbf{Sub}(T)$ is distributive, the representation of subobjects of T using Venn diagrams is sound.

$A \cap (B \cup C) = \{d, e, f\} = (A \cap B) \cup (A \cap C)$

and

$A \cup (B \cap C) = \{a, d, e, f, g\} = (A \cup B) \cap (A \cup C)$



Note that since $\mathbf{Sub}(T)$ might not be a Boolean lattice, not all “zones” in the diagram correspond to subobjects (e.g., a).

Subobject Transformation System

A **Subobject Transformation System** (STS) over an adhesive category $\mathbf{C}$ is $\mathcal{S} = \langle T, P, \pi, S \rangle$, where:

- $T \in \mathbf{C}$ is a **type object**, P are the **production names**,
- $\pi : P \to \mathbf{Sub}(T)^{\cdot \leftarrow \cdot \rightarrow \cdot}$ maps each $p \in P$ to a span $L_p \supseteq K_p \subseteq R_p$ (often denoted $\langle L_p, K_p, R_p \rangle$)
- $S \in \mathbf{Sub}(T)$ is the **start object**.

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A production $\langle L, K, R \rangle$ is pure if $K = L \cap R$

Direct derivations

Given production $\pi(q) = \langle L, K, R \rangle$ and $G \in \mathbf{Sub}(T)$ such that $L \subseteq G$, there is a **direct derivation** $G \Rightarrow^q G'$ if there exists a context $D \in \mathbf{Sub}(T)$ such that:

$$
\begin{array}{llll}
(i) & L \cup D \cong G; & (iii) & D \cup R \cong G'; \\
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Diagrammatically...

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\begin{array}{ccccc}
L & \longleftarrow & L \cap D \cong K \cong D \cap R & \longrightarrow & R \\
\downarrow & & \downarrow & & \downarrow \\
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Yes, this is a double-pushout, but before that...

Elementary Net Systems

An **Elementary Net System** (ENS) is $N = \langle C, E, F, S_{in} \rangle$ where:

1. C and E are disjoint sets of **conditions** and **events**
2. $F \subseteq (C \times E) \cup (E \times C)$ is the **flow relation**
3. $S_{in} \subseteq C$ is the **initial configuration**

As usual, for $x \in C \cup E$, ${}^{\bullet}x = \{y \in C \cup E \mid \langle y, x \rangle \in F\}$
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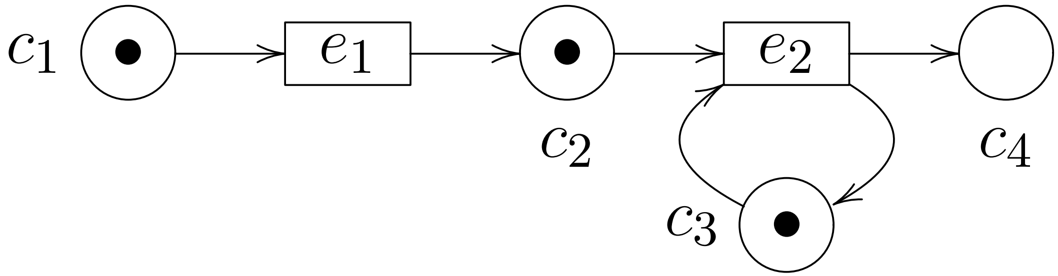
An event $e \in E$ **is enabled at** S if

$$ {}^\bullet e \subseteq S \qquad \wedge \qquad (e^\bullet \setminus {}^\bullet e) \cap S = \varnothing \qquad (\dagger) $$

In this case, e **can fire**: $\quad S\ [e\rangle\ (S \setminus {}^\bullet e) \cup e^\bullet$

A sample net



- e_2 is enabled at $\{c_1, c_2, c_3\}$:

$$^\bullet e_2 \subseteq \{c_1, c_2, c_3\} \quad \wedge \quad (e_2{}^\bullet \setminus {}^\bullet e_2) \cap \{c_1, c_2, c_3\} = \varnothing.$$

- e_1 is not enabled at $\{c_1, c_2, c_3\}$:

$$(e_1{}^\bullet \setminus {}^\bullet e_1) \cap \{c_1, c_2, c_3\} = \{c_2\} \neq \varnothing.$$

This is called a **contact situation**.

