# Flat Coalgebraic Fixed Point Logics

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# Introduction: Modal Logic

Modal logics are a central logical tool in computer science and AI:

- Applications e.g. to
  - reactive systems
  - knowledge representation
  - multi-agent systems
- May be tailored to offer the right expressive means for a given domain
- Often have good computational properties (unlike, e.g., FOL/HOL)

# Introduction: Coalgebra

- Large variety of domain-specific logics of different syntax, semantics, and complexity
- Coalgebra acts as a unifying semantic theory of modal logic and supports
  - generic complete deduction systems
  - generic decidability results
  - generic algorithms and complexity bounds
  - generic implementations
  - systematic logic design

# Modal Logic (Version 1)



Modal logic extends (classical) propositional logic with additional operators, e.g.

$$\phi ::= \bot \mid p \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Box\phi \qquad (p \in P)$$

with $\Box\phi$ read e.g. 'necessarily $\phi$'
(dually: $\Diamond\phi :\equiv \neg\Box\neg\phi$ 'possibly $\phi$')

Modal logic is a logic of relational structures:

- Models $((X, R), V)$, where
  - $(X, R)$ is a Kripke frame, i.e. $R \subseteq X \times X$
  - $V$ is a valuation $P \to \mathcal{P}(X)$.
- Satisfaction is per state $x \in X$

$$x \models \Box\phi \text{ iff } \forall y.\, xRy \Rightarrow y \models \phi.$$

$\Box$ satisfies normality:

$$\Box\top \qquad \Box(a \wedge b) \leftrightarrow (\Box a \wedge \Box b)$$

# Many Logics are Normal

- Temporal logics
  - $\Box\phi =: G\phi$ = 'Generally/Forever $\phi$'
- Epistemic logics
  - $\Box\phi$ = 'I know that $\phi$'
- Logics of Belief
- Standard Deontic Logic
  - $\Box\phi =: O\phi$ = 'It is obligatory that $\phi$'
- Description logic
  - $\mathsf{Mother} = \mathsf{Woman} \wedge \exists \mathsf{hasChild}.\top$

# ...and Many Are Not

- Graded modal logic: $\Box_k \phi$ = 'with at most $k$ exceptions $\phi$'
  - Does have relational semantics, but better: multigraphs $\bullet \xrightarrow{n} \bullet$
- Probabilistic modal logic: $L_p \phi$ = 'with probability $\geq p$, $\phi$'
- Agent logics $E_a \phi$ = 'agent $a$ brings it about that $\phi$'

$$\neg E_a \top$$

- Deontic logics for dilemmas:

$$O\neg\text{leave} \wedge O\text{leave} \not\rightarrow O\text{suicide}$$

- Conditional logic: $a \Rightarrow b$ 'if $a$, then normally $b$'

$$(\text{monday} \Rightarrow \text{bus}) \not\rightarrow (\text{monday} \wedge \text{strike} \Rightarrow \text{bus})$$

# Modal Logic, Version 2

Neighbourhood semantics:

- Models $(X, R, V)$ where
  - $(X, R)$ neighbourhood frame, i.e.

$$R \subseteq X \times \mathcal{P}(X)$$

- $x \models \Box\phi$ iff $xR[\![\phi]\!]$ where $[\![\phi]\!] = \{y \in X \mid y \models \phi\}$.

This does cover nearly everything, but

- is unintuitive
- does not capture the intended semantics
  - and in fact gives up nearly all semantic structure
- is often unsuitable for metatheory and efficient reasoning.

$\rightarrow$ Look for a general framework that retains semantic structure

# A Reformulation of Kripke Semantics

- Frames are $\mathcal{P}$-coalgebras

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$$\xi(x) \in \{A \in \mathcal{P}(X) \mid A \subseteq [\![\phi]\!]\} =: \underbrace{[\![\Box]\!]_X}_{\text{Predicate Lifting}}([\![\phi]\!])$$

# Modal Logic, Version 3: Coalgebraic Modal Logic

- General modal similarity types (collections of modal operators)
- Abstract over the type of systems:
  - Set functor (parametrised datatype) $T : \mathbf{Set} \to \mathbf{Set}$
  - Systems = $T$-coalgebras

$$\xi : X \to TX$$

- Abstract over the interpretation of modal operators $L$:
  - Predicate liftings $[\![L]\!]_X : \mathcal{P}(X) \to \mathcal{P}(TX)$, natural in $X$
  - $x \models L\phi$ iff

