Arc complex and strip deformations of decorated polygons

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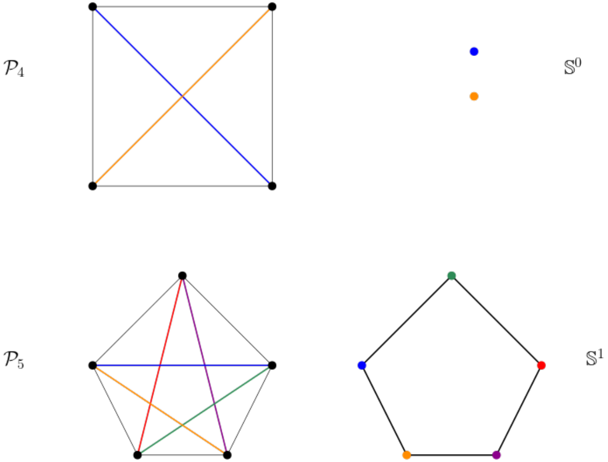
Arc complex

$\mathcal{P}_n$: a convex Euclidean polygon with $n \geq 4$ vertices.

$\mathcal{A}(\mathcal{P}_n)$: a flag, pure simplicial complex constructed in the following way:

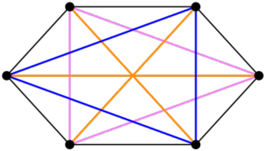
- 0-simplices $\longleftrightarrow$ diagonals,
- For $k \geq 1$, k-simplices $\longleftrightarrow$ $(k+1)$ pairwise disjoint and distinct diagonals.

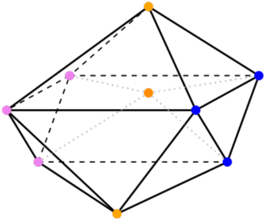
Examples



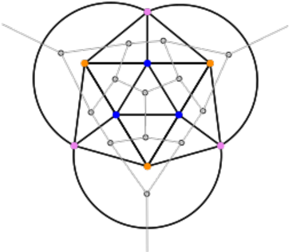
Examples

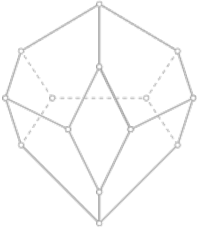
$\mathcal{P}_6$





$\mathbb{S}^2$





A classical result

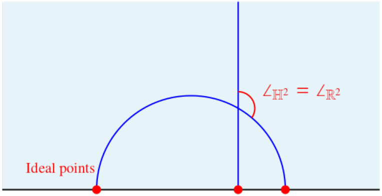
A classical result from combinatorics about the topology of the arc complex of a polygon.

Theorem

The arc complex $\mathcal{A}(\mathcal{P}_n)$ $(n \geq 4)$ is a sphere of dimension $n-4$.

Crash course on hyperbolic 2-space

The upper half plane model



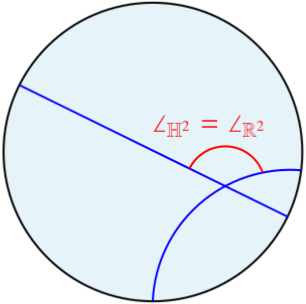
$\mathbb{H}^2 = \{z \in \mathbb{C} \mid \Im\mathrm{m} z > 0\}$

- Hyperbolic metric: $ds^2 = \frac{dx^2+dy^2}{y^2}$
- The boundary: $\partial_\infty \mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$
- Orientation-preserving isometry group:

$$\mathrm{PSL}(2,\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid ad - bc = \pm 1 \right\}, \quad z \mapsto \frac{az+b}{cz+d}.$$

Other models

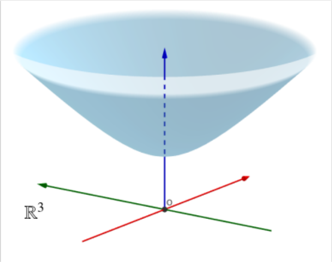
Poincaré disk model



$$\mathbb{H}^2 := \{z \in \mathbb{C} \mid \|z\| < 1\}$$

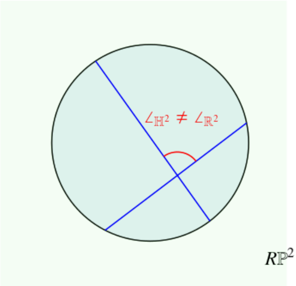
Other models

Hyperboloid model



$$\mathbb{H}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = -1,\ z > 0\}$$