ENS as STS, firing as direct derivation

An ENS is an STS over **Set**, where productions have empty interface. The operational behaviour is the same.

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An ENS is an STS over $\mathbf{Set}$, where productions have empty interface. The operational behaviour is the same.

Given $N = \langle C, E, F, S_{in} \rangle$, consider the STS over $\mathbf{Set}$ $\mathcal{S}(N) = \langle C, E, \pi_N, S_{in} \rangle$, where

$$\text{for all } e \in E,\ \pi_N(e) = ({}^\bullet e \ \supseteq \ \varnothing \ \subseteq \ e^\bullet)$$

Then, $S[e\rangle S'$ **if and only if** $S \Rightarrow^e S'$.

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($\Rightarrow$) Let $D \stackrel{def}{=} S \setminus {}^\bullet e$. Since $\pi_N(e) = \langle {}^\bullet e, \varnothing, e^\bullet \rangle$, conditions $(i) - (iv)$ reduce to (i) $S \cong {}^\bullet e \cup (S \setminus {}^\bullet e)$, (ii) $S' \cong (S \setminus {}^\bullet e) \cup e^\bullet$, (iii) ${}^\bullet e \cap (S \setminus {}^\bullet e) = \varnothing$, and (iv) $(S \setminus {}^\bullet e) \cap e^\bullet = \varnothing$. Now, (i) and (iii) are tautologies, (ii) holds by the definition of firing, and (iv) is equivalent to $S \cap (e^\bullet \setminus {}^\bullet e) = \varnothing$, which is implied by (†).

($\Leftarrow$) Let $\langle L_e, K_e, R_e \rangle \stackrel{def}{=} \langle {}^\bullet e, \varnothing, e^\bullet \rangle$. The first conjunct of (†) is implied by condition (ii). The second one is equivalent to $S \cap R_e \subseteq L_e$, which is shown as follows:

$$S \cap R_e \stackrel{(i)}{\cong} (L_e \cup D) \cap R_e \cong (L_e \cap R_e) \cup (D \cap R_e) \stackrel{(iv)}{\cong} (L_e \cap R_e) \cup K_e \subseteq L_e.$$

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Then, $S[e\rangle S'$ **if and only if** $S \Rightarrow^e S'$.

Interestingly: $S \Rightarrow^e S'$ **implies absence of contact**

$$(e^{\bullet} \setminus {}^{\bullet}e) \cap S = \varnothing \quad \equiv \quad (R_e \setminus L_e) \cap S = \varnothing \quad \equiv \quad S \cap R_e \subseteq L_e$$

A methodological *intermezzo*...

- Relation between Place/Transitions nets and Graph Transformation Systems well understood, and exploited in several ways:
  - concurrent semantics (processes, unfoldings, ...)
  - verification based on approximations (Petri graphs)
  - from zero-safe nets to transactional GTS

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  - concurrent semantics (processes, unfoldings, ...)
  - verification based on approximations (Petri graphs)
  - from zero-safe nets to transactional GTS
- Claim: $$\frac{\textbf{GTS}}{\textbf{P/T nets}} = \frac{\textbf{STS}}{\textbf{ENS}}$$

  We start a new research thread: generalize results about ENS to arbitrary STS
  - analysis of structural properties of systems (contact-freeness, free choice, ...)
  - construction of contact-free system by complementation (???)

Back to foundations: a handy lemma

Given $\mathbf{C}$, adhesive, and $T \in \mathbf{C}$, the following are equivalent:

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & (1) & \downarrow \\ C & \rightarrowtail & D \end{array}$$

(1) Square (1) in $\mathbf{Sub}(T)$ is a pushout in $\mathbf{C}$

(2) $B \cap C \cong A$ and $D \cong B \cup C$

(3) $B \cap C \subseteq A$ and $D \subseteq B \cup C$.

