ETH zürich

Combinatorial Depth Measures

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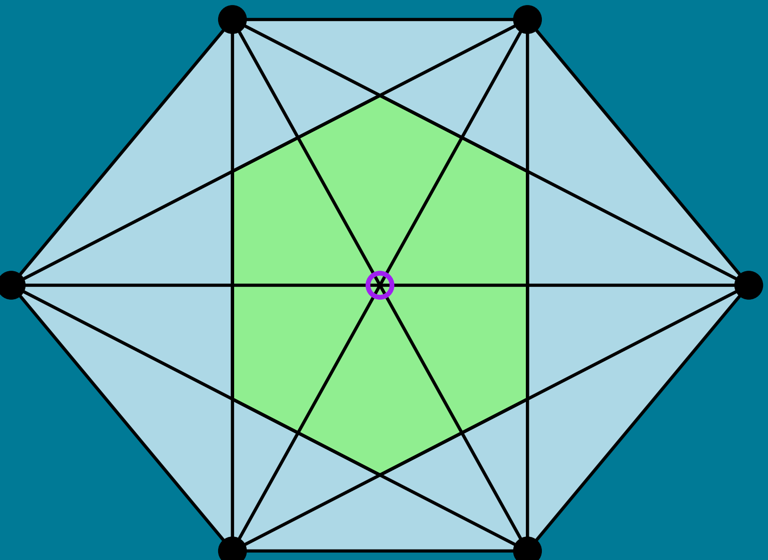
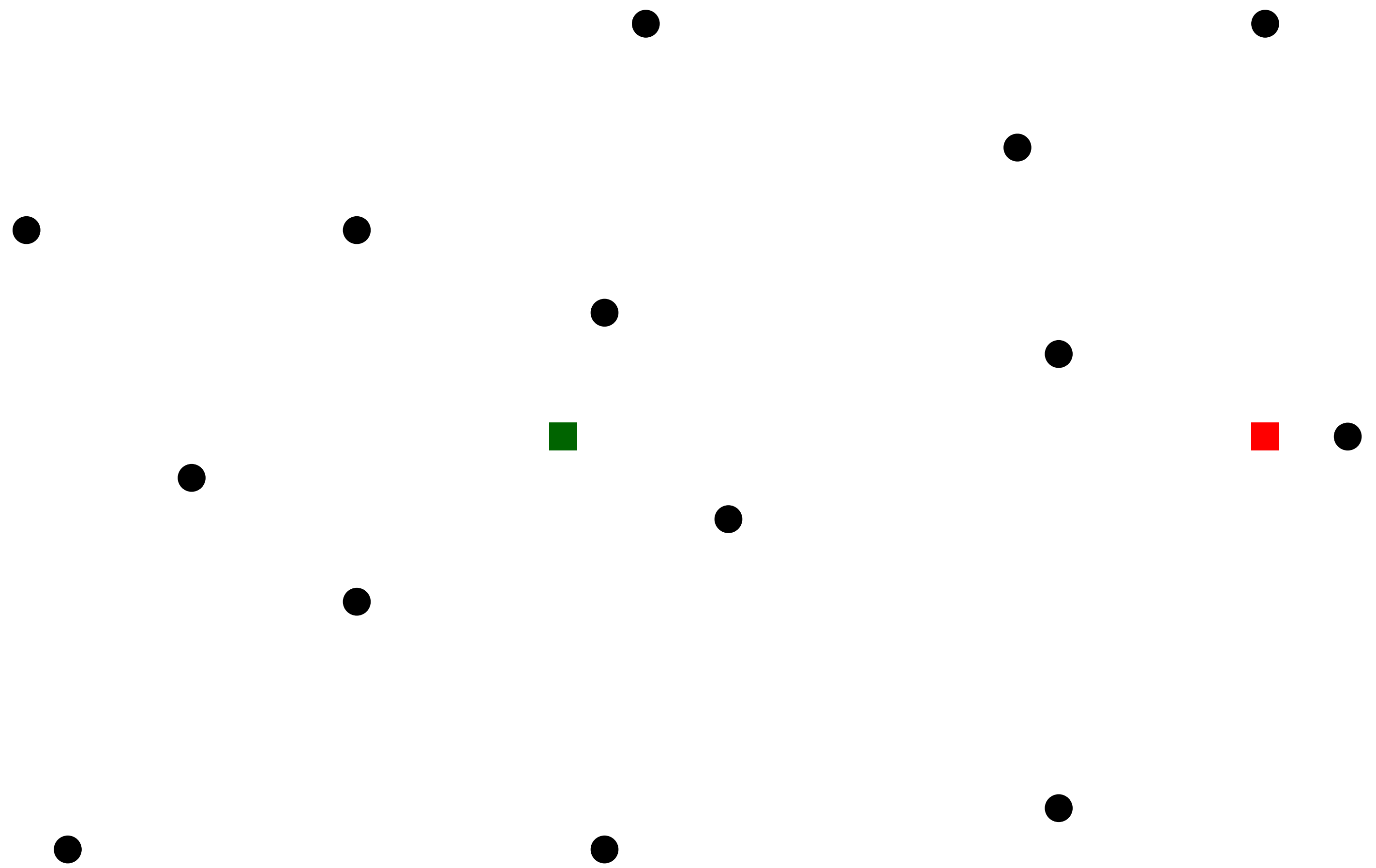
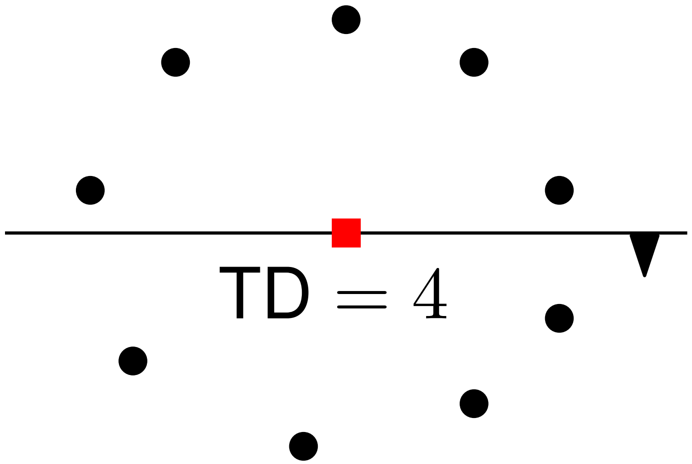


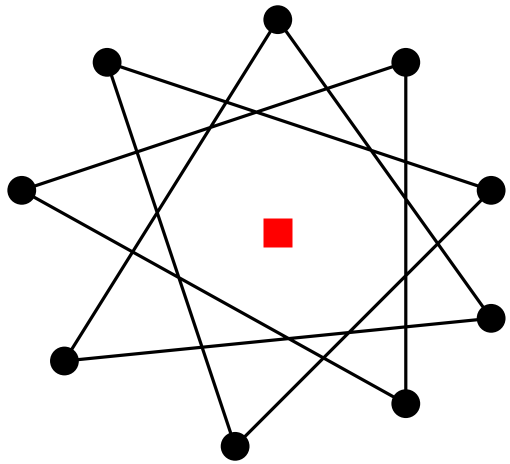
Introduction



Which colored point would you rather call a ”median”?

Tukey and Tverberg





Tukey depth:
Minimum number of data points in any closed half-space containing query point q

Tverberg depth:
Max. number of vertex disjoint simplices whose intersection contains q

Centerpoint theorem:
$\forall S \exists q$: TD$(S, q) \geq \frac{|S|}{d+1}$

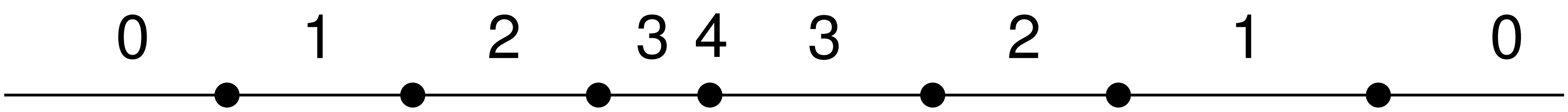
Tverbergs theorem:
$\forall S \exists q$: TvD$(S, q) \geq \frac{|S|}{d+1}$

Combinatorial Depth Measures

$$\rho : S^{\mathbb{R}^d} \times \mathbb{R}^d \to \mathbb{R}_{\geq 0}$$
$$(S, q) \mapsto \rho(S, q)$$