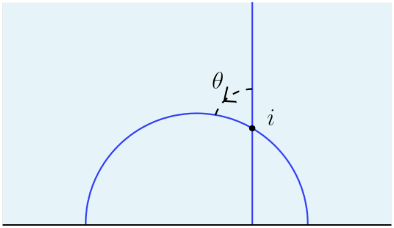
Klein's projective model

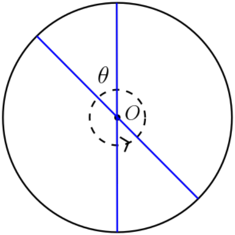


$$\mathbb{H}^2 = \mathbb{P}\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 < 0\}$$

Types of Isometries: Elliptic

Elliptic transformations

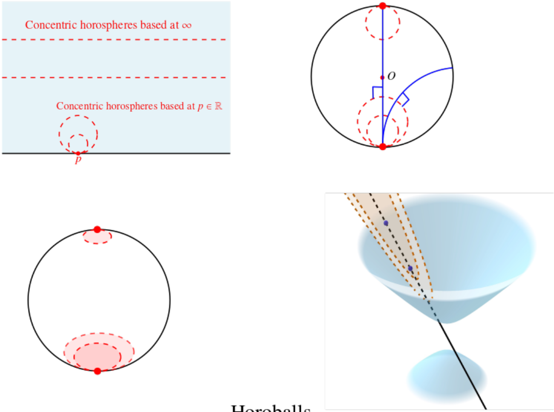




$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Types of Isometries: Parabolic

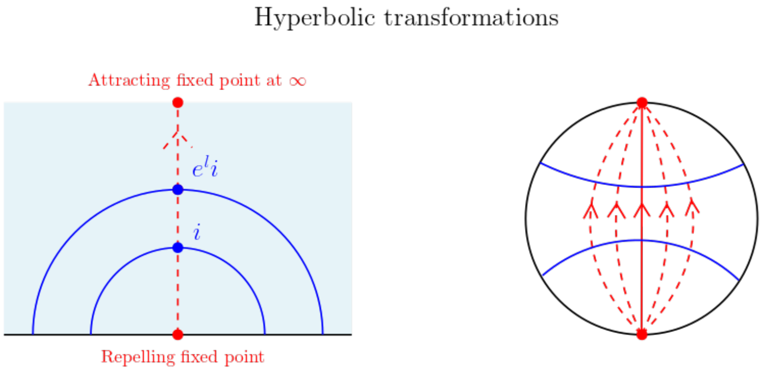
Parabolic Transformations and their orbits



Horoballs

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, t \in \mathbb{R}$$

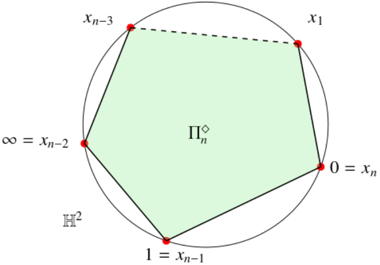
Types of Isometries: Hyperbolic



$$\begin{bmatrix} \exp^{l/2} & 0 \\ 0 & \exp^{-l/2} \end{bmatrix}, l > 0$$

Ideal polygons

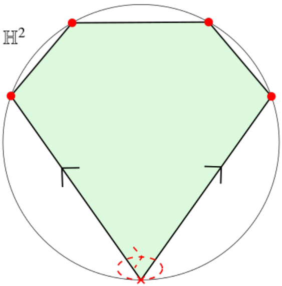
An **ideal n-gon** $\Pi_n^\diamond$, $n \geq 3$ is the convex hull in $\mathbb{H}^2$ of n points in $\partial_\infty \mathbb{H}^2$.