This allows one to switch between diagrammatical and set-theoretical notation

Direct derivations as double pushouts

Recall: $G \Rightarrow^{q} G'$ if there exists a context $D \in \mathbf{Sub}(T)$ such that:

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Then $G \Rightarrow^{q} G'$ **if and only if**

- $G \cap R \subseteq L \in \mathbf{Sub}(T)$ (**no contact**), and
- there is a D such that (1) and (2) are pushouts in $\mathbf{C}$.

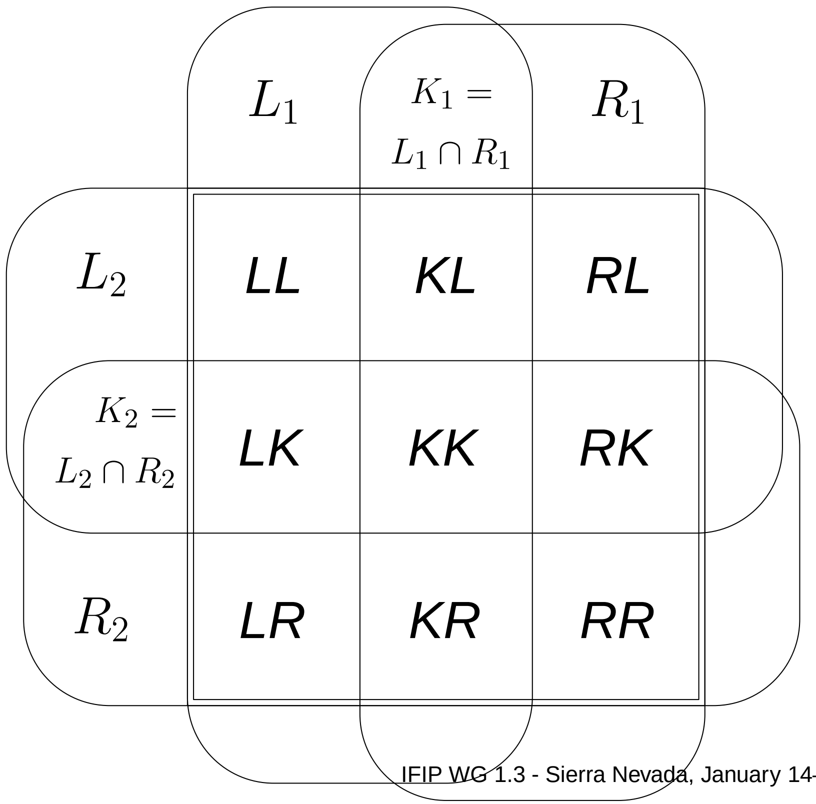
$$
\begin{array}{ccccc}
L & \xleftarrow{\;l\;} & K & \xrightarrow{\;r\;} & R \\
{\scriptstyle m}\downarrow & (1) & \downarrow{\scriptstyle k} & (2) & \downarrow{\scriptstyle n} \\
G & \xleftarrow{\;f\;} & D & \xrightarrow{\;g\;} & G'
\end{array}
$$

Relations among productions of an STS

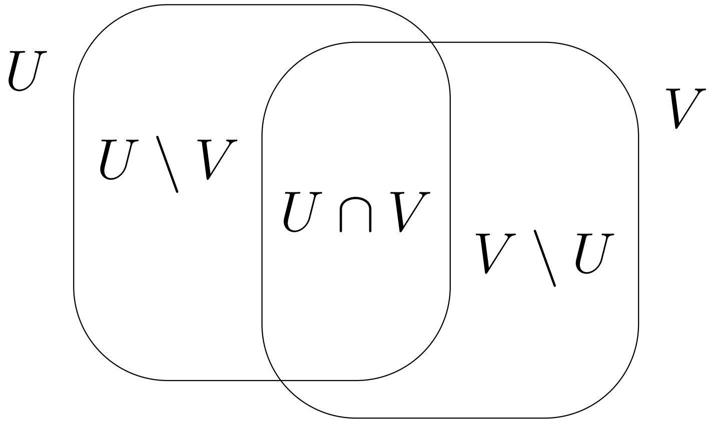
The intersection of two *productions* has nine "disjoint zones".