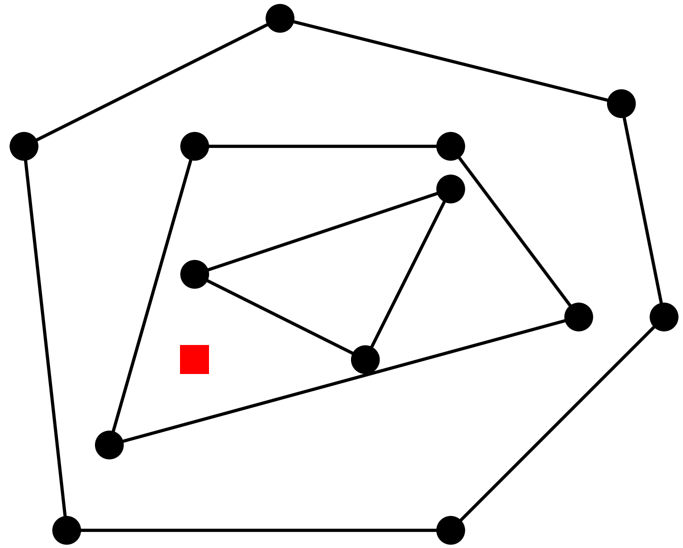
”combinatorial”: depends only on relative position of S and q (order type), not on distances

Standard depth in $\mathbb{R}^1$:

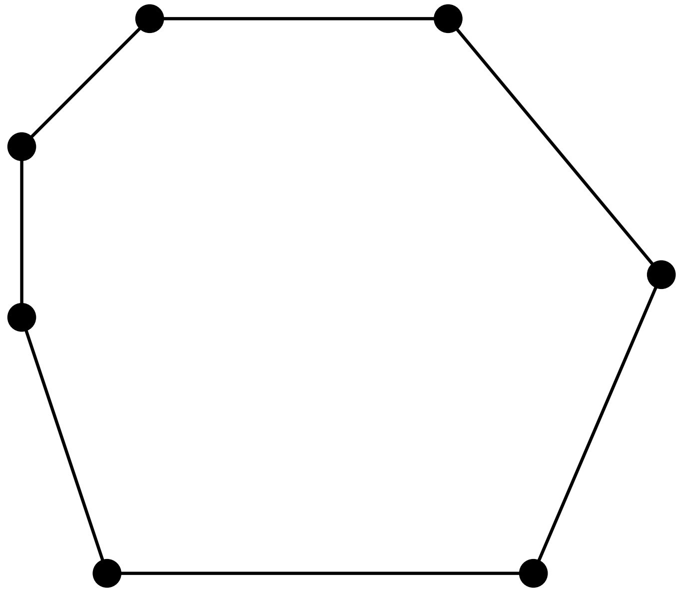


A ”bad” measure

Convex hull peeling depth:



depth 2



no deep query point

Super-additive Depth Measures

ρ is *super-additive* if

(1) $\forall S, q, p : |\rho(S, q) - \rho(S \cup \{p\}, q)| \leq 1$

(2) $\forall S, q : \rho(S, q) = 0$ if $q \notin conv(S)$

(3) $\forall S, q : \rho(S, q) \geq 1$ if $q \in conv(S)$

(4) $\forall S_1 \sqcup S_2 = S : \rho(S, q) \geq \rho(S_1, q) + \rho(S_2, q)$

Theorem [S', '23]:
Let ρ be a super-additive depth measure. Then $\forall S, q$
TD$(S, q) \geq \rho(S, q) \geq$ TvD$(S, q) \geq \frac{1}{d}$TD(S, q).

Central Depth Measures

ρ is *central* if

(1) $\forall S, q, p : |\rho(S, q) - \rho(S \cup \{p\}, q)| \leq 1$

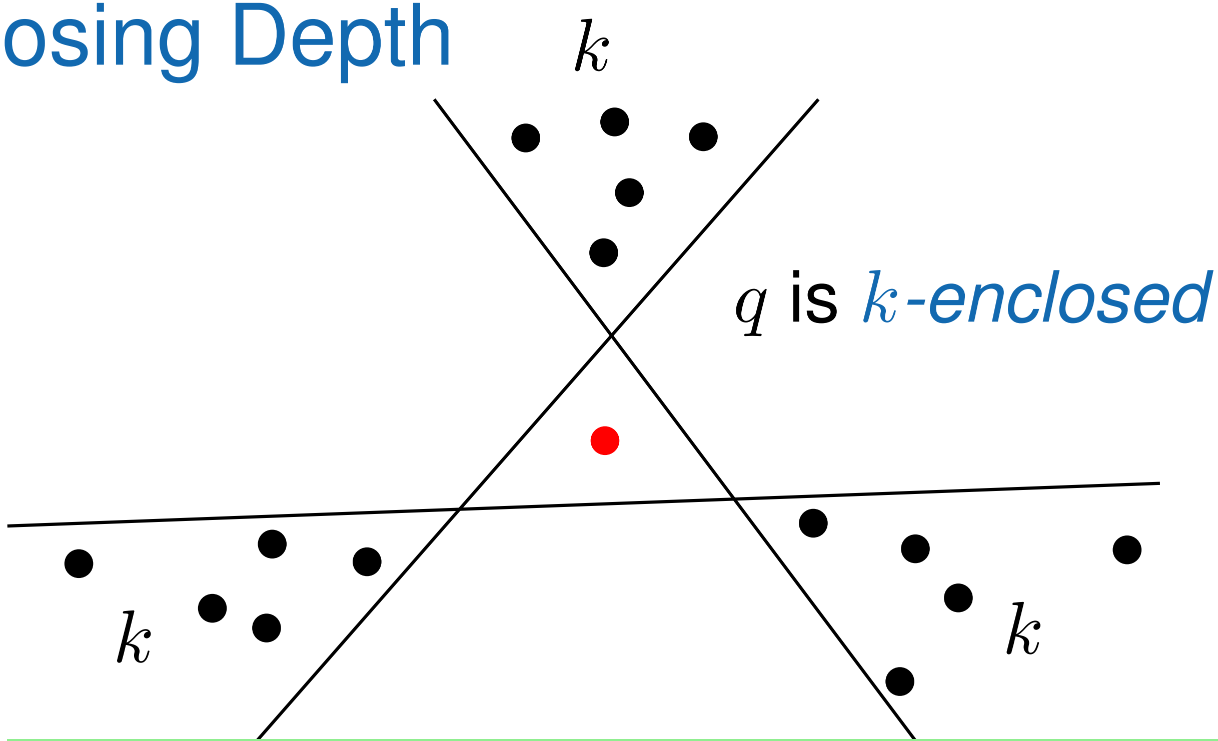
(2) $\forall S, q : \rho(S, q) = 0$ if $q \notin conv(S)$

(3) $\forall S \exists q : \rho(S, q) \geq \frac{|S|}{d+1}$

(4) $\forall S, p, q : \rho(S \cup \{p\}, q) \geq \rho(S, q)$

Theorem [S', '23]:
Let ρ be a central measure. Then $\exists c(d)$ s.t. $\forall S, q$
TD$(S, q) \geq \rho(S, q) \geq c(d) \cdot$ TD(S, q).

Enclosing Depth



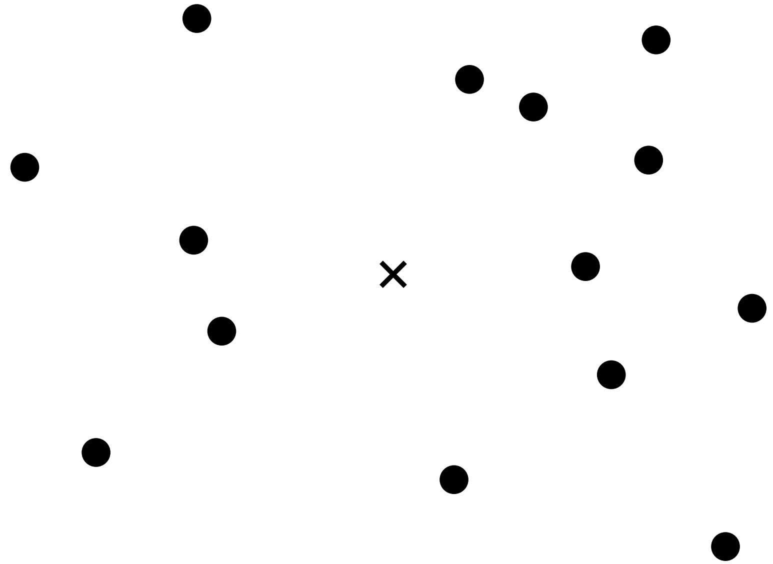
Enclosing depth:
$\text{ED}(S, q) = \max k$ s.t. q is k-enclosed

Lemma:
ρ central. Then
$\rho(S, q) \geq \text{ED}(S, q) - (d + 1)$.