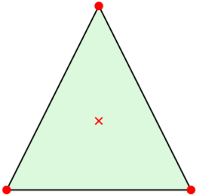
Fact: G acts triply transitively on $\partial_\infty \mathbb{H}^2$.
Thus, the deformation space

$$\mathfrak{D}(\Pi_n^\diamond) \simeq \{\text{(induced) complete finite-area metrics}\}$$
$$\simeq \mathbb{B}^{n-3}$$

Once-punctured polygons



A fundamental domain



$\Pi_3^\diamond$

A **once-punctured** ***n*-gon** $\Pi_n^\diamond$: Glue two consecutive edges of a $(n+2)$-gon, using a parabolic isometry. Thus, $\mathfrak{D}(\Pi_n^\diamond) \simeq \mathbb{B}^{n-1}$.

Arcs and arc complex

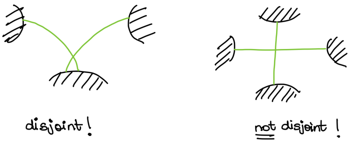
Definition

Arcs (up to isotopy): $\alpha : [0,1] \hookrightarrow S$ s.t $\alpha(0), \alpha(1) \in \partial S$.

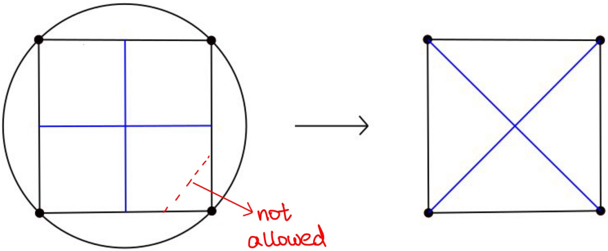
Definition

The *arc complex* is a simplicial complex $\mathcal{A}(S)$:

- $\mathcal{A}(S)^{(0)} = \{$isotopy classes of arcs$\}$
- $\mathcal{A}(S)^{(k)} = \{(k+1)$-tuple of pairwise disjoint isotopy classes$\}$

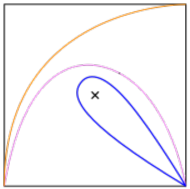


Ideal polygons to Euclidean polygons



$\Pi_4^\diamond \subset \mathbb{H}^2$ $\qquad$ $\mathcal{P}_4$

Arc complex of $\Pi_n^{\diamond}$



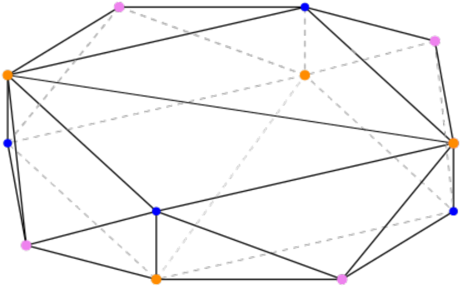


Figure: The arcs and the arc complex of $\Pi_4^{\diamond}$

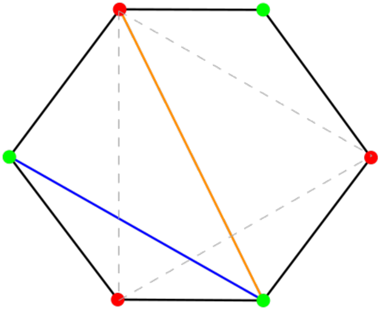
Penner's sphericity results

Theorem (Penner)

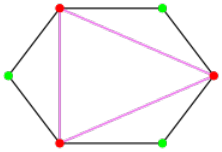
- *The arc complex* $\mathscr{A}\left(\Pi_n^{\diamond}\right)$ *of an ideal polygon* $\Pi_n^{\diamond}$ *($n \geq 4$) is a PL-sphere of dimension* $n-4$.
- *The arc complex* $\mathscr{A}\left(\Pi_n^{\diamond}\right)$ *of an once-punctured polygon* $\Pi_n^{\diamond}$ *($n \geq 2$) is a PL-sphere of dimension* $n-2$.

New Rule: Coloured vertices and permissible arcs

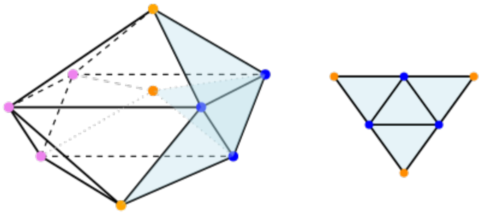
Consider the subcomplex $\mathcal{Y}(\mathcal{P}_n)$ generated by $G - G$, $R - G$ diagonals.



