LIPN 9 Avril

Pyramides et diamants

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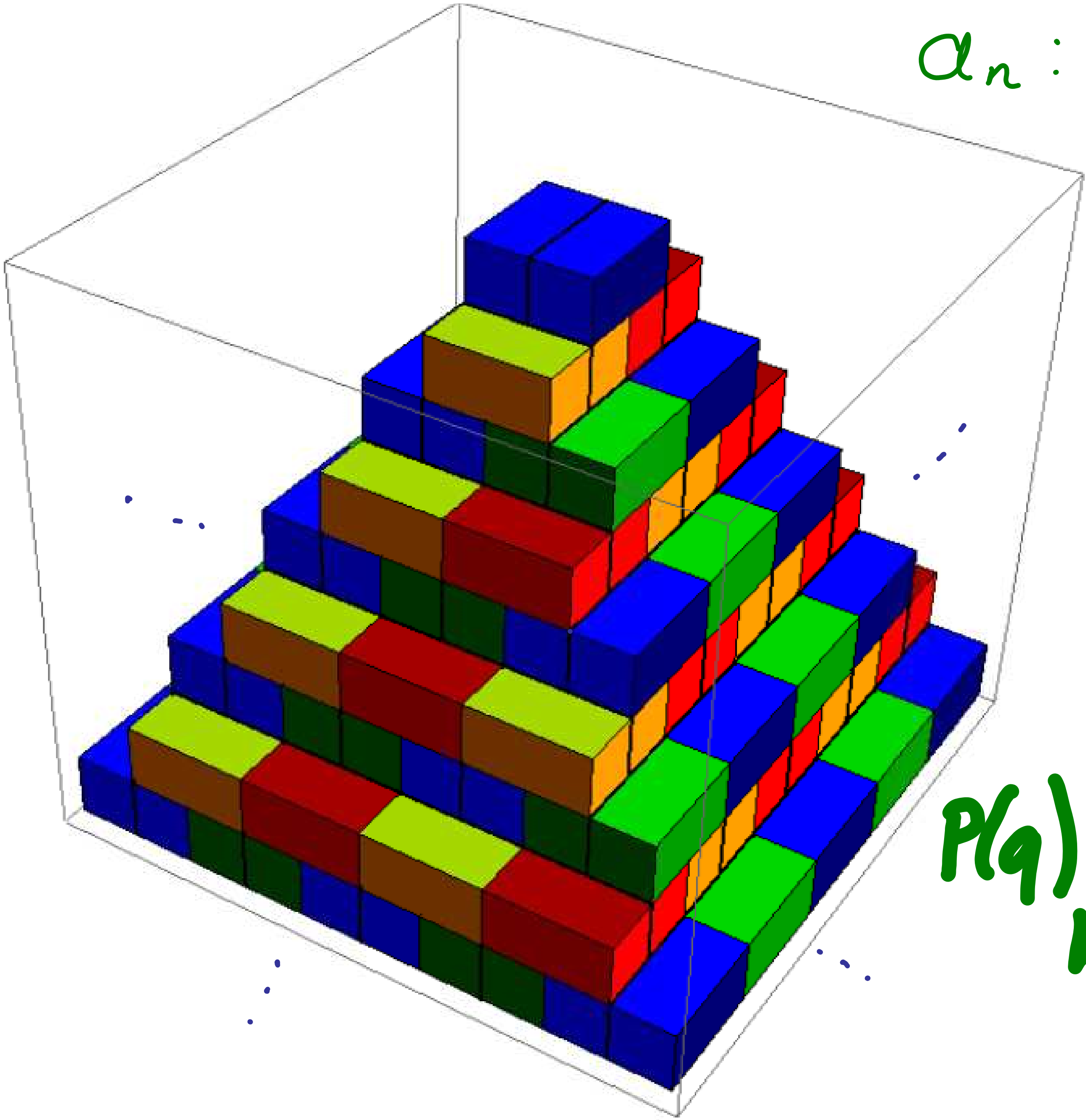
avec J. Bouttier et G. Chapuy

Pyramid partitions (Young 05)



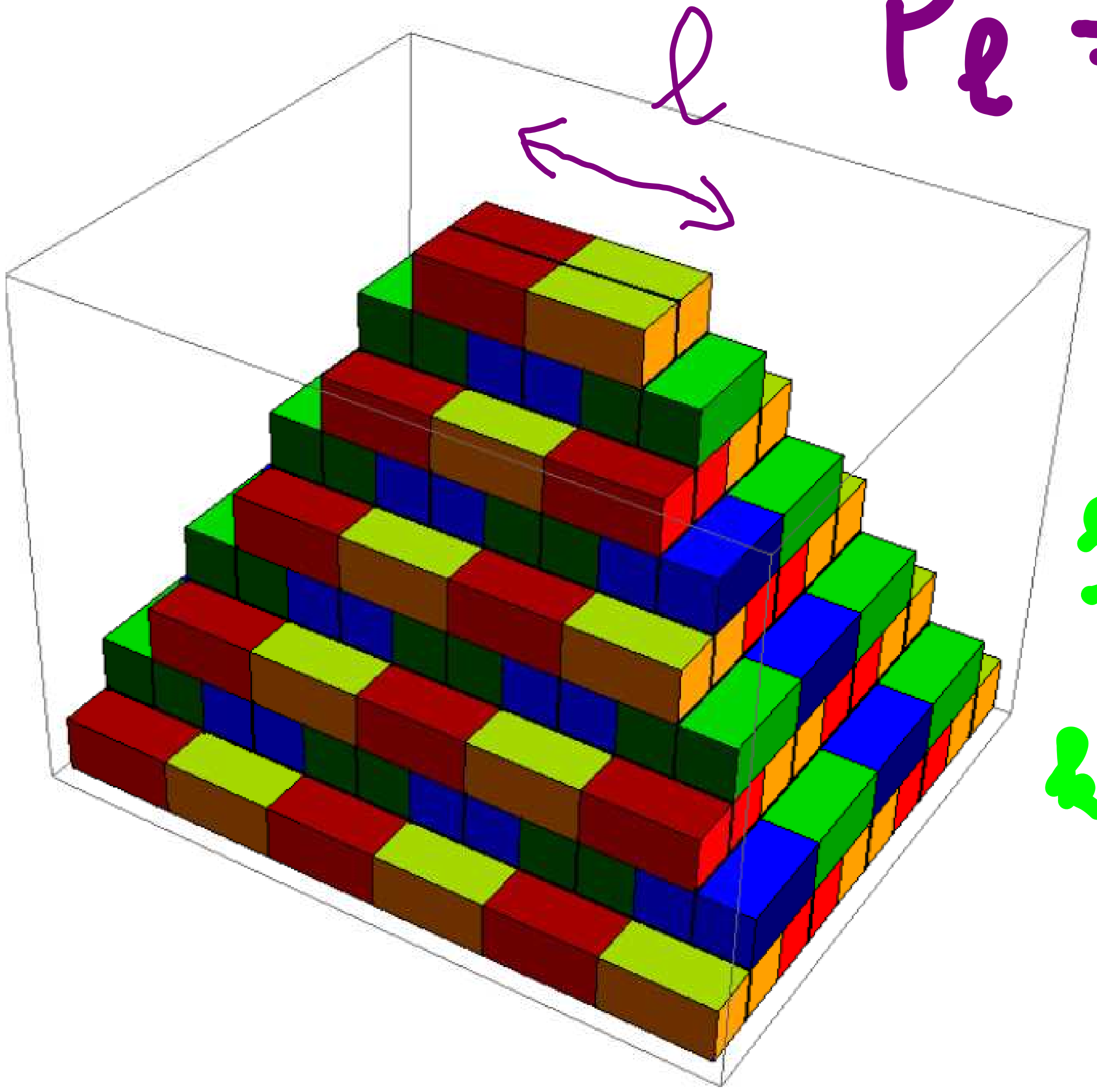
a_n : # de façons de retirer n duplos

$$P(q) = \sum_{n \geq 0} a_n q^n$$

$$P(q) = \prod_{k \geq 1} \frac{(1+q^{2k-1})^{2k-1}}{(1-q^{2k})^{2k}}$$

ℓ-pyramids (Young)