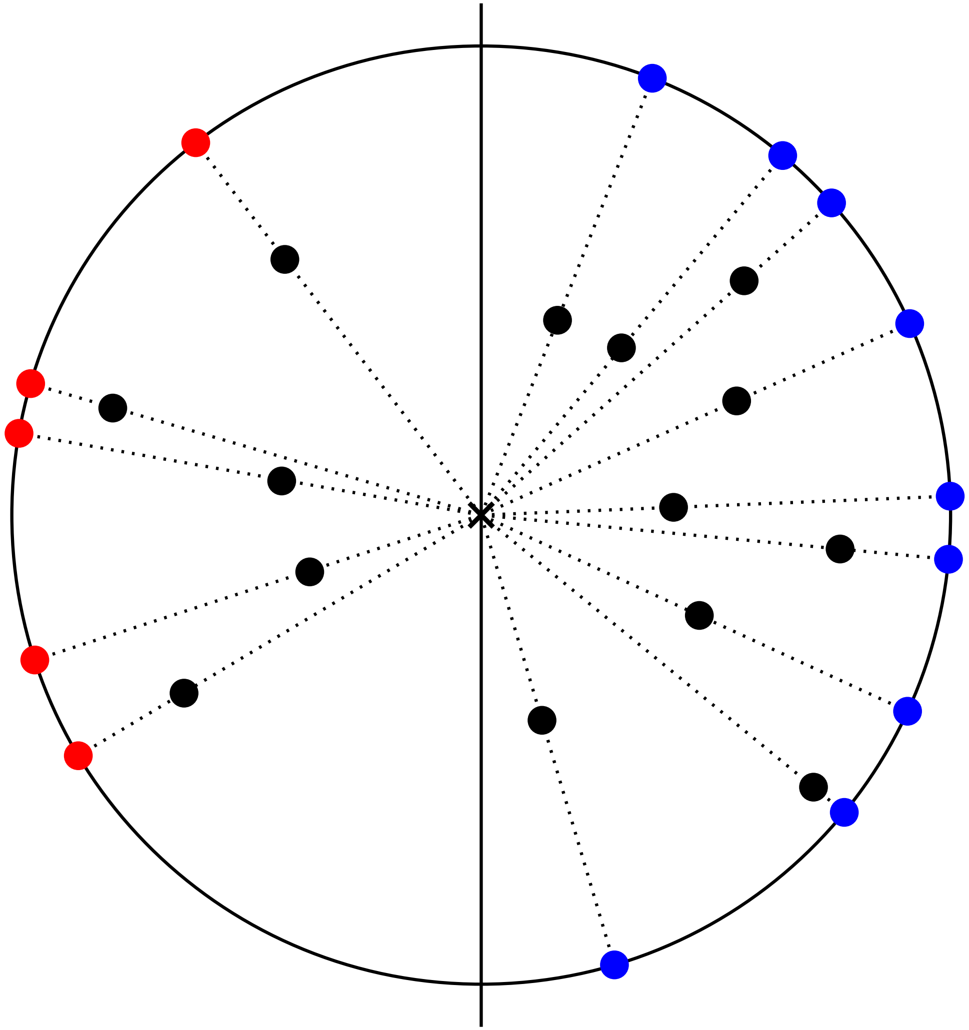
Lemma:
$\exists c(d)$ s.t.
$\text{ED}(S, q) \geq c(d) \cdot \text{TD}(S, q)$.