Examples

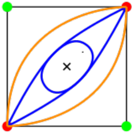


Rejected R-R diagonals

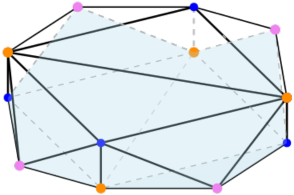


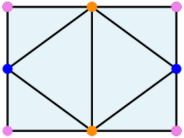
The subcomplex $\mathcal{Y}(\mathcal{P}_6)$

Examples



Rejected R-R diagonals





The subcomplex $\mathcal{Y}(\mathcal{P}_4^{\times})$

Conjecture

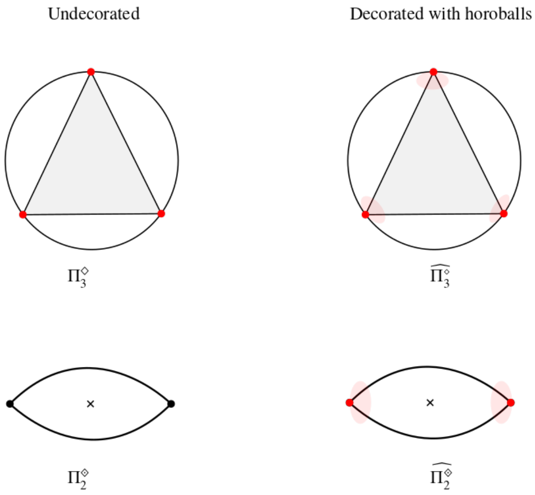
Conjecture

Let $\mathcal{P}_n$ (resp. $\mathcal{P}_n^{\times}$) be a polygon with bicoloured vertices. Then the sub complex $\mathcal{Y}(\mathcal{P}_n)$ (resp. $\mathcal{Y}(\mathcal{P}_n^{\times})$) is a closed ball of dimension $2n-4$ (resp. $2n-2$).

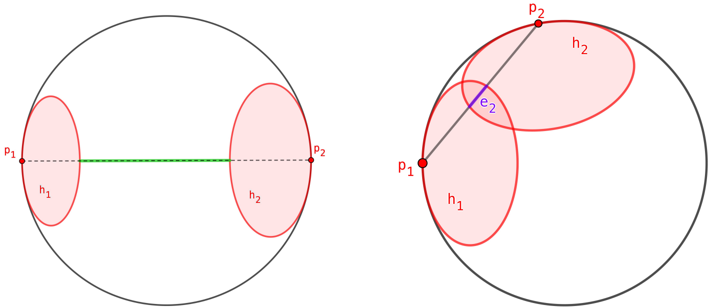
A partial solution: Use *decorated* hyperbolic (once-punctured) polygons to show that the interior is an open ball of right dimension.

Decorated polygons

Decorate each vertex with a *horoball*.



Horoball connections

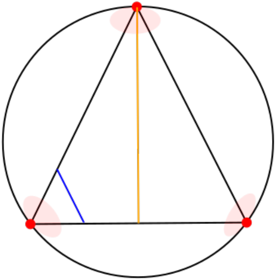


The geodesic joining two decorated vertices $\mathbf{v}_1, \mathbf{v}_2$ is called their horoball connection. Its length is given by the hyperbolic length of the intercepted geodesic segment.

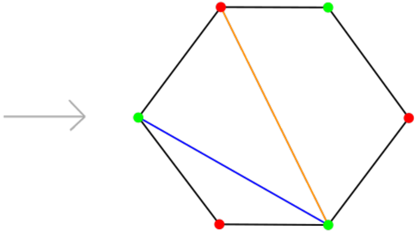
Decorated metrics

- The deformation space of decorated polygons $\mathfrak{D}(\widehat{\Pi_n^{\diamond}}) = \mathfrak{D}(\Pi_n^{\diamond}) \times \mathbb{R}^n = \mathbb{B}^{2n-3}$.
- The deformation space of once-punctured decorated polygons $\mathfrak{D}(\widehat{\Pi_n^{\odot}}) = \mathfrak{D}(\Pi_n^{\odot}) \times \mathbb{R}^n = \mathbb{B}^{2n-1}$.