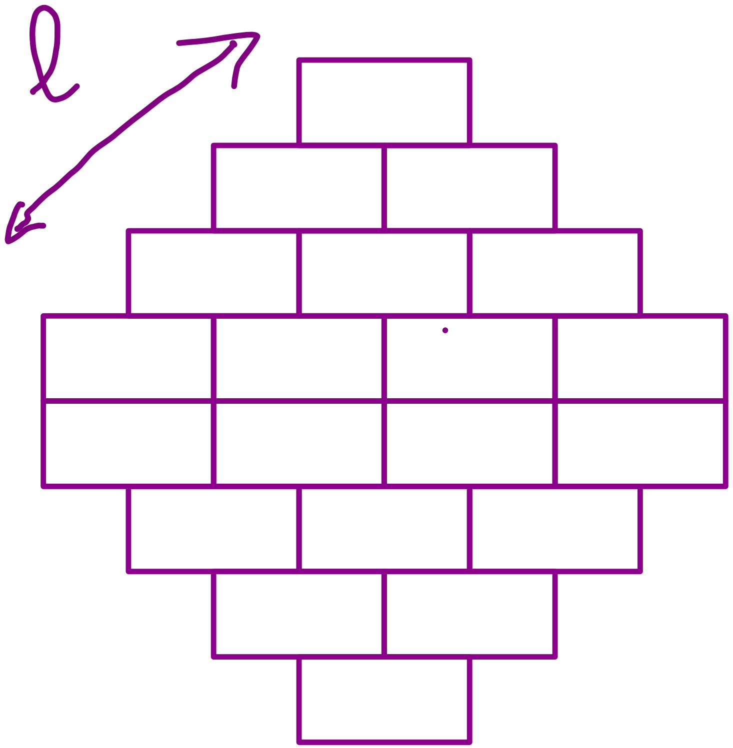
$$P_\ell = P(q) \cdot A_{\ell-1}(q)$$



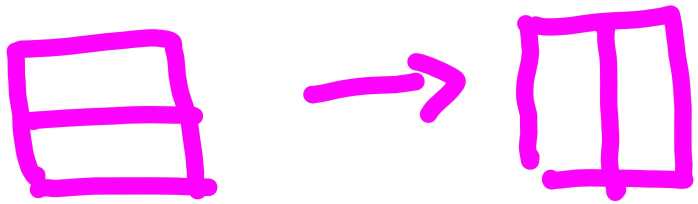
$$\prod_{k=0}^{\ell-1} \left(1+q^{2k+1}\right)^{\ell-k}$$

Aztec diamond (Elkies et al, 1992)

Initial configuration



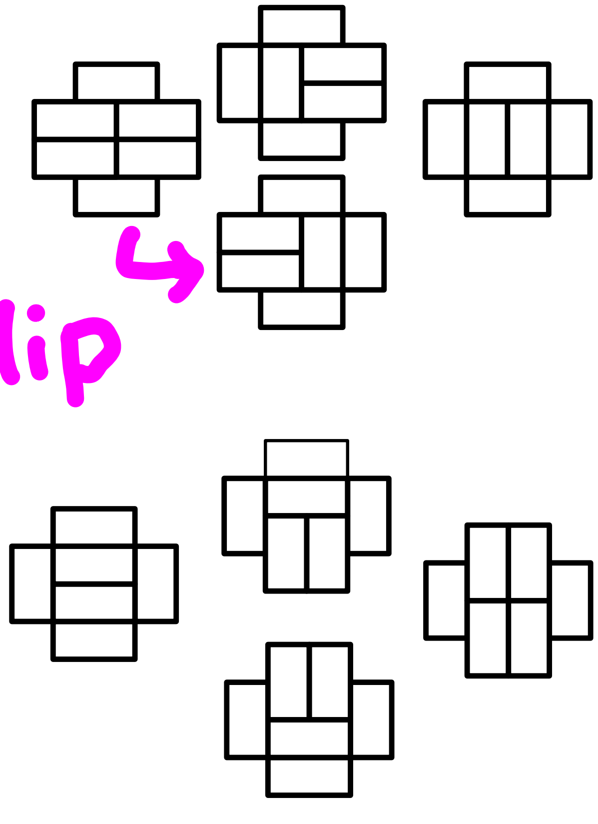
FLIP



Exemple $\ell = 2$

Initial configuration

Flip



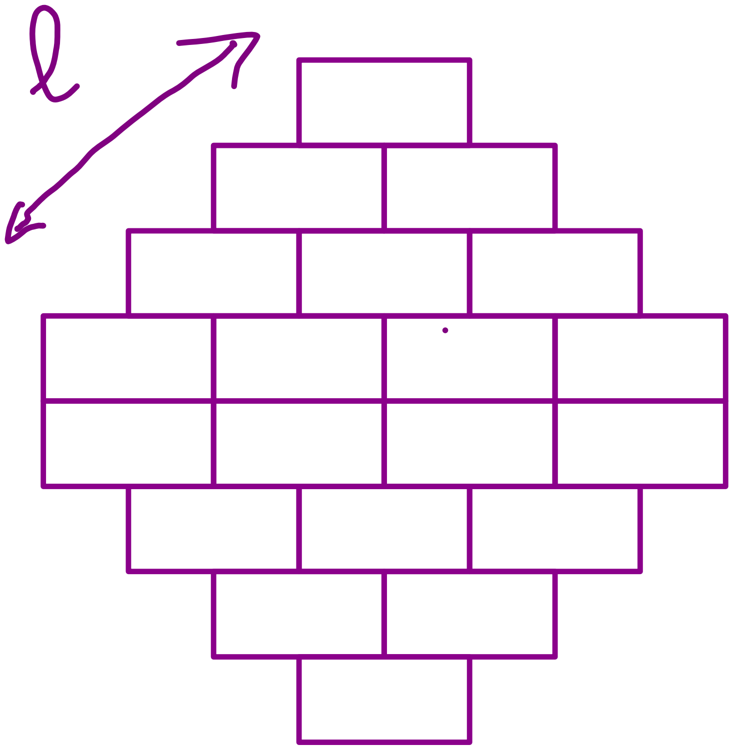
8 tilings

Theorem (92)

$2^{\binom{\ell+1}{2}}$ tilings

Aztec diamond (Elkies et al, 1992)

d_n: # of tilings after n flips



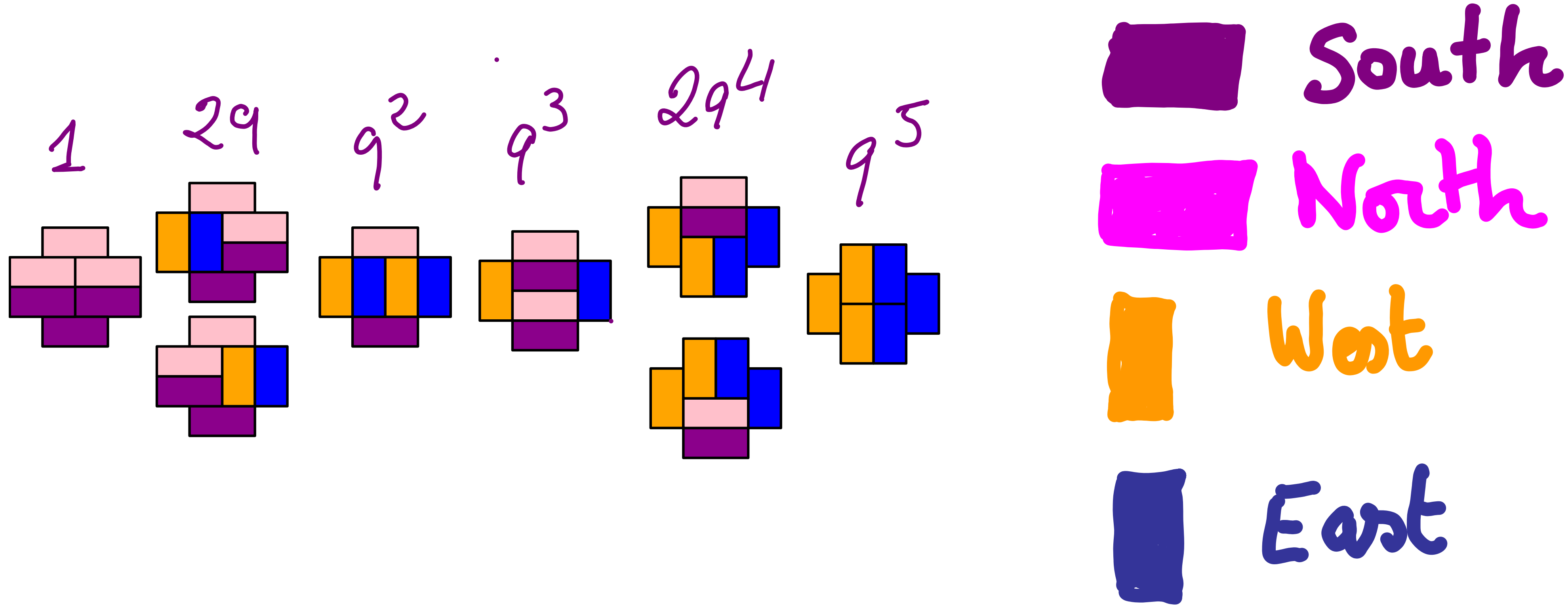
Flip generating func.