The 2D case

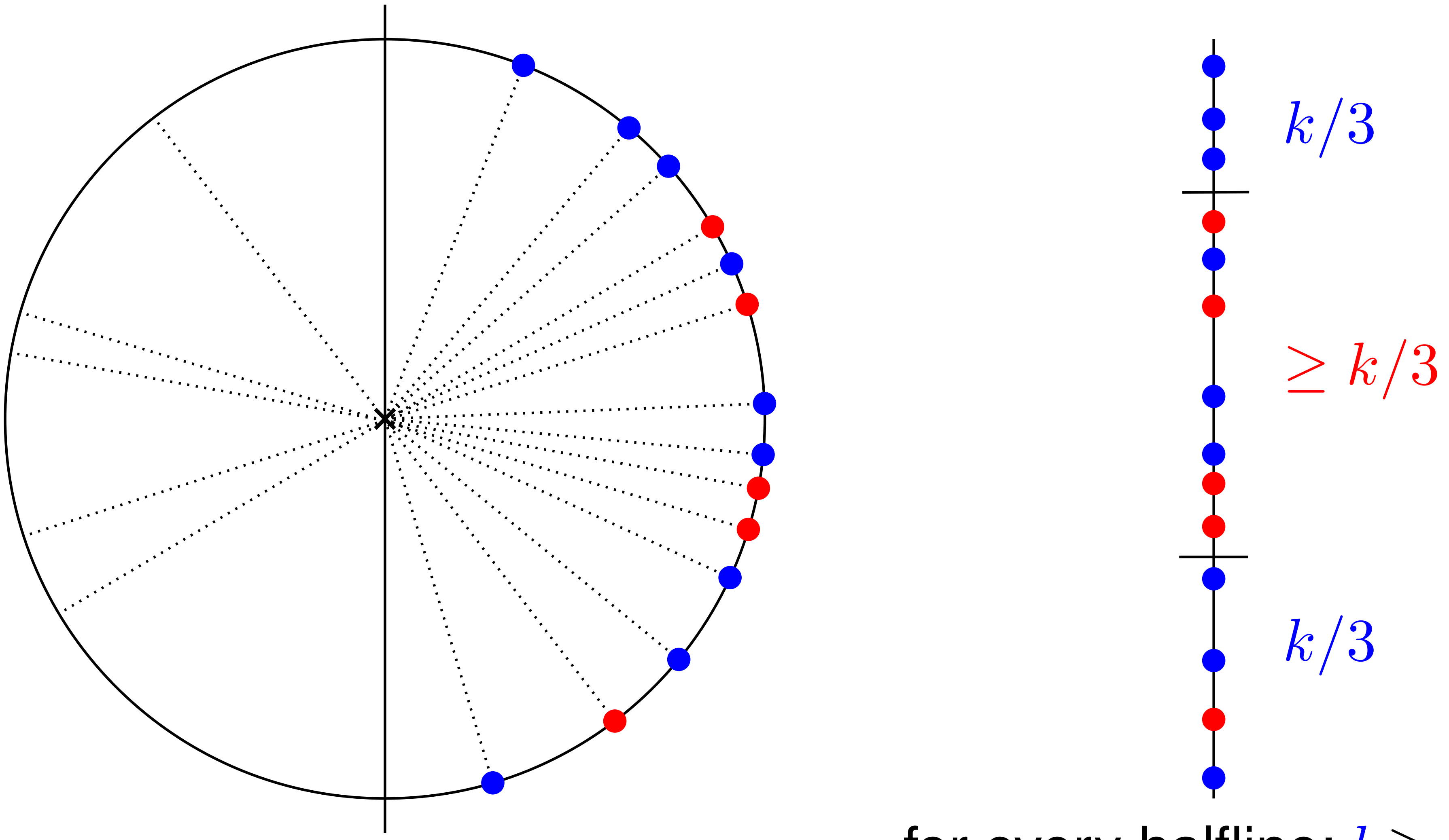
The 2D case



The 2D case

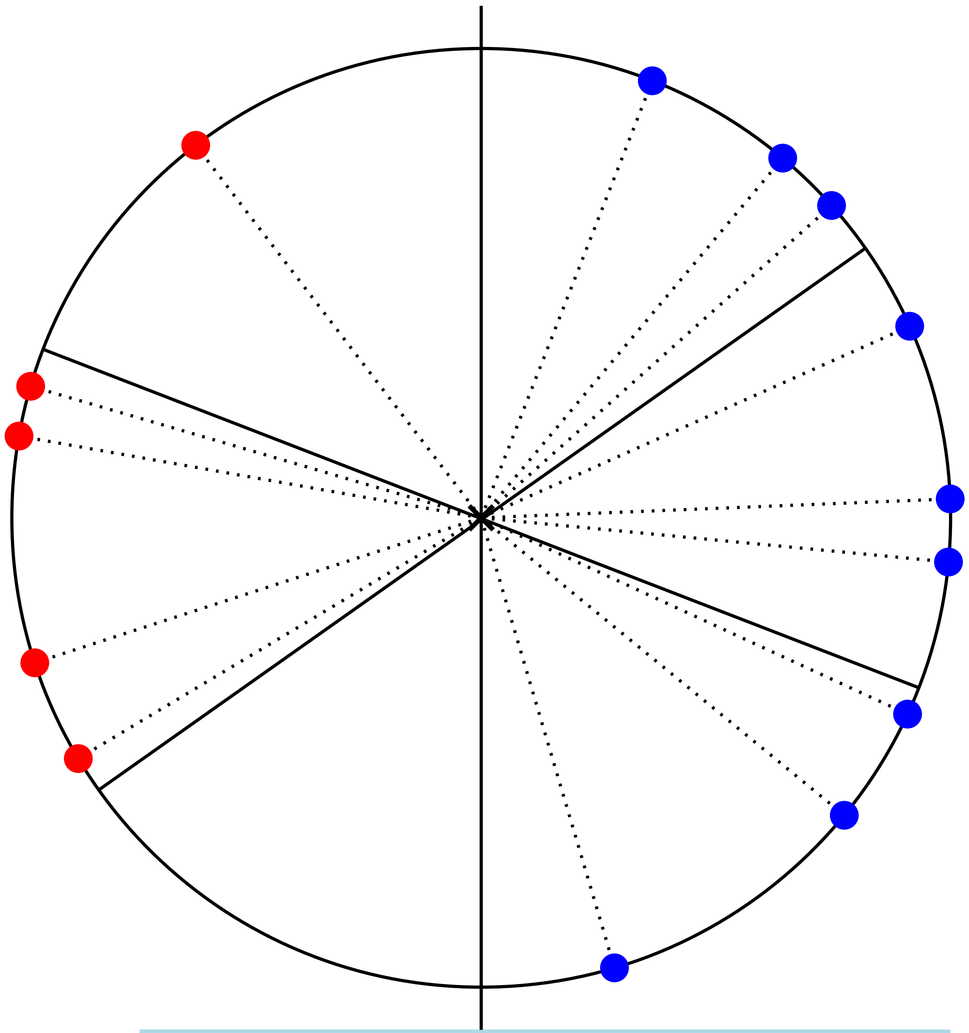


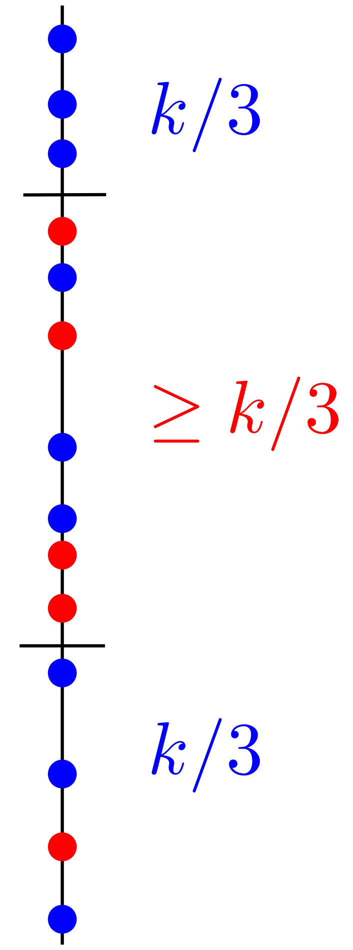
The 2D case



for every halfline: $b \geq r$

The 2D case





$\mathsf{ED}(S, q) \geq \frac{1}{3}\mathsf{TD}(S, q)$.

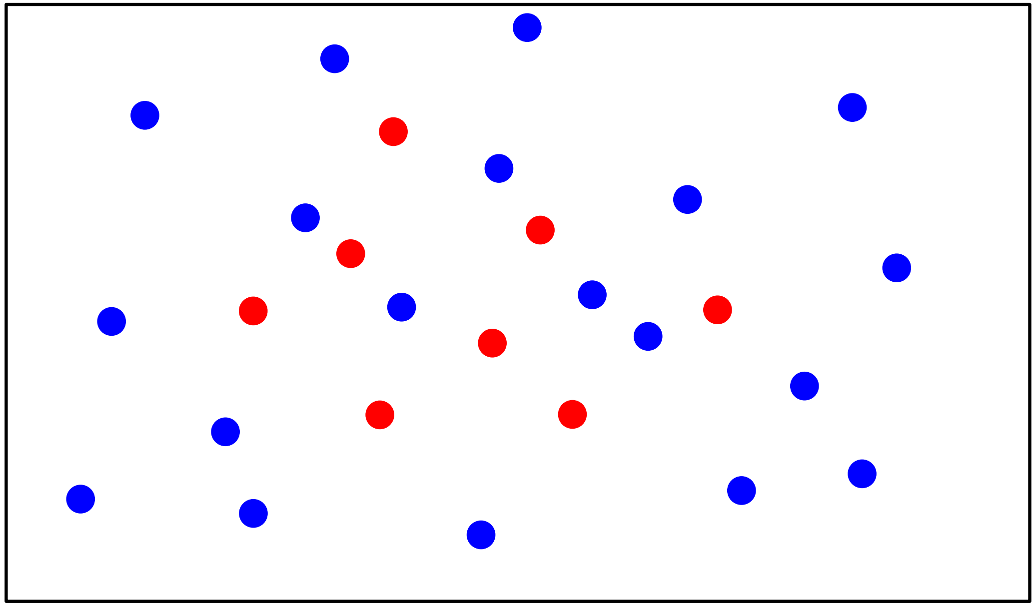