Two productions are completely independent if their intersection is preserved by both, i.e., $(L_1 \cup R_1) \cap (L_2 \cup R_2) \subseteq K_1 \cap K_2$

Each zone (but KK) determines a certain kind of dependency between the productions. For example, "non-emptiness" of LL means that they are in conflict.



Subobject difference as “regions”



Note that $U \setminus V$ and $U \setminus (U \cap V)$ denote the same zone.

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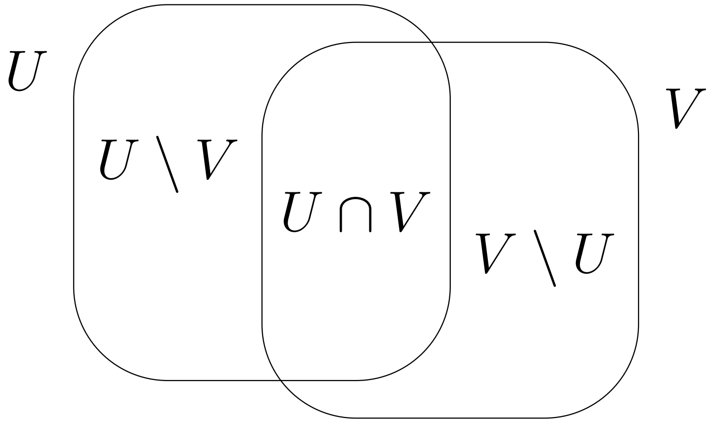

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Given subobjects U, V, W such that $W \cap U \subseteq V$, and Z such that $Z \subseteq U \cup V \cup W$, let

$$(U, V) \equiv (U \cup Z, V \cup W)$$

A **region** $U \setminus V$ is an equivalence class $[U, V]$.

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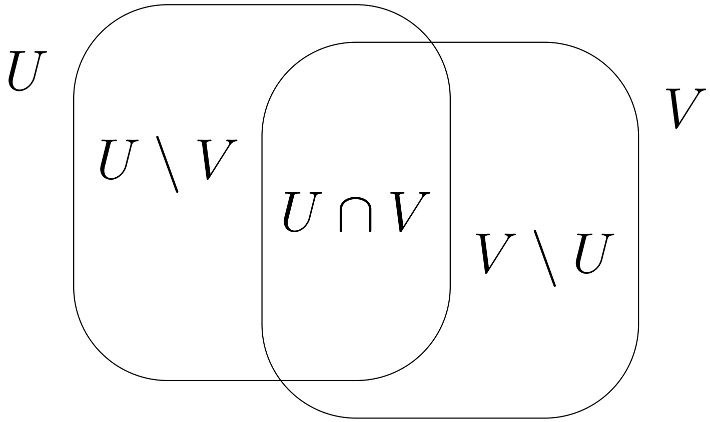

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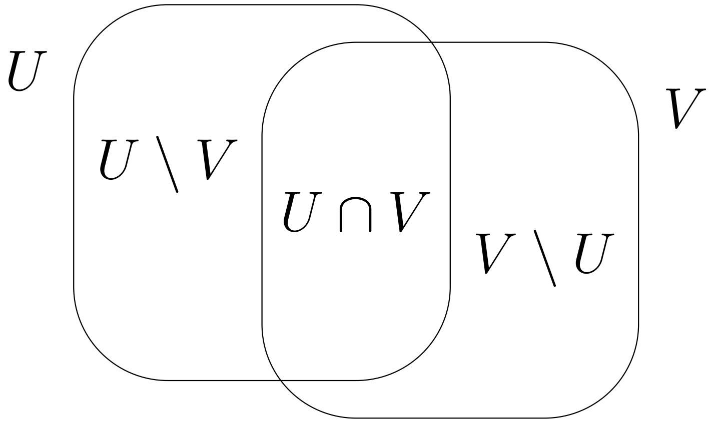