$$A_\ell(q) = \sum_n d_n q^n$$

$$A_\ell(q) = \prod_{j=0}^{\ell-1} (1+q^{2j+1})^{\ell-j}$$

Example $l = 2$

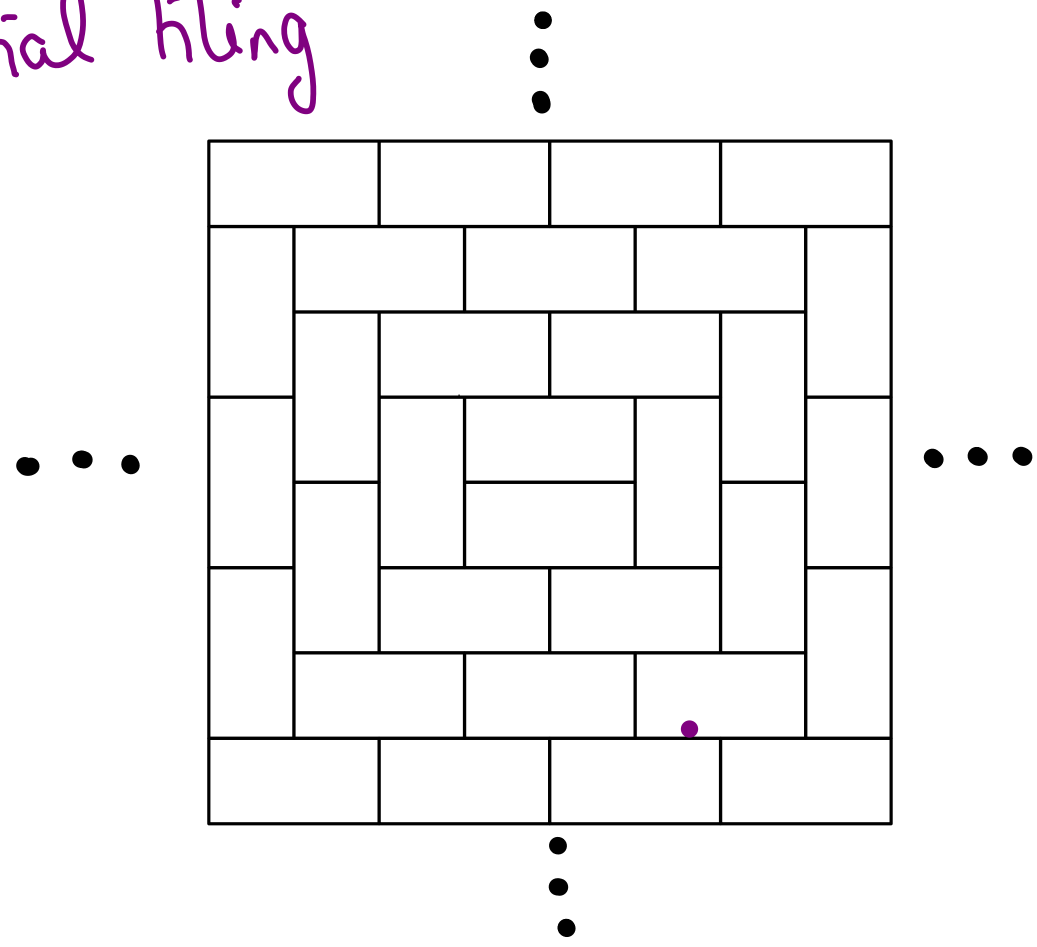
$$A_2(q) = (1+q)^2(1+q^3)$$
$$= 1+2q+q^2+q^3+2q^4+q^5$$



Pyramid partitions

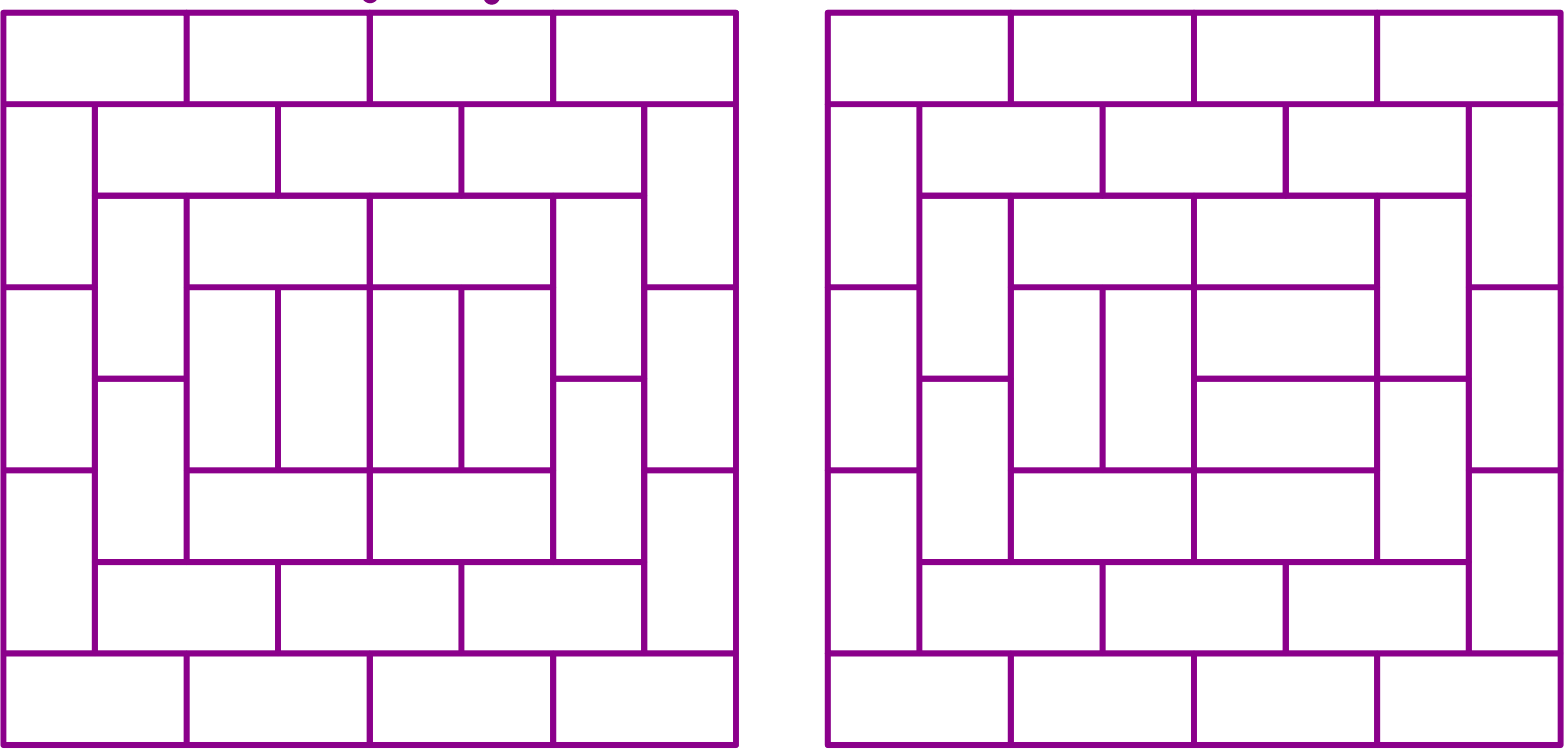
$\longleftrightarrow$ Domino Tilings on the entire plane

Initial tiling



Pyramid partitions ⟷ Tilings

1 flip 2 flips



a_n: # tilings after n flips

$$\sum_n a_n q^n = \prod_k \frac{(1+q^{2k-1})^{2k-1}}{(1-q^{2k})^{2k}}$$

Goal

- Understand the global picture

Are those families of tilings part of the same families?