for every halfline: $b \geq r$

The general case

Assume points are on sphere around q

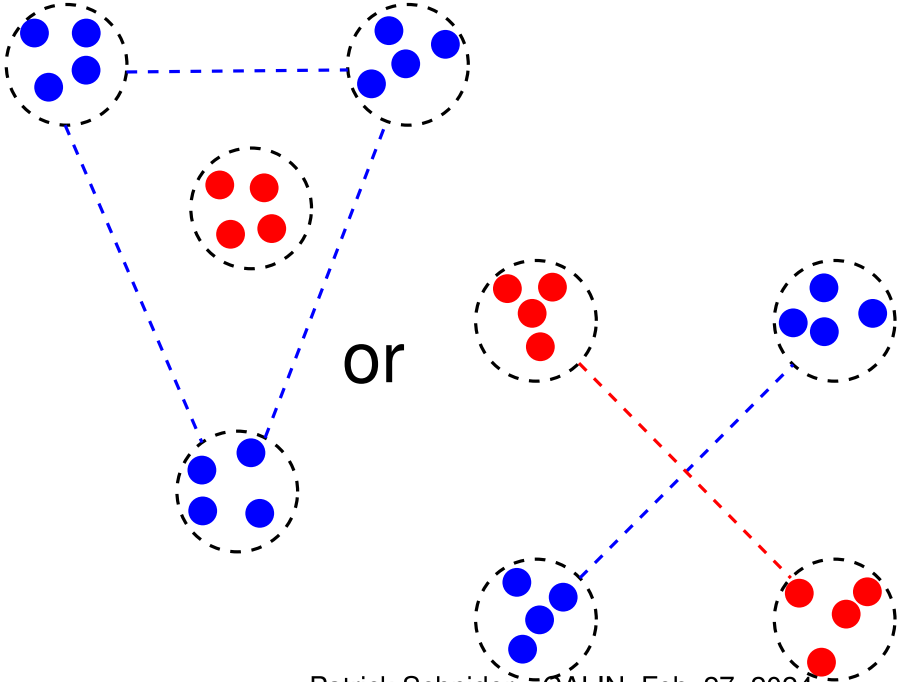
Take witness-hyperplane for Tukey depth k

Project to larger side, color red and blue



for each half-space: $b \geq r$

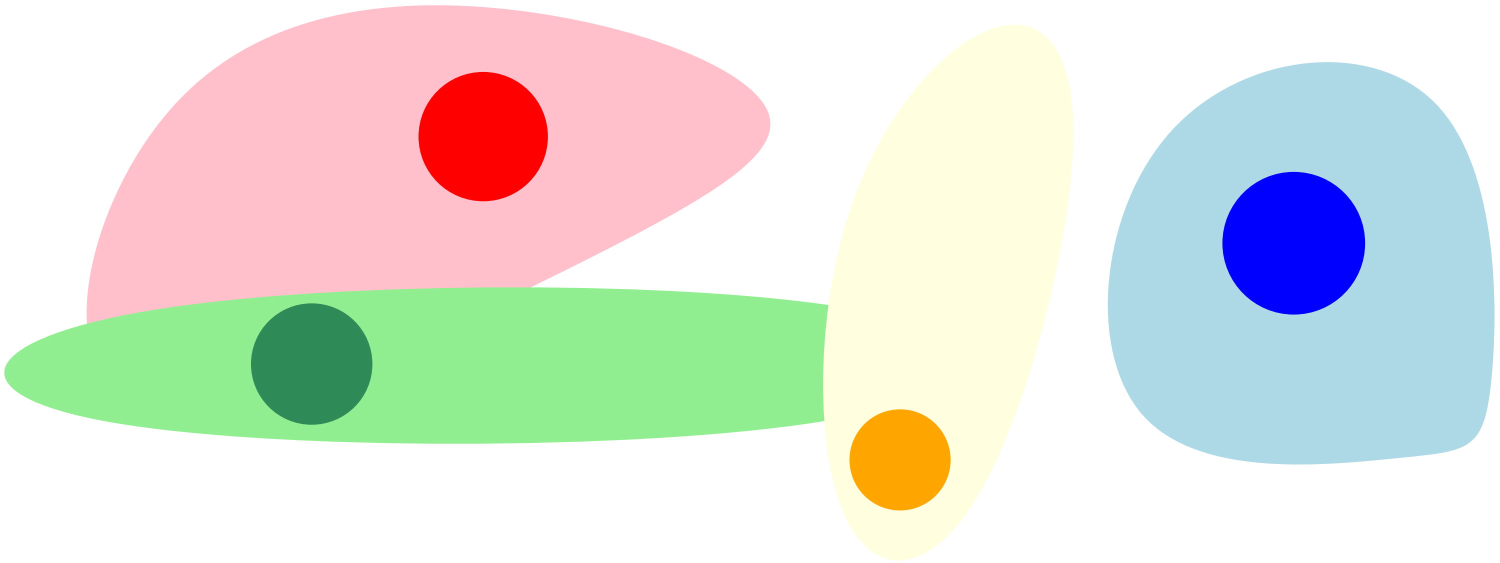
Want:



The Same Type Lemma

Theorem [Bárány, Valtr, '98]:
Let $X_1, \ldots, X_m$ be point sets in $\mathbb{R}^d$. Then there is a constant $c(d, m)$ and subsets $Y_i \subseteq X_i$ s.t.

- $|Y_i| \geq c \cdot |X_i|$ and
- each selection $y_1 \in Y_1, \ldots, y_m \in Y_m$ has the same order type.



Constant Fraction Radon

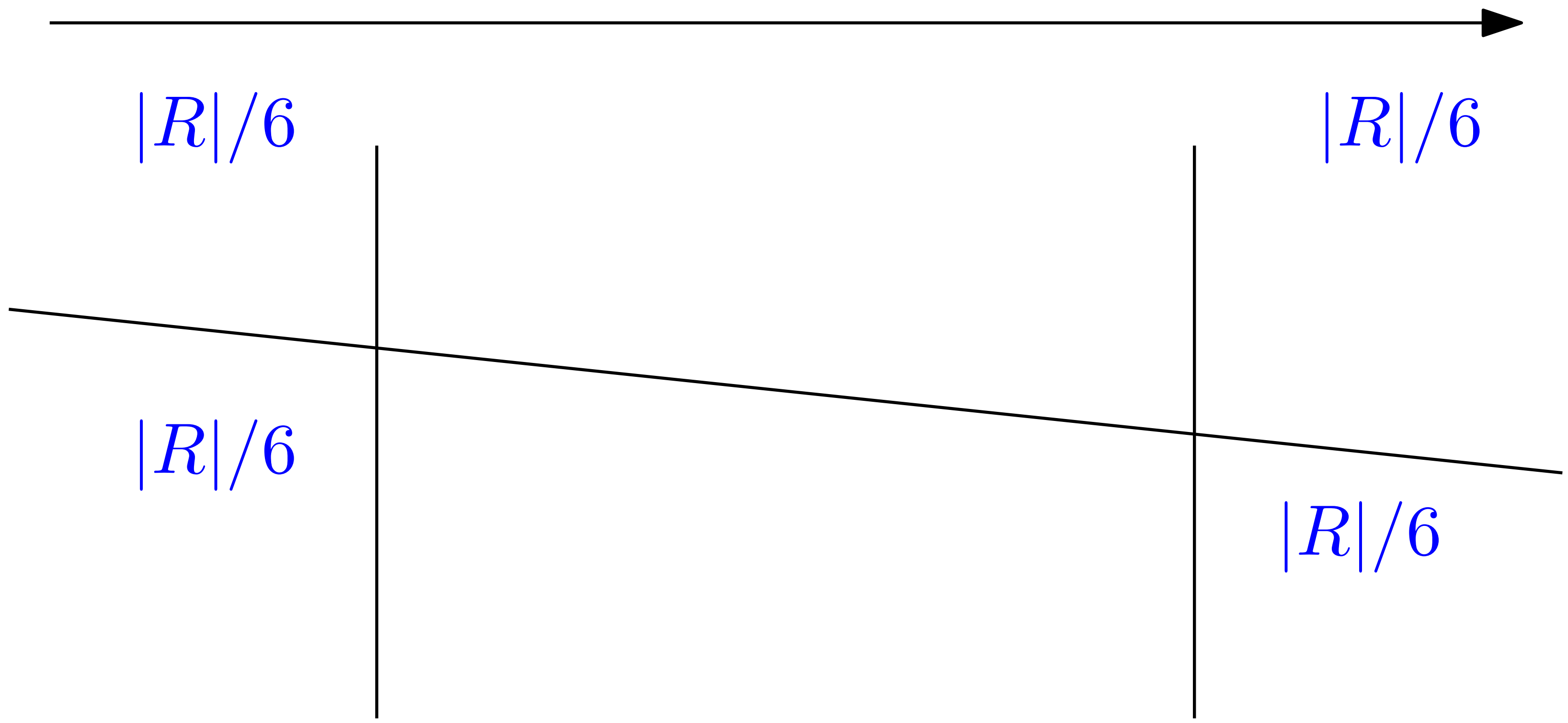
Theorem [S', '23]:
Let $P = R \cup B$ be a point set in $\mathbb{R}^d$ s.t. for every half-space we have $b \geq r$. Then there is a constant $c(d)$ and subsets $R_1, \ldots, R_a \subseteq R$ and $B_1, \ldots, B_b \subseteq B$ s.t.

- $a + b = d + 2$,
- $|R_i| \geq c \cdot |R|$ and $|B_j| \geq c \cdot |R|$ and
- for each selection $r_1 \in R_1, \ldots, r_a \in R_a, b_1 \in B_1, \ldots, b_b \in B_b$ we have the sme order type and conv$(r_1, \ldots, r_a) \cap$ conv$(b_1, \ldots, b_b) \neq \emptyset$.

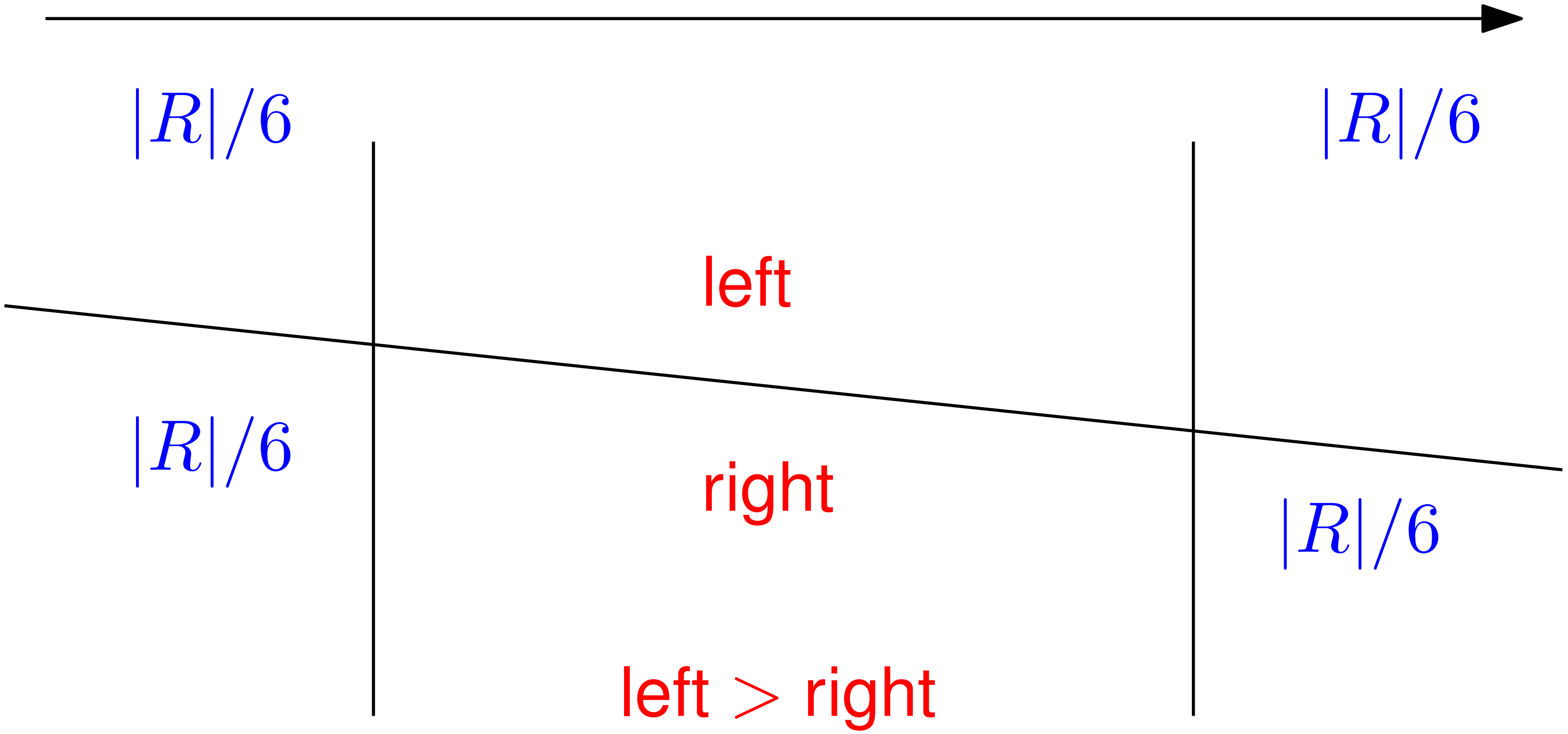
Constant Fraction Radon in $\mathbb{R}^d \Rightarrow$ Enclosing Depth in $\mathbb{R}^{d+1}$