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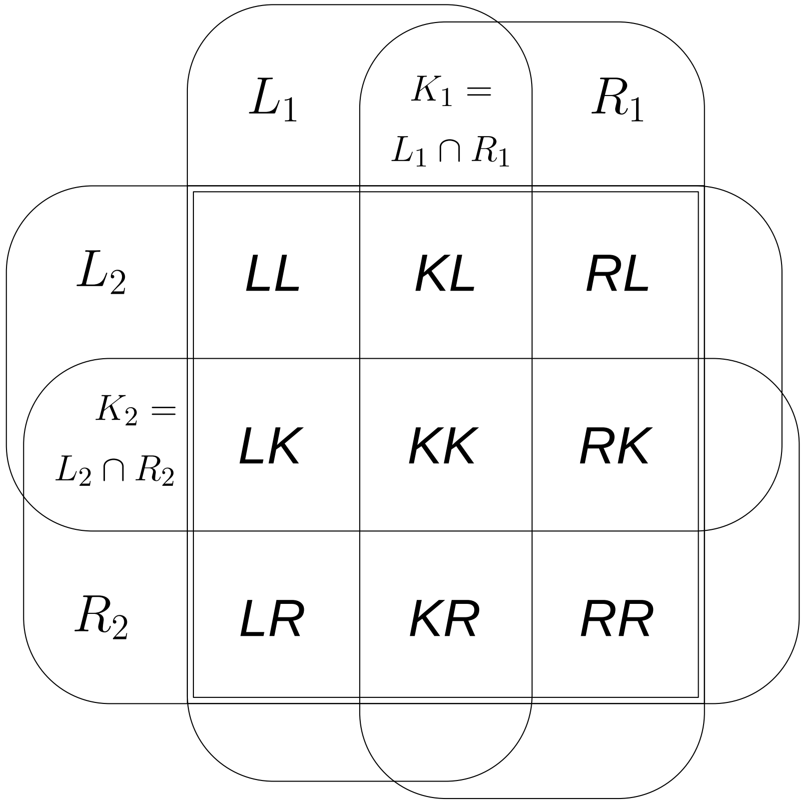
Region $U \setminus V$ is **empty** if $U \subseteq V$.

Useful fact: Given subobjects $U_1 \supseteq U_2 \supseteq U_3$, region $U_1 \setminus U_3$ is empty if and only if both regions $U_1 \setminus U_2$ and $U_2 \setminus U_3$ are empty.

Regions of the intersection of productions

The basic regions of the intersection are, with $X, Y \in \{L, R\}$:

- $XY = X_1 \cap Y_2 \setminus K_1 \cup K_2$,
- $KX = K_1 \cap X_2 \setminus K_2$, and
- $XK = X_1 \cap K_2 \setminus K_1$.



Non-basic regions are for example
RL+RK = $R_1 \cap L_2 \setminus K_1$, and
KL+RK = $(K_1 \cap L_2) \cup (K_2 \cap R_1) \setminus K_1 \cap K_2$.

The five basic relations

Name	Symbol	Inequation	Diagram in $\mathbf{C}$	Non-empty region
Conflict	$q_1 \curlywedge q_2$	$L_1 \cap L_2 \not\subseteq K_1 \cup K_2$	$\begin{array}{ccc} K_1 \cup K_2 & \rightarrowtail & L_1 \cup K_2 \\ \downarrow & \neg\text{PO} & \downarrow \\ K_1 \cup L_2 & \rightarrowtail & L_1 \cup L_2 \end{array}$	*LL*
Deactivation	$q_1 <_d q_2$	$K_1 \cap L_2 \not\subseteq K_2$	$\begin{array}{ccc} K_2 & \rightarrowtail & L_2 \\ \downarrow & \neg\text{PO} & \downarrow \\ K_1 \cup K_2 & \rightarrowtail & K_1 \cup L_2 \end{array}$	*KL*
Write causality	$q_1 <_{wc} q_2$	$R_1 \cap L_2 \not\subseteq K_1 \cup K_2$	$\begin{array}{ccc} K_1 \cup K_2 & \rightarrowtail & R_1 \cup K_2 \\ \downarrow & \neg\text{PO} & \downarrow \\ K_1 \cup L_2 & \rightarrowtail & R_1 \cup L_2 \end{array}$	*RL*
Read causality	$q_1 <_{rc} q_2$	$R_1 \cap K_2 \not\subseteq K_1$	$\begin{array}{ccc} K_1 & \rightarrowtail & R_1 \\ \downarrow & \neg\text{PO} & \downarrow \\ K_1 \cup K_2 & \rightarrowtail & R_1 \cup K_2 \end{array}$	*RK*
Backward conflict	$q_1 \curlyvee q_2$	$R_1 \cap R_2 \not\subseteq K_1 \cup K_2$	$\begin{array}{ccc} K_1 \cup K_2 & \rightarrowtail & R_1 \cup K_2 \\ \downarrow & \neg\text{PO} & \downarrow \\ K_1 \cup R_2 & \rightarrowtail & R_1 \cup R_2 \end{array}$	*RR*