Result (Bouttier, Chapuy, C.)

Given a word on length 2ℓ on the alphabet $\{+,-\}$

$$w = (w_1, \dots, w_{2\ell})$$

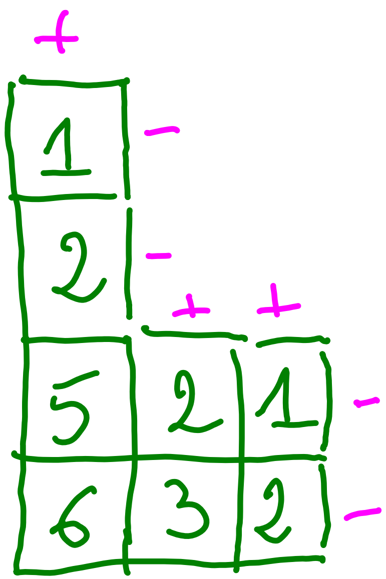
there exists a family of domino tilings whose flip generating function is

$$F_w(q) = \prod_{\substack{i<j \\ w_i=+,\, w_j=-}} \left(1+\varepsilon_{ij}\, q^{j-i}\right)^{\varepsilon_{ij}} \qquad \varepsilon_{ij} = \begin{cases} 1 & j-i \text{ odd} \\ -1 & j-i \text{ even} \end{cases}$$

Remark

This is a hook formula

$$w = (+, -, -, +, +, -, -)$$



$$F_w(q) = \frac{(1+q)^2(1+q^3)(1+q^5)}{(1-q^2)^2\,(1-q^6)}$$

$$w \longmapsto \lambda(w)$$

$$F_w(q) = \prod_{\substack{x\in\lambda \\ h(x)\ \text{odd}}} (1+q^{h(x)}) \prod_{\substack{x\in\lambda \\ h(x)\ \text{even}}} \frac{1}{1-q^{h(x)}}$$

Special cases

- $w = (\underbrace{+, -, \ldots, +-}_{2\ell})$

  Aztec diamond

- $w = (\ldots, +, +, -, -, \ldots)$

  Pyramid partitions

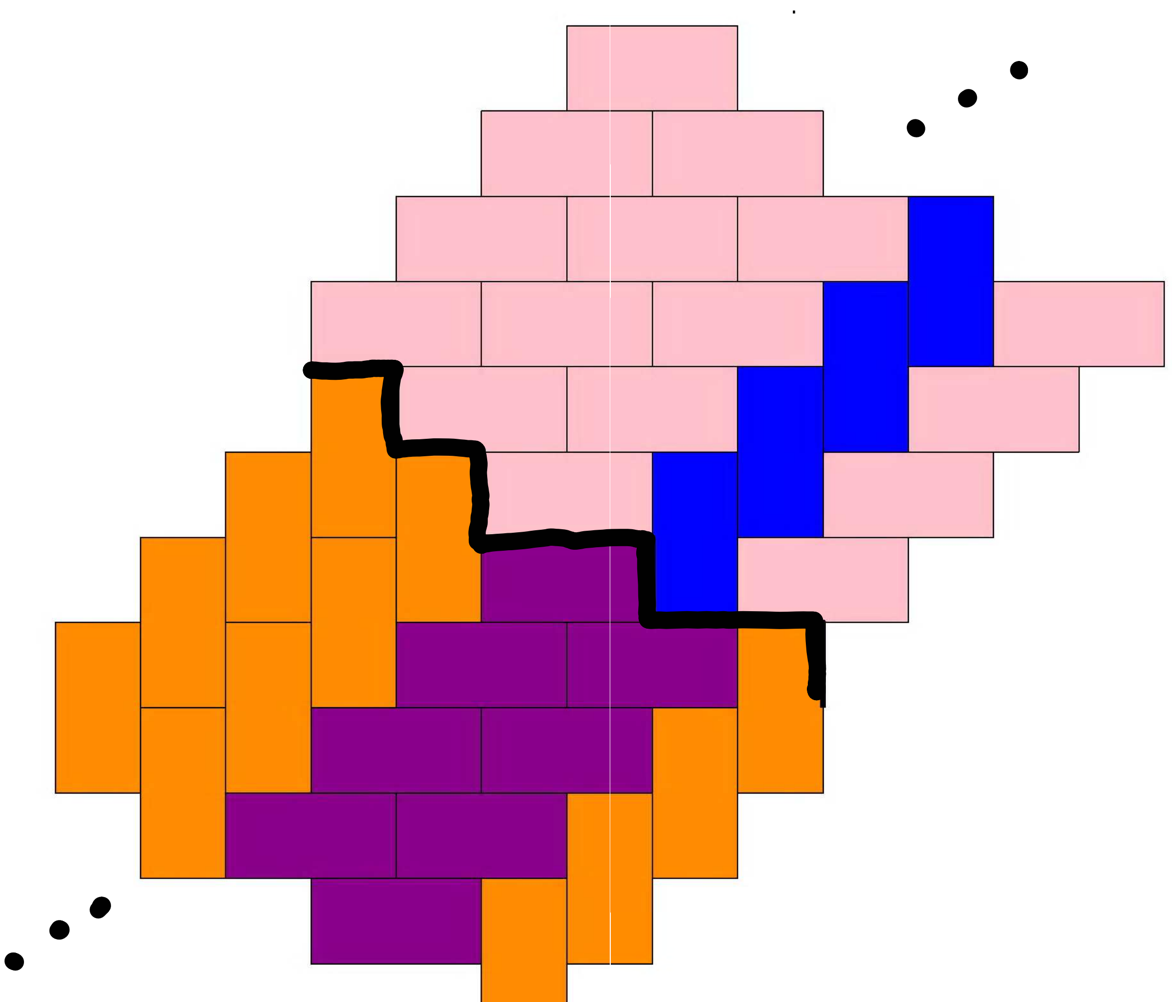
- $w = (\ldots, +, +, (+, -)^{\ell}, -, -, \ldots)$

  ℓ - pyramid partitions

1		
3	1	
5	3	1

What are those tilings?