Permitted arcs in decorated polygons



Arcs in a decorated ideal triangle $\widehat{\Pi_3^\diamond}$

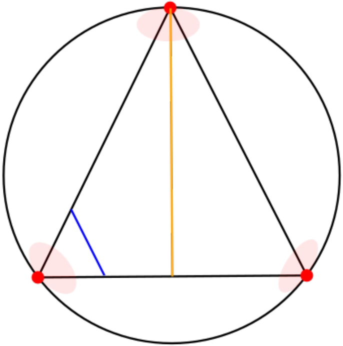


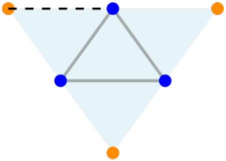
G-G and R-G diagonals in $\mathcal{P}_6$ with alternate G,R partitioning

$$\mathcal{A}\left(\Pi_n^\diamond\right) = \mathcal{Y}(\mathcal{P}_{2n})$$

Pruned arc complex

- The interior of the arc complex is called the *pruned* arc complex.
- Simplices not contained in the boundary decompose the polygon into disks with at most one decorated vertex. These are called *filling* simplices.





Admissible cone

Definition

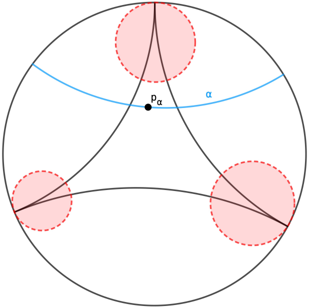
The *admissible cone* of a decorated possibly punctured polygon Π is defined to be the set of all infinitesimal deformations of the decorated metric $m \in \mathfrak{D}(\Pi)$ that uniformly lengthen all horoball connections. It is denoted by $\Lambda(m)$.

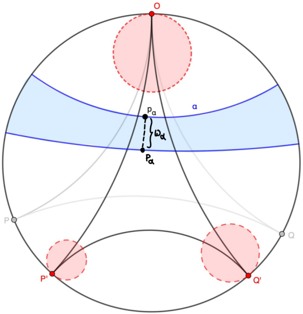
Lemma

The admissible cone of a decorated (possibly punctured) polygon Π*, endowed with a metric* m*, is an open convex subset of* $T_m\mathfrak{D}(\Pi)$.

Hyperbolic strip deformations

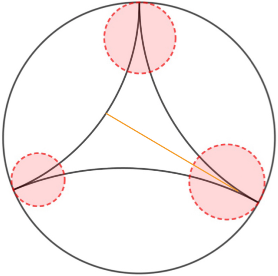
Hyperbolic strip deformation along a finite arc α with strip template $(\alpha_g, p_\alpha, w_\alpha)$.

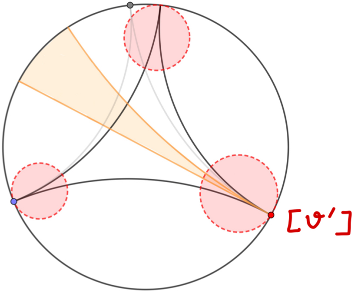




Parabolic strip deformations

Parabolic strip deformation along an infinite arc.





The strip map

Infinitesimal strip deformation:

$$\begin{array}{rcccl} f_\alpha & : & \mathfrak{D}(\Pi) & \longrightarrow & T\mathfrak{D}(\Pi) \\ & & m & \mapsto & f_\alpha(m) \in T_m\mathfrak{D}(\Pi) \end{array}$$

The strip map

Infinitesimal strip deformation:

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The **strip map**:

$$\begin{array}{rccc} f & : & \mathcal{A}(\Pi) & \longrightarrow & T_{[\rho]}\mathfrak{D}(\Pi) \\ & & \sum_{i=1}^{N} c_i\alpha_i & \mapsto & \sum_{i=1}^{\dim \mathfrak{D}(\Pi)} c_i f_{\alpha_i}(m) \end{array}$$

The strip map

Infinitesimal strip deformation:

$$\begin{array}{rcccl} f_\alpha & : & \mathfrak{D}(\Pi) & \longrightarrow & T\mathfrak{D}(\Pi) \\ & & m & \mapsto & f_\alpha(m) \in T_m\mathfrak{D}(\Pi) \end{array}$$

The **projectivised strip map**:

$$\begin{array}{rcccccc} \mathbb{P}f & : & \mathcal{A}(\Pi) & \xrightarrow{f} & T_m\mathfrak{D}(\Pi) & \xrightarrow{\mathbb{P}^+} & \mathbb{P}^+T_m\mathfrak{D}(\Pi) \\ & & x = \sum_{i=1}^{N} c_i\alpha_i & \mapsto & \sum_{i=1}^{\dim \mathfrak{D}(\Pi)} c_i f_{\alpha_i}(m) & \mapsto & [f(x)] \end{array}$$

Main theorems

Theorem (P.)