Radon in 2D

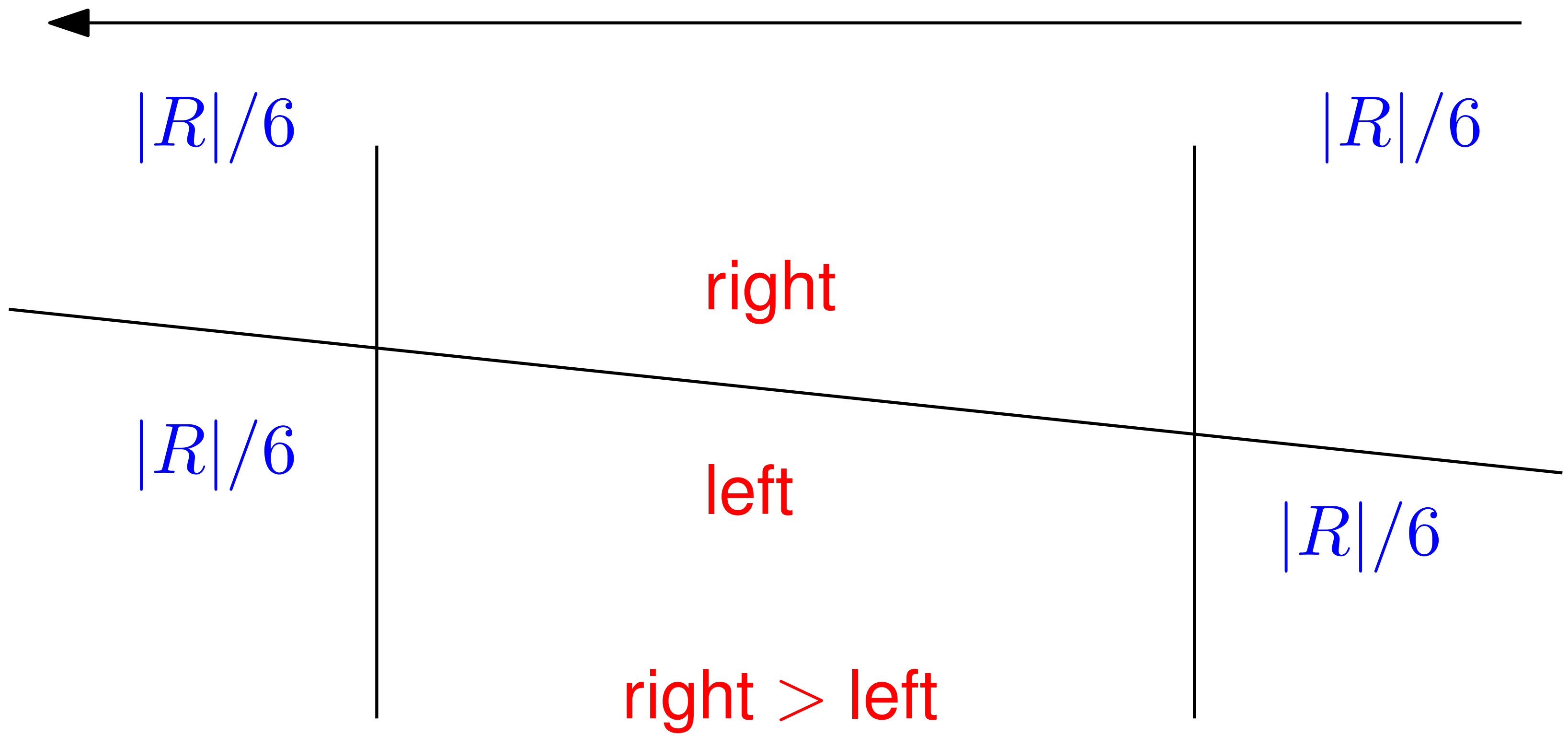
Radon in 2D



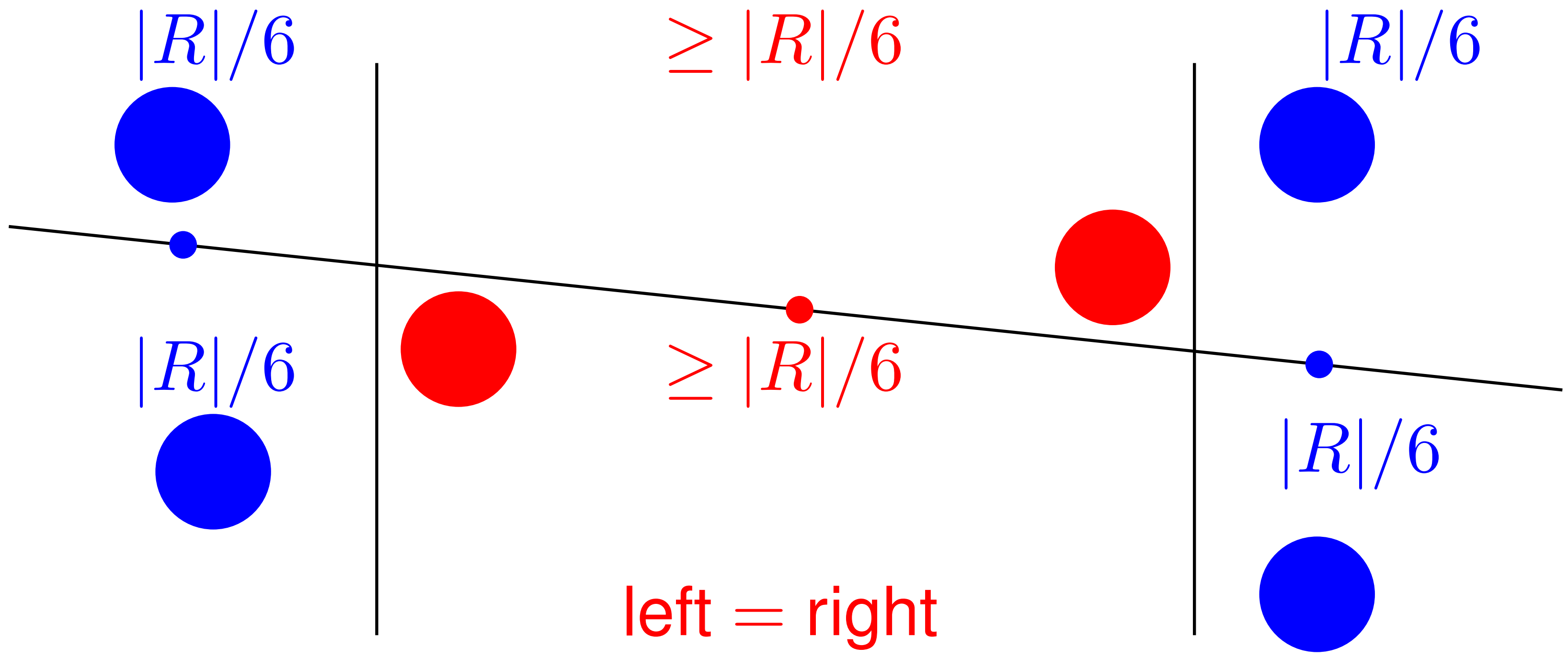
Radon in 2D



Radon in 2D

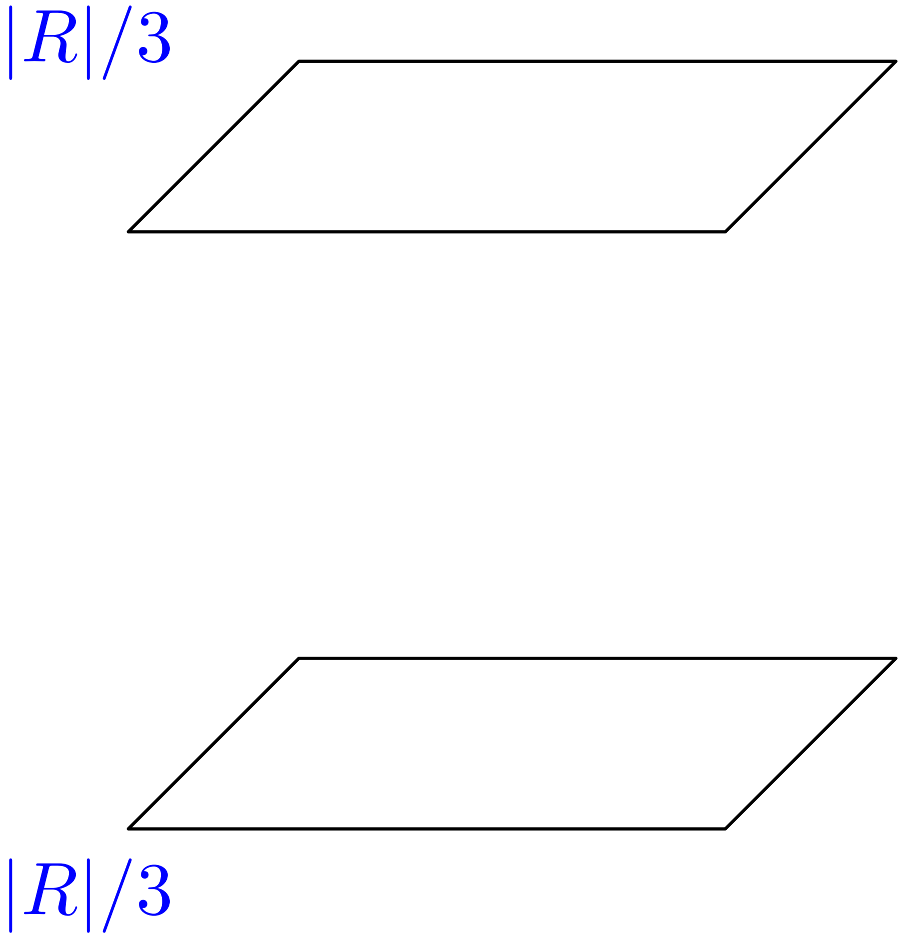


Radon in 2D

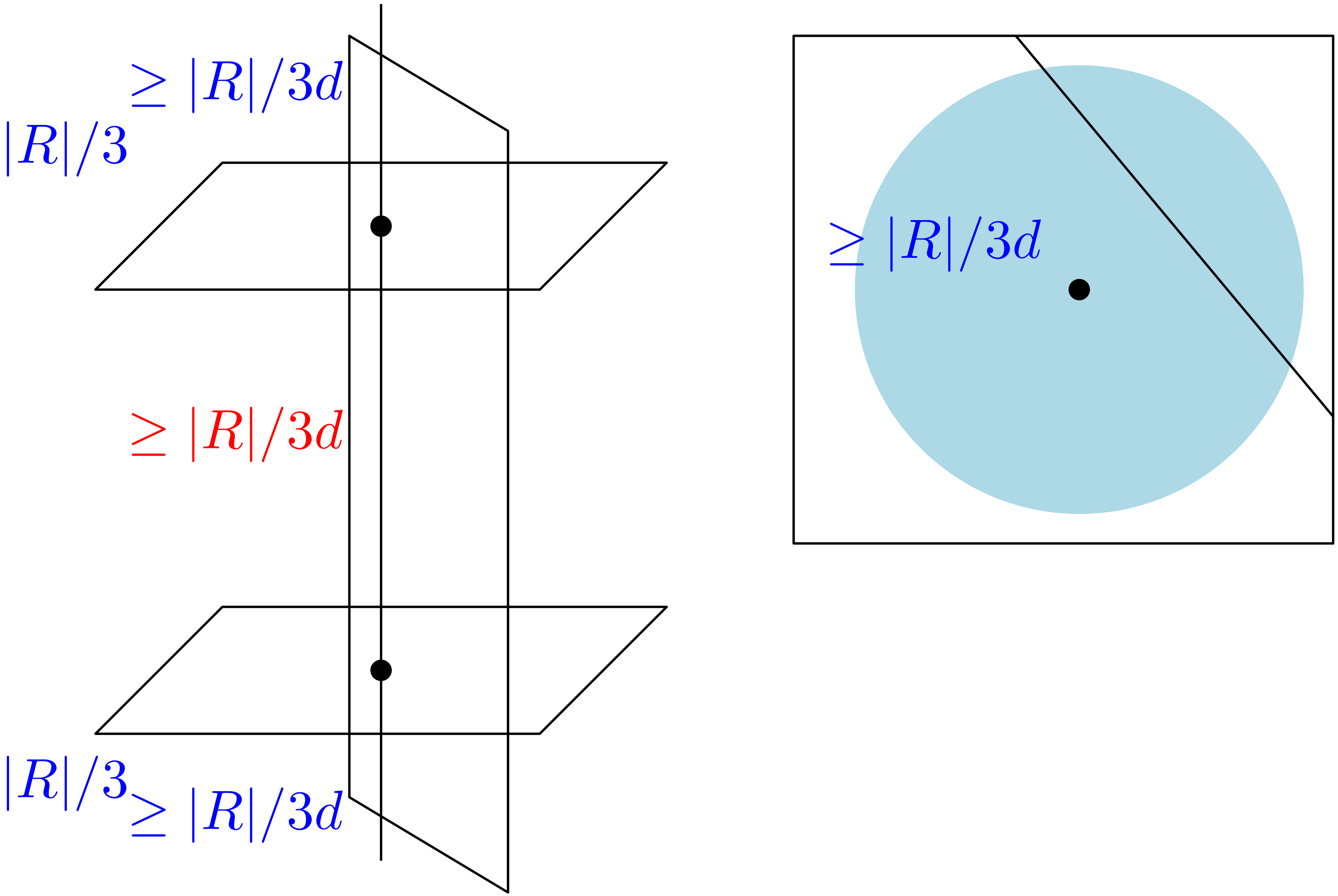


Kirchberger: if the colors have a common intersection, then some $d+2$ elements do.