$w = (+,+,+,+,+,-,-,-,+,+)$



① Path

$w_i = +$

i odd

i even

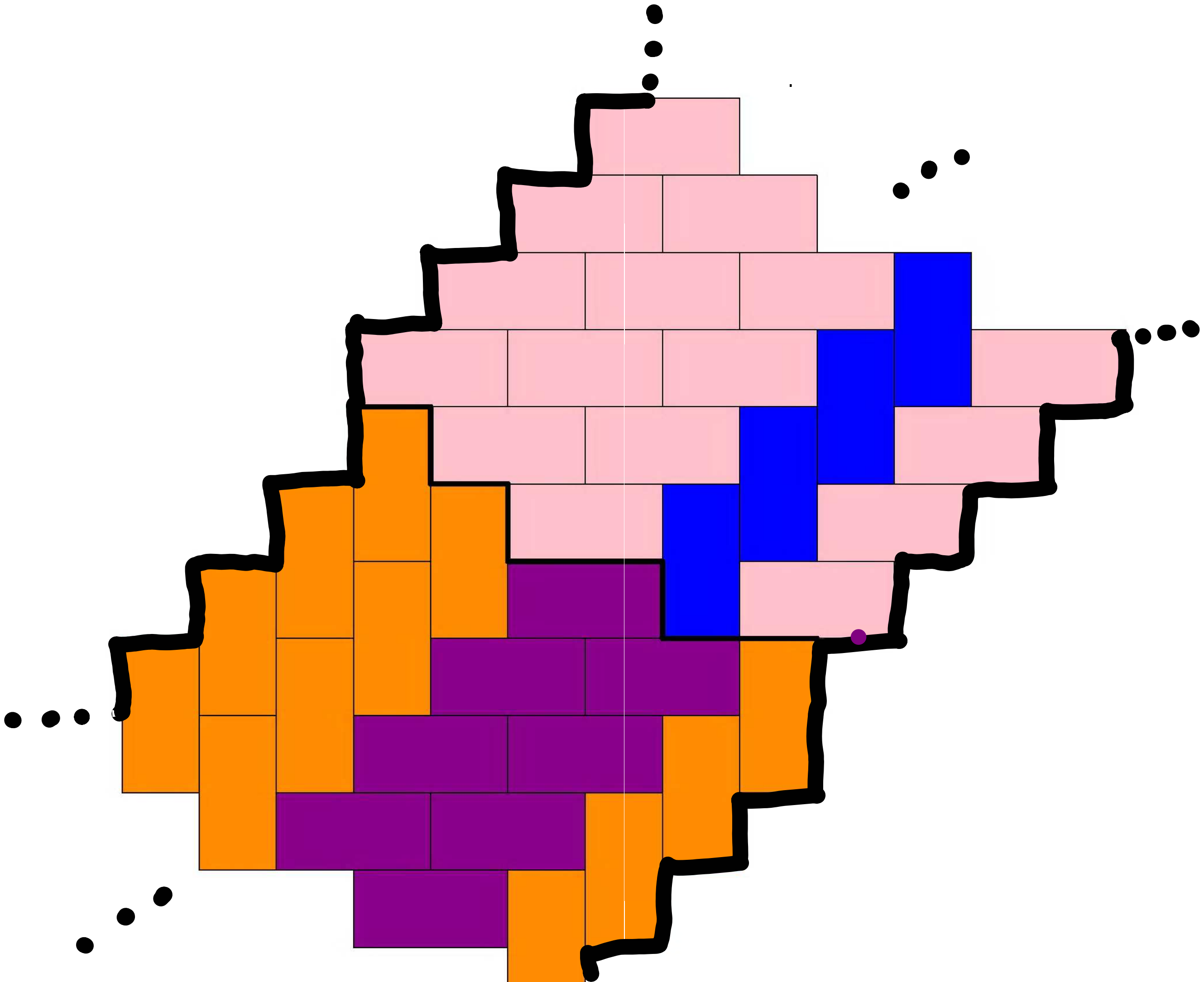
$w_i = -$

i even

i odd

$w = (+,+,+,+,+,-,-,-,+,+)$

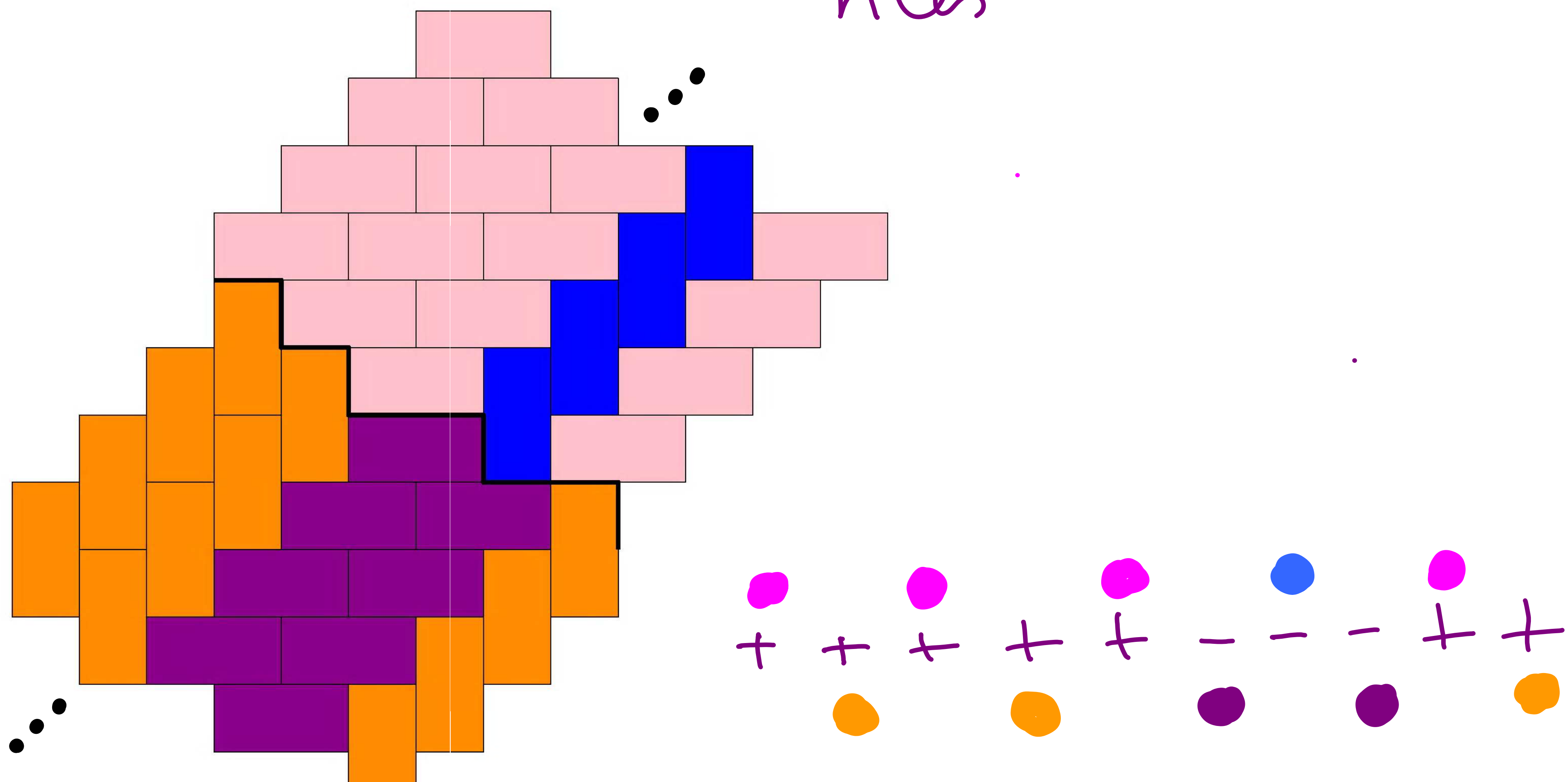
① Path



② Region

③ Minimal tiling

above the path : pink and blue tiles
under the path : orange and purple tiles

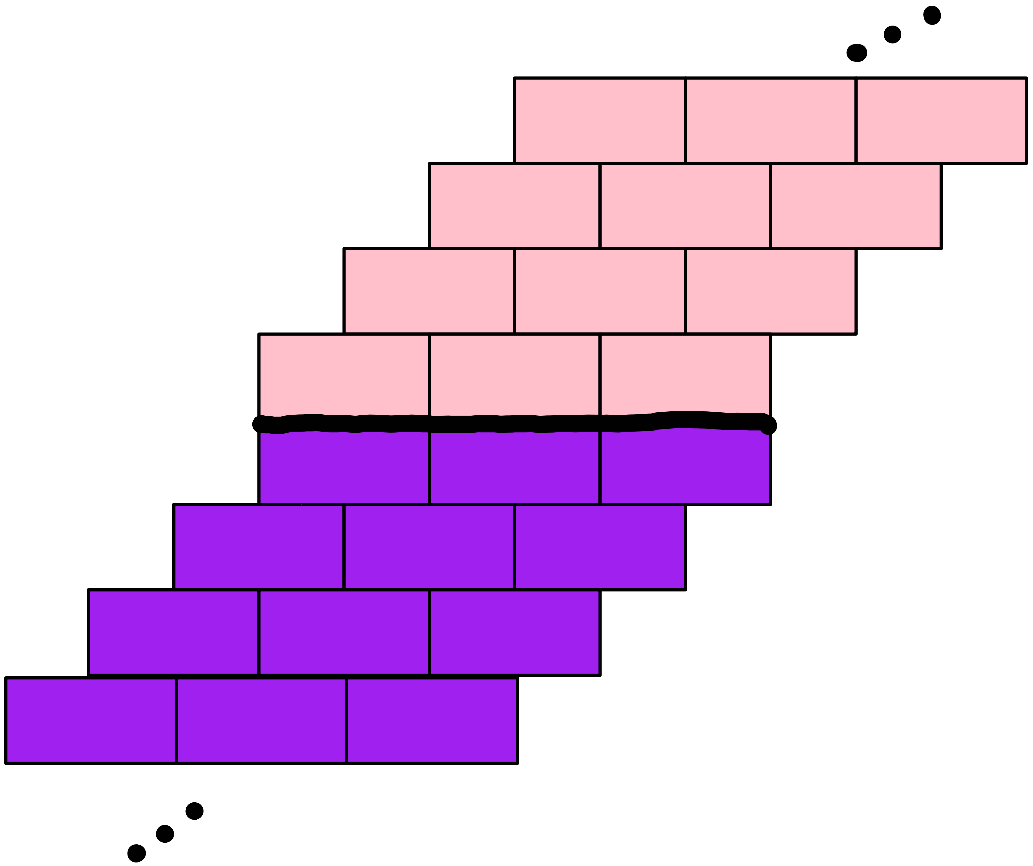