Let $\widehat{\Pi_n^\diamond}$ ($n \geq 3$) be a decorated n-gon with a metric $m \in \mathfrak{D}(\widehat{\Pi_n^\diamond})$. Fix a choice of strip template. Then the projectivised strip map $\mathbb{P}f$, when restricted to the pruned arc complex $\mathcal{PA}(\Pi)$, is a homeomorphism onto the projectivised admissible cone $\mathbb{P}^+(\Lambda(m))$.

Theorem (P.)

Let $\widehat{\Pi_n^\diamond}$ ($n \geq 2$) be a decorated once-punctured polygon with a metric $m \in \mathfrak{D}(\widehat{\Pi_n^\diamond})$. Fix a choice of strip template. Then the infinitesimal strip map $\mathbb{P}f$, when restricted to the pruned arc complex $\mathcal{PA}(\widehat{\Pi_n^\diamond})$, is a homeomorphism onto the projectivised admissible cone $\mathbb{P}^+(\Lambda(m))$.

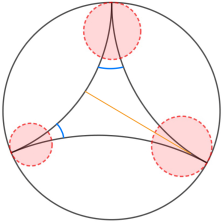
Motivation

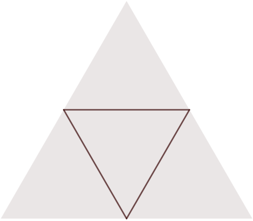
Theorem (Danciger-Guéritaud-Kassel)

Let $S = S_{g,n}$ or $T_{h,n}$ be a compact hyperbolic surface with totally geodesic boundary. Let $m = ([\rho]) \in \mathfrak{D}(S)$ be a metric. Fix a choice of strip template $\{(\alpha_g, p_\alpha, w_\alpha)\}_{\alpha \in \mathcal{K}}$ with respect to m. Then the restriction of the projectivised infinitesimal strip map $\mathbb{P}f : \mathcal{PA}(S) \longrightarrow \mathbb{P}^+(T_m\mathfrak{D}(S))$ is a homeomorphism on its image $\mathbb{P}^+(\Lambda(m))$.

Here the admissible cone $\Lambda(m)$ consists of all infinitesimal deformations that uniformly lengthen every non-trivial closed geodesic.

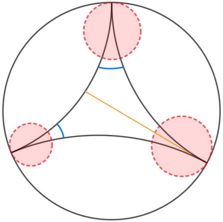
Illustration

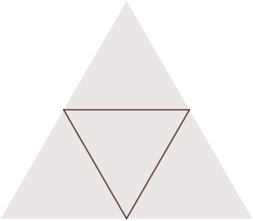




$\widehat{\Pi_n^\diamond}$: a decorated ideal triangle.

Illustration

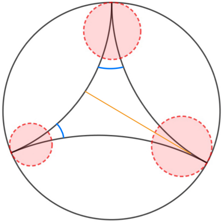


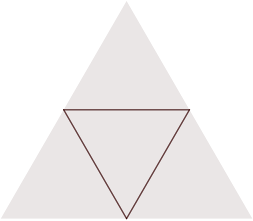


$\widehat{\Pi_n^\diamond}$: a decorated ideal triangle.

- $\mathfrak{D}(\widehat{\Pi_3^\diamond}) \simeq \mathbb{B}^3$,

Illustration

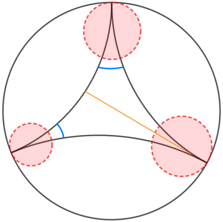


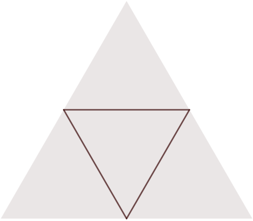


$\widehat{\Pi_n^\diamond}$: a decorated ideal triangle.