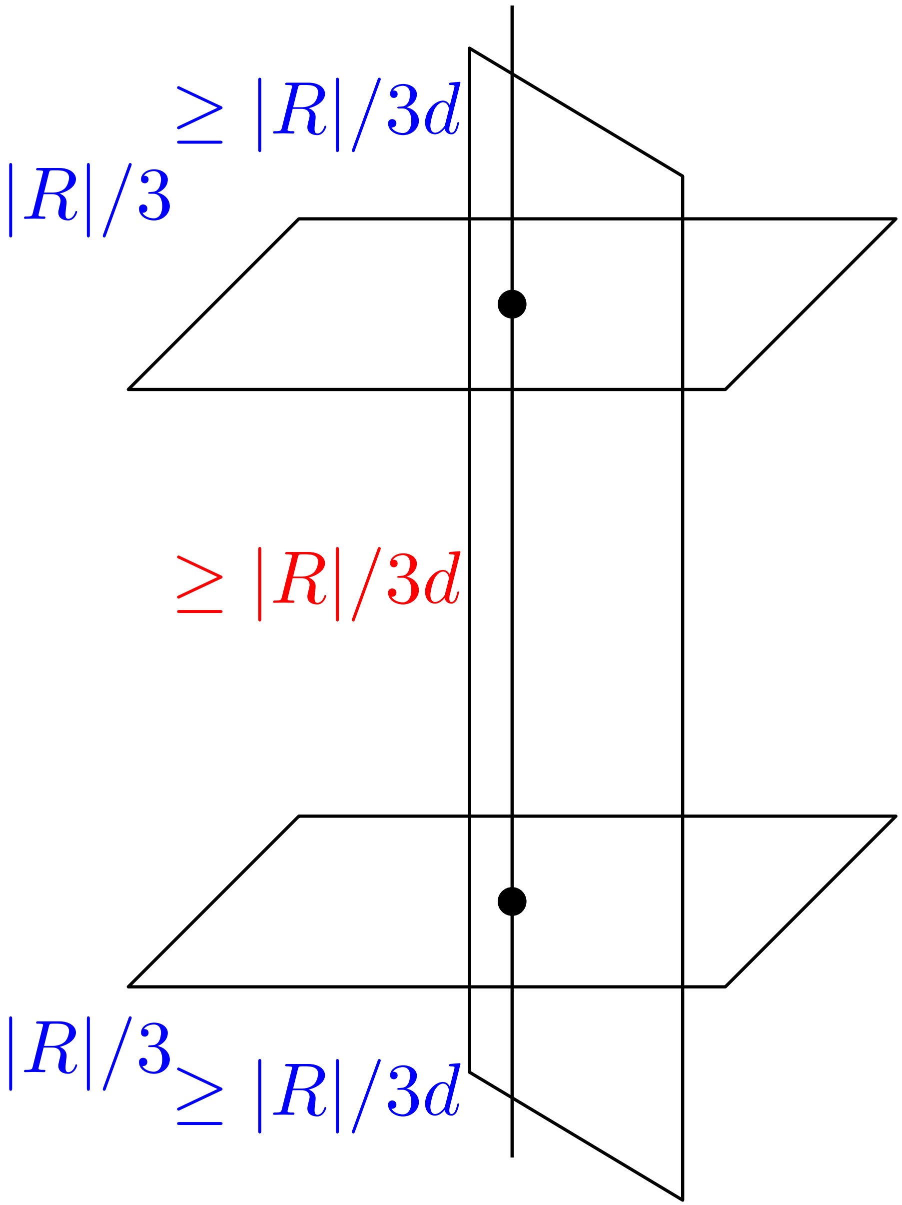
Radon: the general case

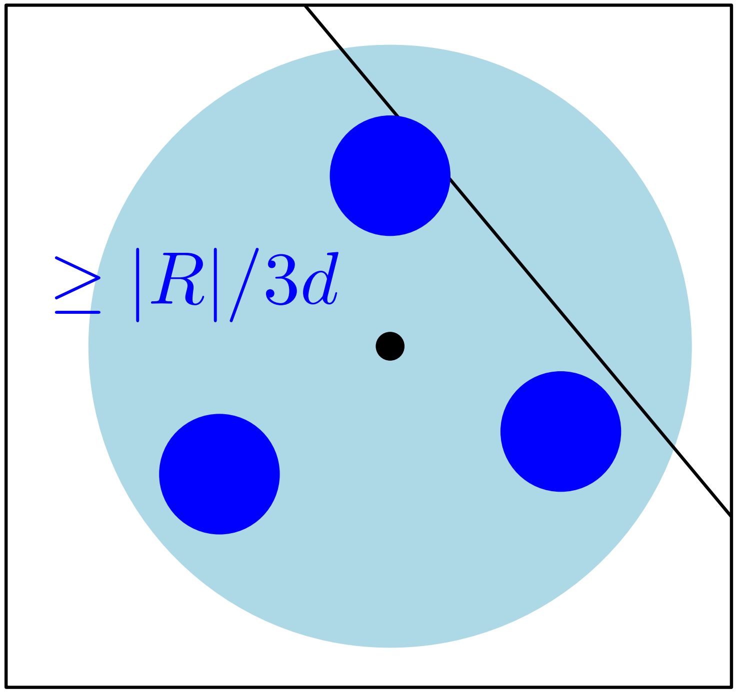


Radon: the general case



Radon: the general case

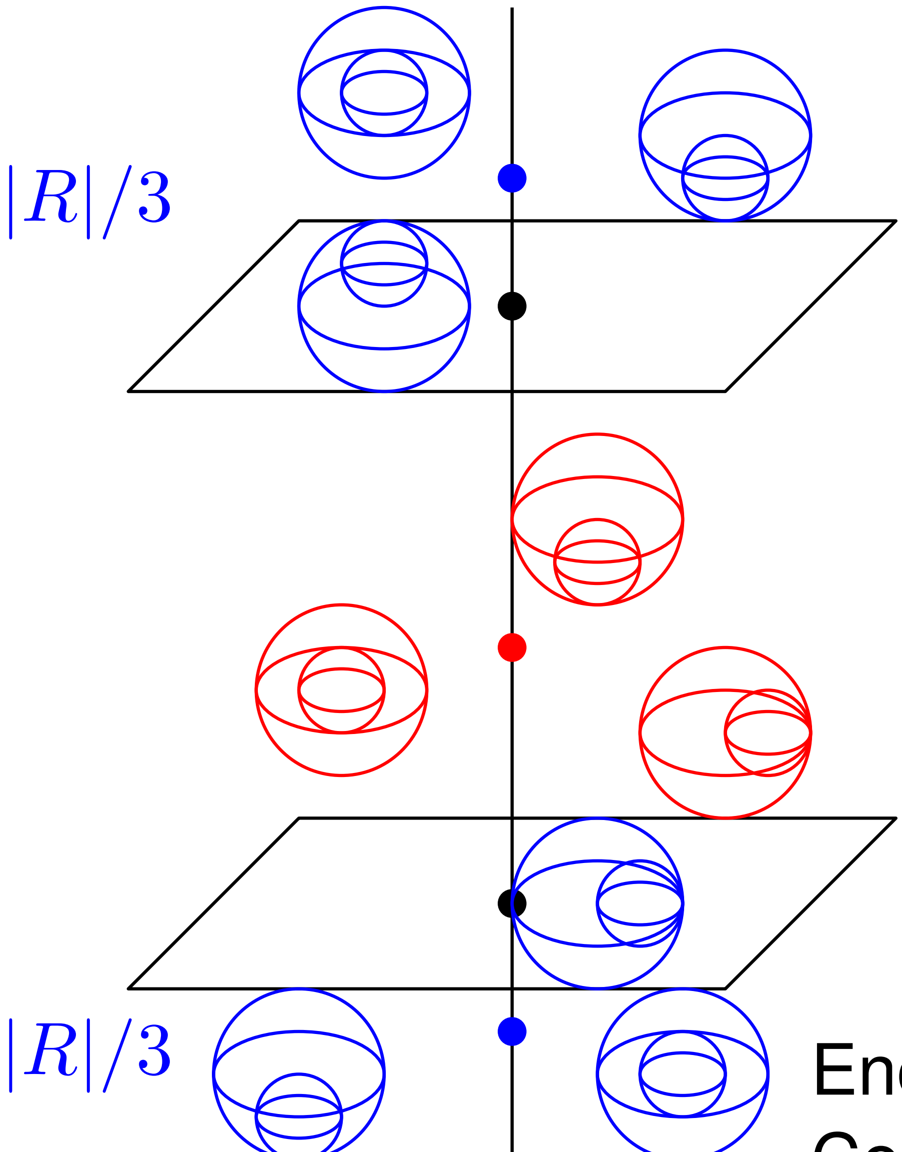


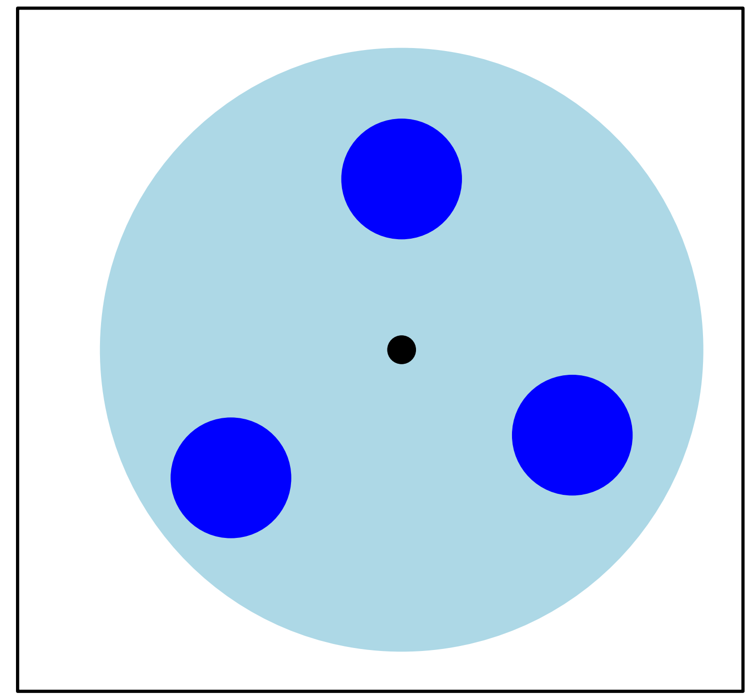


TD $\geq |R|/3d$

ED $\geq c \cdot$ TD

Radon: the general case





TD $\geq |R|/3d$

ED $\geq c \cdot$ TD

Enclosing depth in $\mathbb{R}^{d-1} \Rightarrow$
Constant Fraction Radon in $\mathbb{R}^d$

Conclusion

- Combinatorial depth measures are intimately related to fundamental results in discrete geometry
- Many related algorithmic questions:
  - Complexity of finding a Centerpoint/Tverberg point
  - Complexity of computing Enclosing depth in general dimension? (Known $O(n^{d^2})$)
- Many depth measures are an approximation of Tukey depth
  - Improve factors?

Thank you!