Example

$w = (+, -, +, -, +, -)$

Aztec diamond of size 3

Path $= (- - - - - -)$

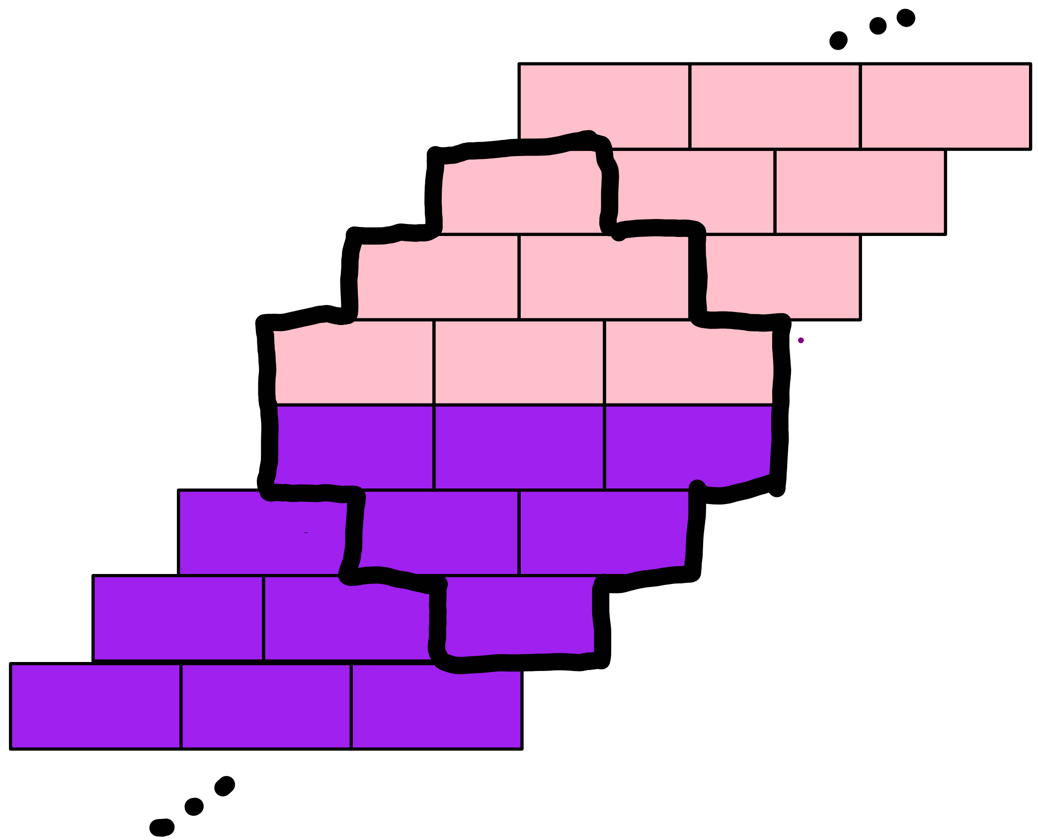


Example

$w = (+, -, +, -, +, -)$

Aztec diamond of size 3

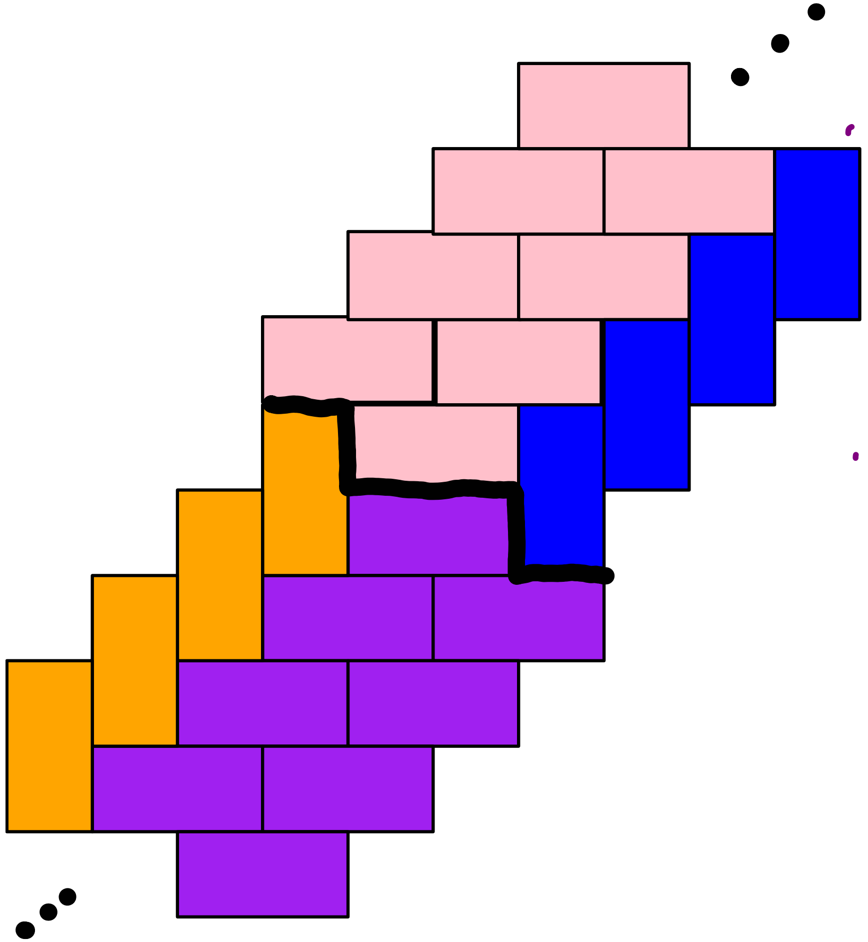
Path $= (- - - - - -)$



Example 2

$w = (+, +, +, -, -, -)$

Semi pyramid partitions