$$\xi(x) \in [\![L]\!]_X([\![\phi]\!])$$

# Nearly Everything is Coalgebraic

| Logic | Systems | Syntax | Functor |
|---|---|---|---|
| Normal modal logics | Kripke frames | $\Box\phi$ | powerset $\mathcal{P}(X)$ |
| Probabilistic modal logics | Markov chains | $L_p\phi$ <br> $\sum a_i P(\phi_i) \geq b$ | distributions $D(X)$ |
| Graded modal logics | Multigraphs | $\geq nR.\phi$ <br> $\sum a_i \#(\phi_i) \geq b$ | multisets $\mathcal{B}(X) = X \to \mathbb{N}_\infty$ |
| Conditional logics | Conditional frames | $\phi \Rightarrow \psi$ | selection functions $\mathcal{P}(X) \to \mathcal{P}(X)$ |
| Classical modal logics | Neighbourhood frames | $\Box\phi$ | neighbourhoods $\mathcal{P}(\mathcal{P}(X))$ |
| Coalition logic | Game frames | $[C]\phi$ | Games $\exists(S_i).(\prod S_i \to X)$ |

(Schröder/Pattinson/Cirstea/Kurz/Venema et al. 2004–2008)

# Example: Alternating Temporal Logic (ATL)

(Alur et al. JACM 2002)

- Signature: $[C]$ for $C \subseteq N$ coalition
  - $[C]\phi$ '$C$ can force $\phi$ in the next step'
- Temporal operators such as $\langle\langle C \rangle\rangle F \phi$ '$C$ can eventually force $\phi$', e.g.
  - $\langle\langle \{a\} \rangle\rangle F\,$access
    'Agent $a$ can access some resource autonomously'
  - $\langle\langle \{a, s\} \rangle\rangle F\,$access
    'Agent $a$ can access some resource in collaboration with server $s$'
- Functor:

$$F(X) = \exists S_1, \dots, S_N. \left(\prod_{i \in N} S_i\right) \to X$$

- Predicate liftings

$$[\![ [C] ]\!]_X(A) = \{f \in F(X) \mid \exists \sigma_C. \forall \sigma_{N-C}. f(\sigma_C, \sigma_{N-C}) \in A\}$$

# One-Step Rules

One-step logic: $V$ set of prop. var., $\Sigma(V) = \{La \mid a \in V, L \in \Sigma\}$.
Given $\tau : V \to \mathcal{P}(X)$, interpret

- propositional formulas $\varphi$ over $V$ as $[\![\varphi]\!]\tau \subseteq X$
- propositional formulas $\psi$ over $\Sigma(V)$ as $[\![\psi]\!]\tau \subseteq TX$ by

$$[\![La]\!]\tau = [\![L]\!]_X \tau(a)$$

One-step rules $\dfrac{\varphi}{\psi}$ over $V$:

$\varphi$ propositional over $V$

$\psi$ clause over $\Sigma(V)$

$\varphi/\psi$ one-step sound if $[\![\varphi]\!]\tau = X \implies [\![\psi]\!]\tau = TX$.

# The Cut Rule

$$\frac{A \rightarrow C \qquad C \rightarrow B}{A \rightarrow B}$$

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— not so good for proof search.

# One-Step Cut-Free Completeness



A set $\mathcal{R}$ of one-step rules is one-step cut-free complete (OSCC) if for every clause $\chi$ over $\Sigma(V)$,

$$[\![\chi]\!]\tau = TX \implies \exists \varphi/\psi \in \mathcal{R}, \sigma : V \to V.$$
$$[\![\varphi\sigma]\!]\tau = X, \quad \psi\sigma \text{ contracted}, \quad \psi\sigma \subseteq \chi.$$

- $\mathcal{R}$ one-step cut-free complete $\iff$ $\mathcal{R}$ absorbs cut and contraction
- OSCC rule sets
  - induce model constructions, in which states are demands, i.e. conj. clauses of conclusions of dual tableau rules $\bar{\psi}/\bar{\varphi}$
  - yield cut-free complete modal deduction systems $\to$ proof search

(Schröder/Pattinson LICS 06, ACM TOCL 09; Pattinson/Schröder I&C)

# Examples: OSCC Rule Sets

$K$ ($\Box$ with Kripke semantics):

$$\frac{\bigwedge_{i=1}^{n} a_i \to b}{\bigwedge_{i=1}^{n} \Box a_i \to \Box b} \quad (n \geq 0)$$

Alternating-time logic:

$$\frac{\bigvee_{i=1}^{n} \neg a_j}{\bigvee_{i=1}^{n} \neg [C_j] a_j} \qquad \frac{\bigwedge_{i=1}^{n} a_j \to (b \vee \bigvee_{j=1}^{m} c_j)}{\bigwedge_{i=1}^{n} [C_j] a_j \to ([D] b \vee \bigvee_{j=1}^{m} [N] c_j)} \qquad \begin{array}{l}(m, n \geq 0, C_j \subseteq D, \\ C_i \cap C_j = \emptyset \text{ for } i \neq j)\end{array}$$

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Sichere Kognitive Systeme

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# Generic Algorithms via OSCC Rule Sets