Intuitive meaning of relations

Conflict: $q_1 \curlywedge q_2$ when there is an “item” consumed by both q_1 and q_2

Deactivation: $q_1 <_d q_2$ when there is an item preserved by q_1 and consumed by q_2; the firing of q_2 *deactivates*

Write causality: $q_1 <_{wc} q_2$ when there is an item produced by q_1 and consumed by q_2

Read causality: $q_1 <_{rc} q_2$ when there is an item produced by q_1 and preserved by q_2

Backwards conflict: $q_1 \curlyvee q_2$ when there is an item produced by both q_1 and q_2

Laws on relations

Given production q with $\pi(q) = \langle L, K, R \rangle$, let

$$q^{\mathrm{op}} \equiv \langle R, K, L \rangle$$

The following equivalences follow from the definitions:

1. $q_1 <_d q_2 \quad \Leftrightarrow \quad q_2^{\mathrm{op}} <_{rc} q_1^{\mathrm{op}}$;
2. $q_1 \curlyvee q_2 \quad \Leftrightarrow \quad q_2^{\mathrm{op}} \curlywedge q_1^{\mathrm{op}}$.
3. $q_1^{\mathrm{op}} <_{wc} q_2^{\mathrm{op}} \quad \Leftrightarrow \quad q_2 <_{wc} q_1$
4. $q_1 \curlywedge q_2 \quad \Leftrightarrow \quad q_1^{\mathrm{op}} <_{wc} q_2 \quad \Leftrightarrow \quad q_2^{\mathrm{op}} <_{wc} q_1$;
5. $q_1 \curlyvee q_2 \quad \Leftrightarrow \quad q_1 <_{wc} q_2^{\mathrm{op}} \quad \Leftrightarrow \quad q_2 <_{wc} q_1^{\mathrm{op}}$;
6. $q_1 <_{rc} q_2 \quad \Leftrightarrow \quad q_1 <_{rc} q_2^{\mathrm{op}}$;
7. $q_1 <_d q_2 \quad \Leftrightarrow \quad q_1^{\mathrm{op}} <_d q_2$;

Compound relations

Name	Symbol	Inequation	Diagram	Non-empty region
Causality	$q_1 <_c q_2$	$R_1 \cap L_2 \not\subseteq K_1$	$\begin{array}{ccc} K_1 & \rightarrowtail & R_1 \\ \downarrow & \neg\text{PO} & \downarrow \\ K_1 \cup L_2 & \rightarrowtail & R_1 \cup L_2 \end{array}$	*RL*+*RK*
Disabling	$q_1 \ll_d q_2$	$L_1 \cap L_2 \not\subseteq K_2$	$\begin{array}{ccc} K_2 & \rightarrowtail & L_2 \\ \downarrow & \neg\text{PO} & \downarrow \\ L_1 \cup K_2 & \rightarrowtail & L_1 \cup L_2 \end{array}$	*LL*+*KL*
Co-causality	$q_1 <^c q_2$	$L_2 \cap R_1 \not\subseteq K_2$	$\begin{array}{ccc} K_2 & \rightarrowtail & L_2 \\ \downarrow & \neg\text{PO} & \downarrow \\ R_1 \cup K_2 & \rightarrowtail & R_1 \cup L_2 \end{array}$	*KL*+*RL*
Co-disabling	$q_1 \ll^d q_2$	$R_1 \cap R_2 \not\subseteq K_1$	$\begin{array}{ccc} K_1 & \rightarrowtail & R_1 \\ \downarrow & \neg\text{PO} & \downarrow \\ K_1 \cup R_2 & \rightarrowtail & R_1 \cup R_2 \end{array}$	*RK*+*RR*