- $\mathfrak{D}(\widehat{\Pi_3^\diamond}) \simeq \mathbb{B}^3$,
- Finite arc complex; $\mathcal{PA}(\widehat{\Pi_3^\diamond}) \simeq \mathbb{B}^2$,

Illustration





$\widehat{\Pi_n^\diamond}$: a decorated ideal triangle.

- $\mathfrak{D}(\widehat{\Pi_3^\diamond}) \simeq \mathbb{B}^3$,
- Finite arc complex; $\mathcal{PA}(\widehat{\Pi_3^\diamond}) \simeq \mathbb{B}^2$,
- three edges of the triangle $\leftrightarrow$ 3 horoball connections

Theorem

The projectivised strip map $\mathbb{P}f : \mathcal{P}\mathcal{A}(\widehat{\Pi_3^{\diamond}}) \longrightarrow \mathbb{P}^{+}\Lambda(m)$ *is a homeomorphism.*

Idea of the proof:

Steps of the proof

Theorem

The projectivised strip map $\mathbb{P}f : \mathcal{PA}(\widehat{\Pi_3^\diamond}) \longrightarrow \mathbb{P}^+\Lambda(m)$ *is a homeomorphism.*

Idea of the proof:

1. $\mathbb{P}f$ is a local homeomorphism.

Steps of the proof

Theorem

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Idea of the proof:

1. $\mathbb{P}f$ is a local homeomorphism.
2. $\mathbb{P}f$ is proper.

Steps of the proof

Theorem

The projectivised strip map $\mathbb{P}f : \mathcal{P}\mathcal{A}(\widehat{\Pi_3^{\diamond}}) \longrightarrow \mathbb{P}^{+}\Lambda(m)$ *is a homeomorphism.*

Idea of the proof:

1. $\mathbb{P}f$ is a local homeomorphism.
2. $\mathbb{P}f$ is proper.
3. Steps (1) and (2) imply that $\mathbb{P}f$ is a covering map.

Steps of the proof

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Idea of the proof:

1. $\mathbb{P}f$ is a local homeomorphism.
2. $\mathbb{P}f$ is proper.
3. Steps (1) and (2) imply that $\mathbb{P}f$ is a covering map.
4. Domain, codomain are simply connected. Conclude.

Step 1: Local homeomorphism

Remark: Every point $x \in \mathcal{PA}(\widehat{\Pi_3^{\diamond}})$ is contained in the interior of a unique filling simplex σ_x.

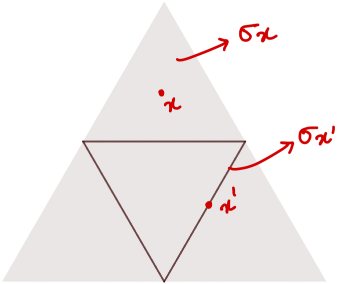
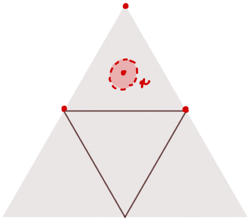


Figure: codim $(\sigma_x) = 0$, codim $(\sigma_{x'}) = 1$

Case 1: Local homeomorphism around x, with $\operatorname{codim}(\sigma_x) = 0$.



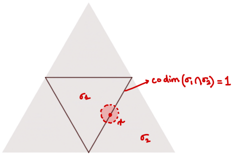
It is enough to show

Theorem

Let σ be a codimension zero simplex of the arc complex. Then the set $B = \{f_\alpha(m) \mid \alpha \in \sigma^{(0)}\}$ forms a basis of $T_m \mathfrak{D}(\widehat{\Pi_3^\diamond})$.

Step 1: Local homeomorphism

Case 2: Local homeomorphism around x, with $\operatorname{codim}(\sigma_x) = 1$.



It is enough to show that

Theorem

Let $\sigma_1, \sigma_2 \in \mathcal{A}\left(\widehat{\Pi_3^\diamond}\right)$ *s.t* $\operatorname{codim}(\sigma_i) = 0$ *for* $i = 1, 2$, $\operatorname{codim}(\sigma_1 \cap \sigma_2) = 1$ *and* $\operatorname{int}(\sigma_1 \cap \sigma_2) \subset \mathcal{PA}(\widehat{\Pi_3^\diamond})$*. Then,*

$$\operatorname{int}(\mathbb{P}f(\sigma_1)) \cap \operatorname{int}(\mathbb{P}f(\sigma_2)) = \varnothing.$$