- PSPACE for next-step-logics
- PSPACE for coalgebraic hybrid logic
- EXPTIME for coalgebraic description logics (i.e. with TBoxes)
- Completeness and EXPTIME global caching for flat fixed point logics via $\mathcal{O}$-adjointness (Schröder/Venema 2010)
  - Alternating $\mu$-calculus (Alur et al. 2002)
  - Graded $\mu$-calculus (Kupferman et al. 2002)

# Flat Coalgebraic Fixed Point Logics

Flat fixed point operators

$$\sharp_\gamma(\varphi) \equiv \mu x.\, \gamma(\varphi, x)$$
$$\flat_\gamma(\varphi) \equiv \nu x.\, \gamma(\varphi, x) \qquad (\gamma \text{ modal})$$

$\rightarrow$ fragments of single-variable coalgebraic $\mu$-calculus.

E.g.

- CTL: $AF\varphi = \sharp_{p \vee \Box x}\varphi$
- $\flat_{p \wedge \Box\Box x}$ not in CTL*
- ATL: $\langle\langle C \rangle\rangle F\phi = \sharp_{p \vee [C]x}\varphi$
- Graded $\mu$-calculus (Kupferman et al. 2002):

$$\sharp_{p \vee \Diamond_2 x}\phi$$

'the current state is the root of a binary tree whose leaves satisfy $\phi$'.

# The Kozen-Park Axioms

Briefly: '$\sharp_\gamma(\phi)$ is a least fixed point', i.e.:

Unfolding:

$$\sharp_\gamma\varphi \leftrightarrow \gamma(\varphi, \sharp_\gamma\varphi)$$

Fixed-point induction:

$$\frac{\gamma(\varphi,\chi) \to \chi}{\sharp_\gamma(\varphi) \to \chi}$$

Are these complete?

- Do imply that $\sharp_\gamma(\phi)$ is a least fixed point in the Lindenbaum algebra

# Strategy for the Completeness Proof

- Show constructivity of the Lindenbaum algebra:

$$\sharp_\gamma(\varphi) = \bigvee_{i<\omega} \gamma(\varphi)^i(\bot)$$

via $\mathcal{O}$-adjointness of $\gamma(\varphi)$: for all $\psi$ there is a finite set $G_{\gamma(\phi)}(\psi)$ s.t.

$$\gamma(\varphi,\rho) \leq \psi \iff \rho \leq \chi \quad \text{for some } \chi \in G_{\gamma(\phi)}(\psi)$$

- Constructivity implies

$$\sharp_\gamma\varphi \wedge \psi \quad \text{consistent} \implies \gamma(\varphi)^i(\bot) \wedge \psi \quad \text{consistent for some } i < \omega.$$

- Tableau construction with time-outs

- Unfolding & guardedness: w.l.o.g. the top level of every formula is modal
- Rigidity lemma: w.l.o.g. proofs of modal clauses end in modal one-step rules

**Example**: Adjointness of $\Box$. Recall rule:

$$\frac{\bigwedge_{i=1}^{n} a_i \to b}{\bigwedge_{i=1}^{n} \Box a_i \to \Box b} \quad (n \geq 0)$$

Calculate:

$$\begin{aligned} & \Box\rho \leq \psi = \Box\psi_1 \vee \cdots \vee \Box\psi_n \\ \iff & \vdash \Box\rho \to \Box\psi_1 \vee \cdots \vee \Box\psi_n \\ \iff & \vdash \rho \to \psi_i \quad \text{for some } i \end{aligned}$$

Thus put $G_{\Box x}(\psi) = \{\psi_1, \ldots, \psi_n\}$

# Conclusions

Sichere Kognitive Systeme

- Coalgebra provides a uniform framework for modal and hybrid logics
  - Graded operators (knowledge representation, redundancy)
  - Probabilistic operators (quantitative uncertainty, reactive systems)
  - Conditional operators (nonmonotonic reasoning)
  - Alternating-time logics, game logic, logics of agency (multi-agent systems)
- One-step cut-free complete rule sets are an important tool
- Completeness (and EXPTIME global caching) for flat coalgebraic fixed point logics
  - Graded $\mu$-calculus
  - Alternating-time $\mu$-calculus (e.g. ATL: Goranko/van Drimmelen TCS 2006)

# Future Work

Sichere Kognitive Systeme

- Alternation-free coalgebraic $\mu$-calculus
- Emerson/Halpern style fragment method
- Manydimensional coalgebraic logics
  - e.g. random Kripke models

$$\mu \in D(X \to \mathcal{P}(X))$$

$\to$ Prob-$\mathcal{ALC}$ (Lutz/Schröder KR 2010)

- Vision: generic, efficient modular reasoning tools
  - CoLoSS (`http://www.informatik.uni-bremen.de/cofi/CoLoSS/`)