Compound relations via basic ones

Causality: $q_1 <_c q_2 \quad \Leftrightarrow \quad q_1 <_{rc} q_2 \vee q_1 <_{wc} q_2$;

Disabling: $q_1 \ll_d q_2 \quad \Leftrightarrow \quad q_1 <_d q_2 \vee q_2 \curlywedgedownarrow q_1$;

Co-causality: $q_1 <^c q_2 \quad \Leftrightarrow \quad q_1 <_d q_2 \vee q_2 <_{wc} q_1$;

Co-disabling: $q_1 \ll^d q_2 \quad \Leftrightarrow \quad q_1 <_{wc} q_2 \vee q_1 \curlyveedownarrow q_2$.

Compound relations via basic ones

Causality: $q_1 <_c q_2 \quad \Leftrightarrow \quad q_1 <_{rc} q_2 \vee q_1 <_{wc} q_2$;

Disabling: $q_1 \lll_d q_2 \quad \Leftrightarrow \quad q_1 <_d q_2 \vee q_2 \curlywedge q_1$;

Co-causality: $q_1 <^c q_2 \quad \Leftrightarrow \quad q_1 <_d q_2 \vee q_2 <_{wc} q_1$;

Co-disabling: $q_1 \lll^d q_2 \quad \Leftrightarrow \quad q_1 <_{wc} q_2 \vee q_1 \curlyvee q_2$.

Sample proof for **Causality**:

In terms of regions, the statement means *region RL+RK is not empty iff either RL or RK is not empty*, and thus *region RL+RK is empty iff RL and RK are empty*. Now let $U_1 = R_1 \cap L_2,\ U_2 = (K_1 \cap L_2) \cup (R_1 \cap K_2)$ and $U_3 = K_1 \cap L_2$. It is straightforward to check that *RL* represents $U_1 \setminus U_2$, *RK* represents $U_2 \setminus U_3$, and *RL+RK* represents $U_1 \setminus U_3$; furthermore since $U_1 \supseteq U_2 \supseteq U_3$, we can conclude.

Independence in STSs

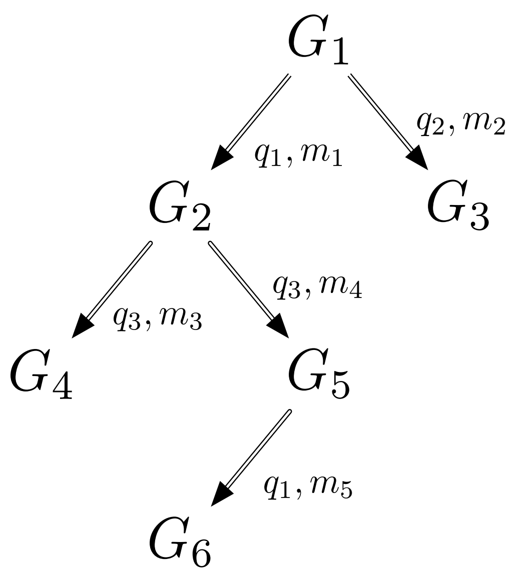
Two productions q_1 and q_2 are *independent*, denoted $q_1 \Diamond q_2$, if

$$(L_1 \cup R_1) \cap (L_2 \cup R_2) \subseteq (K_1 \cap K_2)$$

- It is possible to show that $q_1 \Diamond q_2$ if and only if they are not related by any of the basic relations (reasoning in terms of emptiness of regions)
- Several characterization of independence (similar to parallel and sequential independence)
- Local Church-Rosser theorem for STSs

From derivation trees to STSs

Obtaining an STS from a derivation tree. Generalization of the construction of a process from a given derivation.