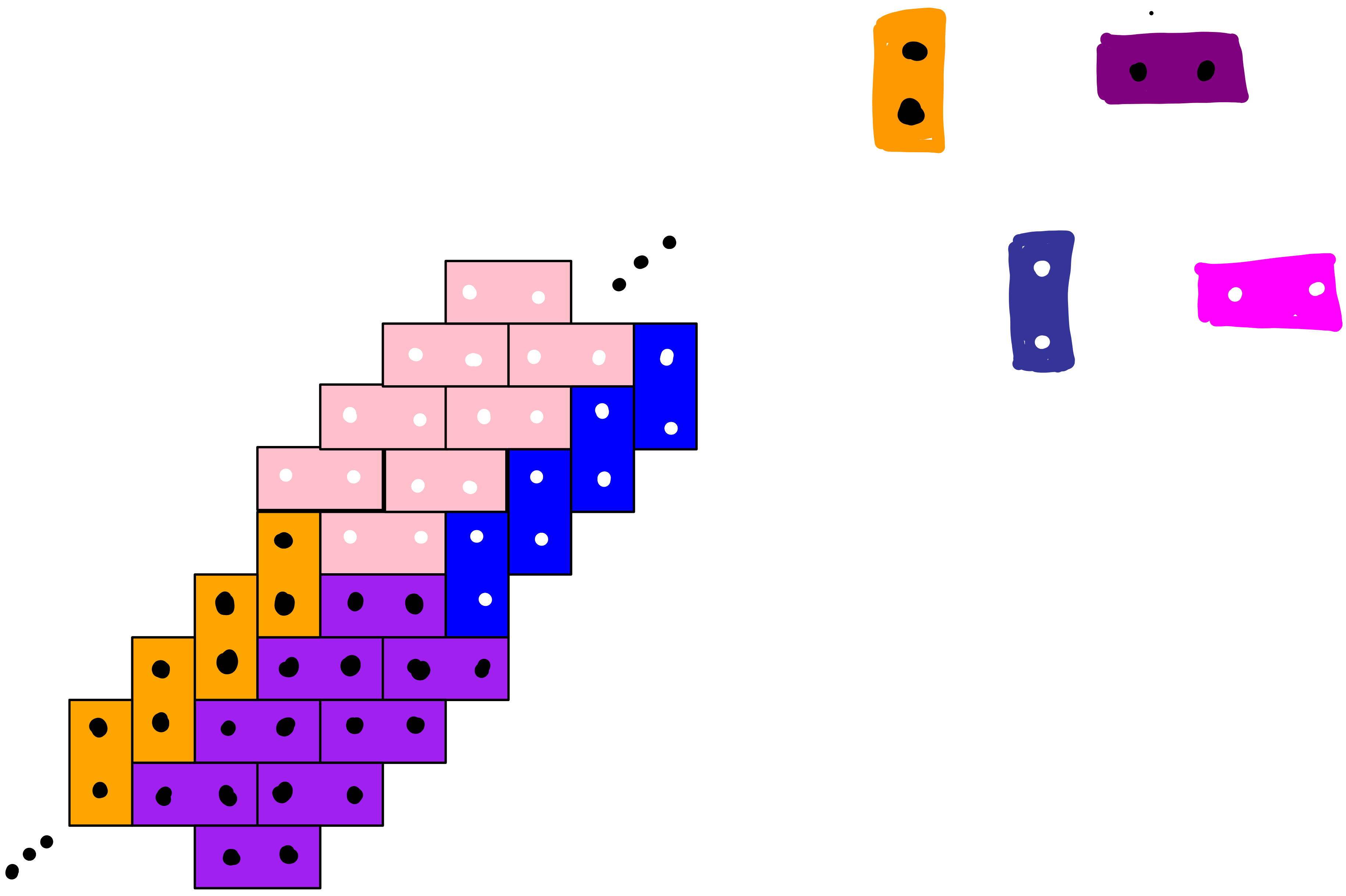
Proposition

There exists a bijection between

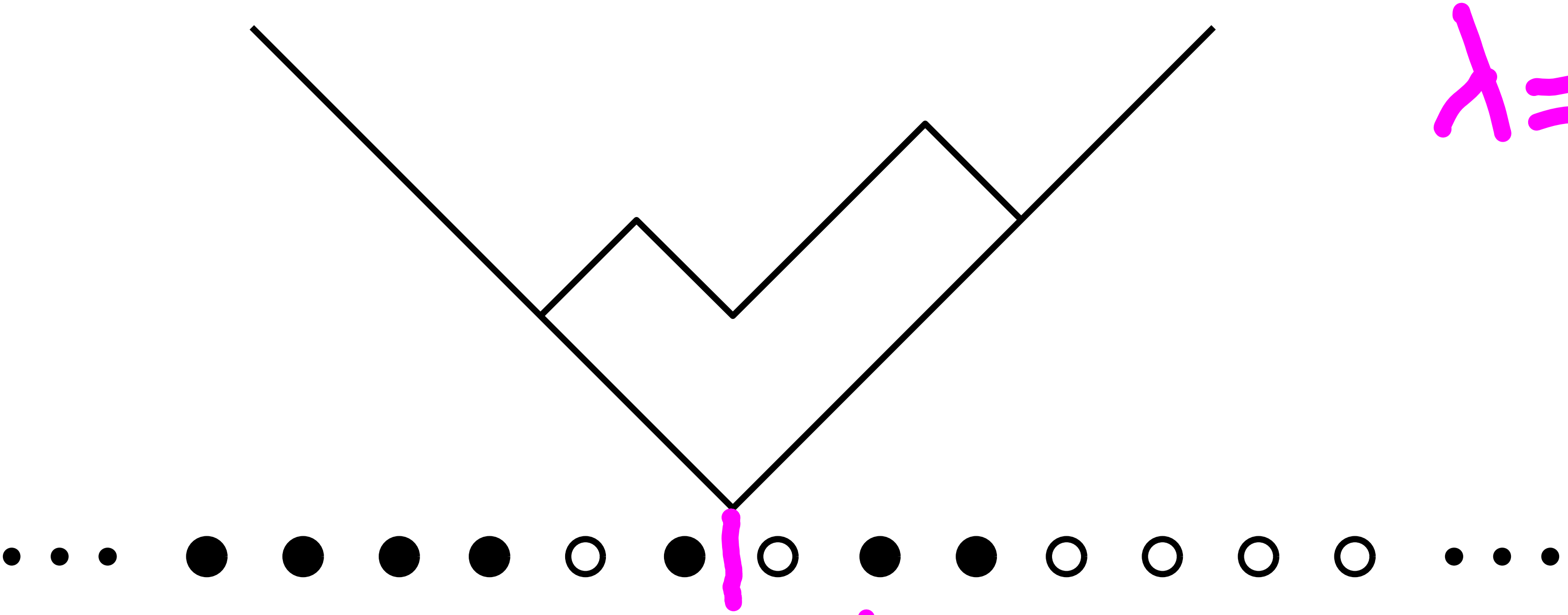
- tilings defined by w of length 2ℓ
- Sequences of integer partitions $\lambda^{(0)}, \dots, \lambda^{(2\ell)}$ with "some conditions"

such that the min number of flips is $|\lambda^{(0)}| + \dots + |\lambda^{(2\ell)}|$.