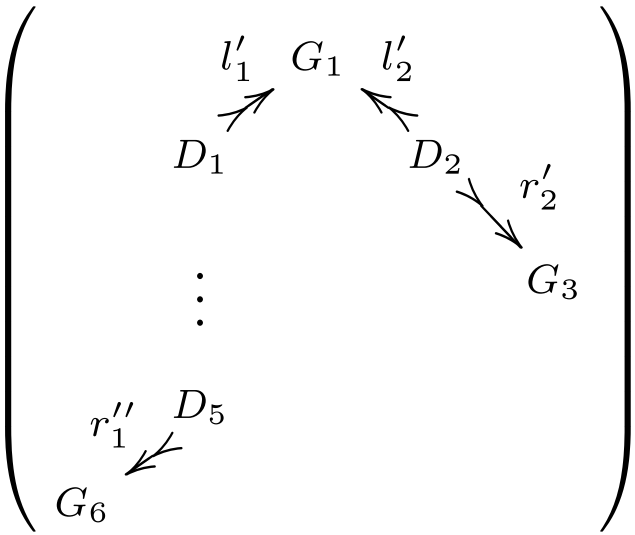


$\longmapsto$

$$\langle T, P, \pi, G_1 \rangle$$

$$P = \{q_{11}, q_{22}, q_{33}, q_{34}, q_{15}\}$$

$T = \text{colim}$



More formally...

- Given an adhesive grammar $\mathcal{G}$ over $\mathbf{C}$ we define the strict monoidal category of derivation trees $\mathbf{DerTree}(\mathcal{G})$:

  Objects: finite words of objects of $\mathbf{C}$

  Arrows: derivation forests

- For a given object $S \in \mathbf{C}$ and a derivation tree rooted at S, we build an STS having as type graph the colimit of the diagram in $\mathbf{C}$ witnessing the derivation tree.
- The construction extends to a functor

$$\mathrm{Prc} : S/\mathbf{DerTree}(\mathcal{G}) \to \mathbf{STS}$$

Analysis of derivations

The dependencies among the steps in a derivation tree can be faithfully analyzed in the generated STS.
Suppose that $\mathcal{G}$ is an adhesive grammar. Let α be a derivation tree in $\mathcal{G}$ with root S ($\alpha \in S/\mathbf{DerTree}(\mathcal{G})$).

1. Let $C_1 \Rightarrow^{q_1} C_2 \Rightarrow^{q_2} C_3$ be two steps in α, and let q'_1 and q'_2 be the corresponding productions in $\mathrm{Prc}(\alpha)$. Then:

   they are sequential independent iff $q'_1 \Diamond q'_2$ iff
   $(q'_1 \not\lessdot_{rc} q'_2) \wedge (q'_1 \not\lessdot_{wc} q'_2) \wedge (q'_1 \not\lessdot_{d} q'_2)$.

2. Let $C_1 \Rightarrow^{q_1} C_2$, $C_1 \Rightarrow^{q_2} C_3$ be two steps in α, and let q'_1 and q'_2 be the corresponding productions in $\mathrm{Prc}(\alpha)$. Then:

   they are parallel independent iff $q'_1 \Diamond q'_2$ iff
   $\neg(q'_1 \curlywedge q'_2) \wedge (q'_1 \not\lessdot_{d} q'_2) \wedge (q'_2 \not\lessdot_{d} q'_1)$.

Conclusions

- We introduced Subobject Transformation Systems as DPO in the lattice of subobjects of an object of an adhesive category.
- STS provide a formal framework for the analysis of relationships among production occurrences in the derivation space of a DPO system
- They provide an alternative syntax (set-theoretical, using Venn diagrams) w.r.t. to the standard one based on diagram chasing
- They generalize Elementary Net Systems in the same way Adhesive DPO Transformation Systems generalize Place/Transition nets

Future Work

- Developing an algebra of regions, and exploring its usefulness
- Exploring in depth the relation with ENS systems, generalizing their theory to arbitrary STSs
- Generalizing constructions and results to infinite derivation trees