From tilings to particles



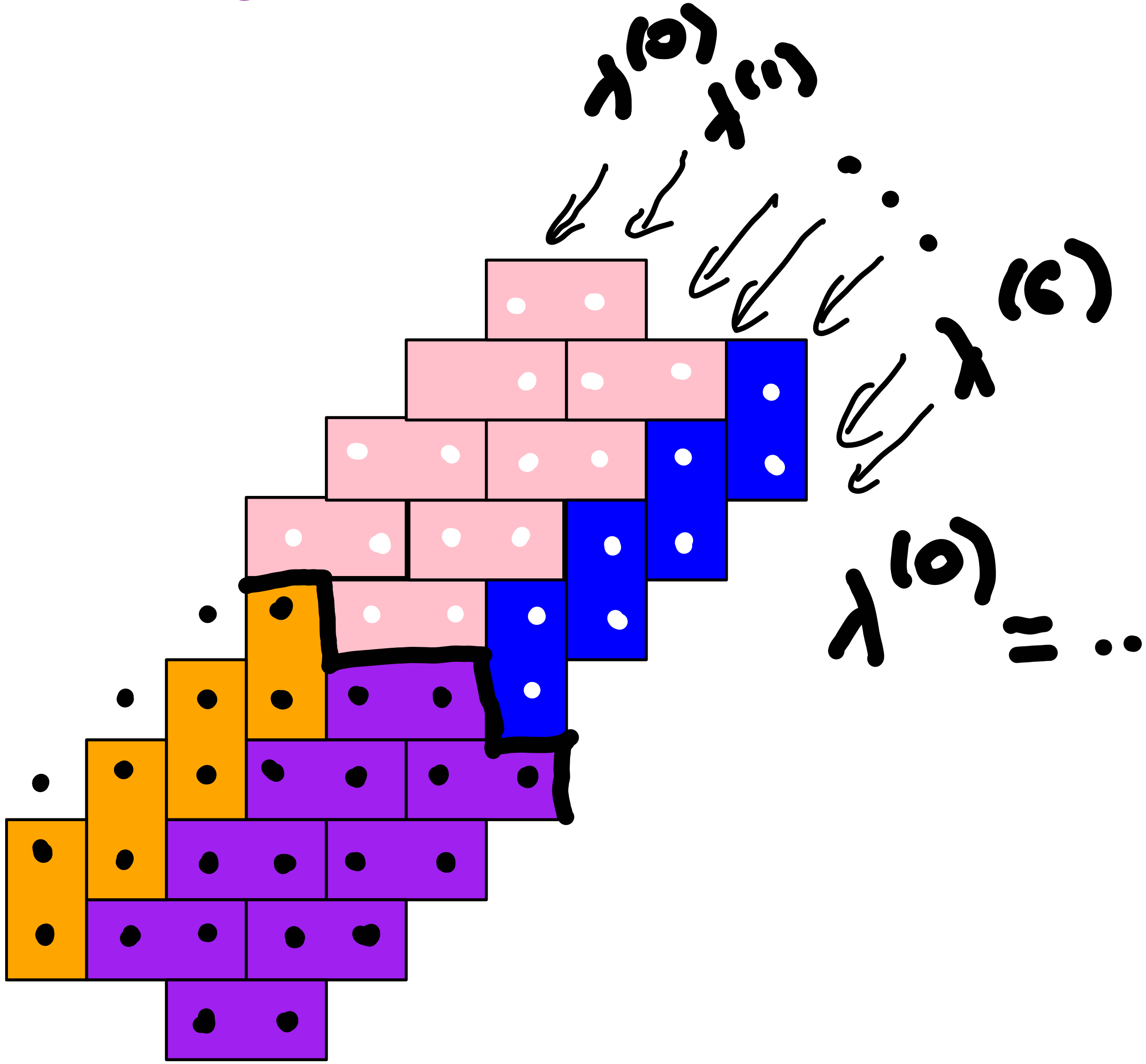
From particles to integer partitions



$\lambda = (3,1)$

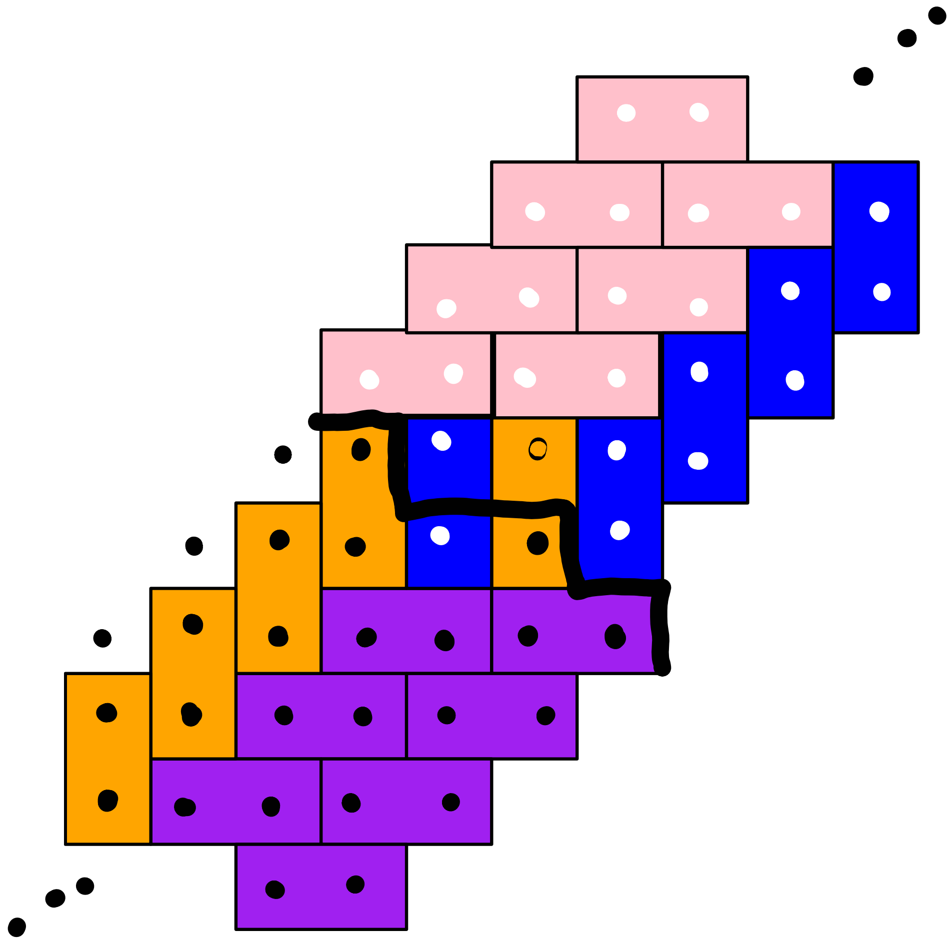
Maya diagram

From tilings to sequences of integer partitions



$$\lambda^{(0)} = \dots = \lambda^{(6)} = \emptyset$$

One flip

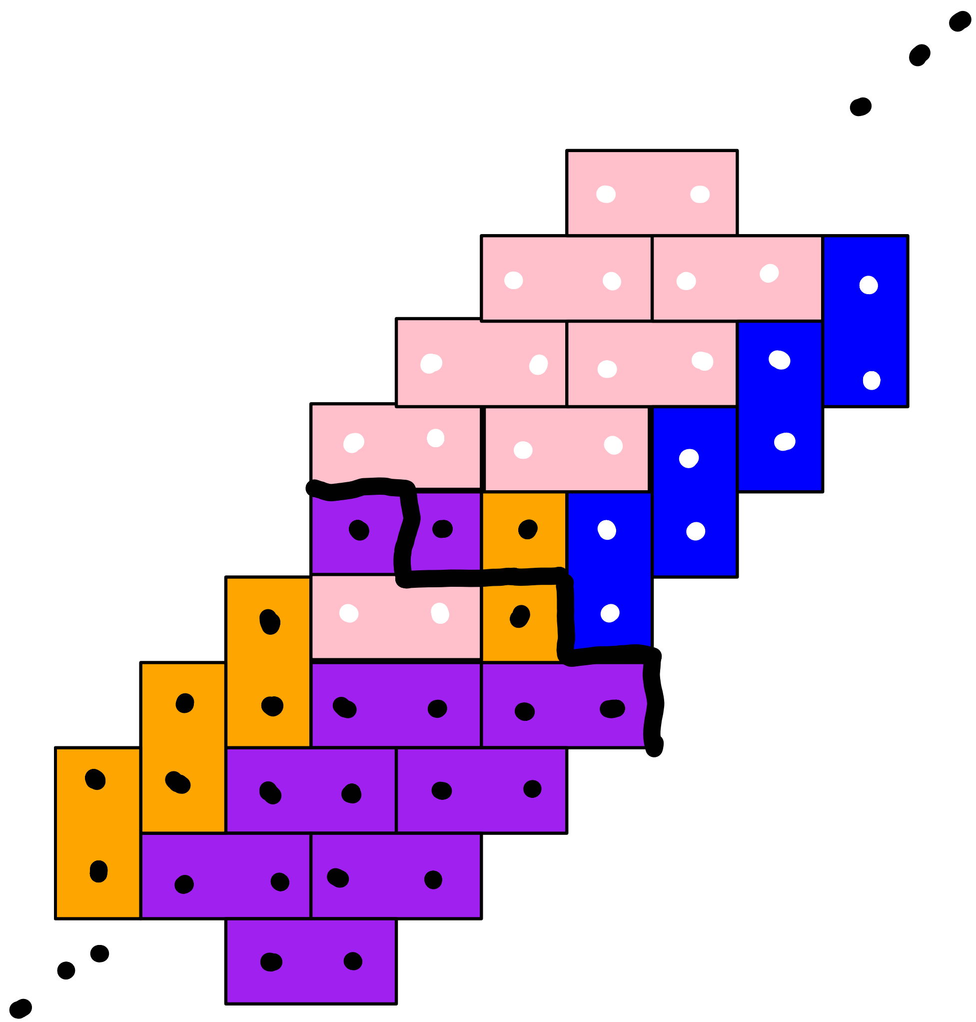


$$\lambda^{(0)} = \lambda^{(1)} = \emptyset = \lambda^{(2)}$$

$$\lambda^{(3)} = (1)$$

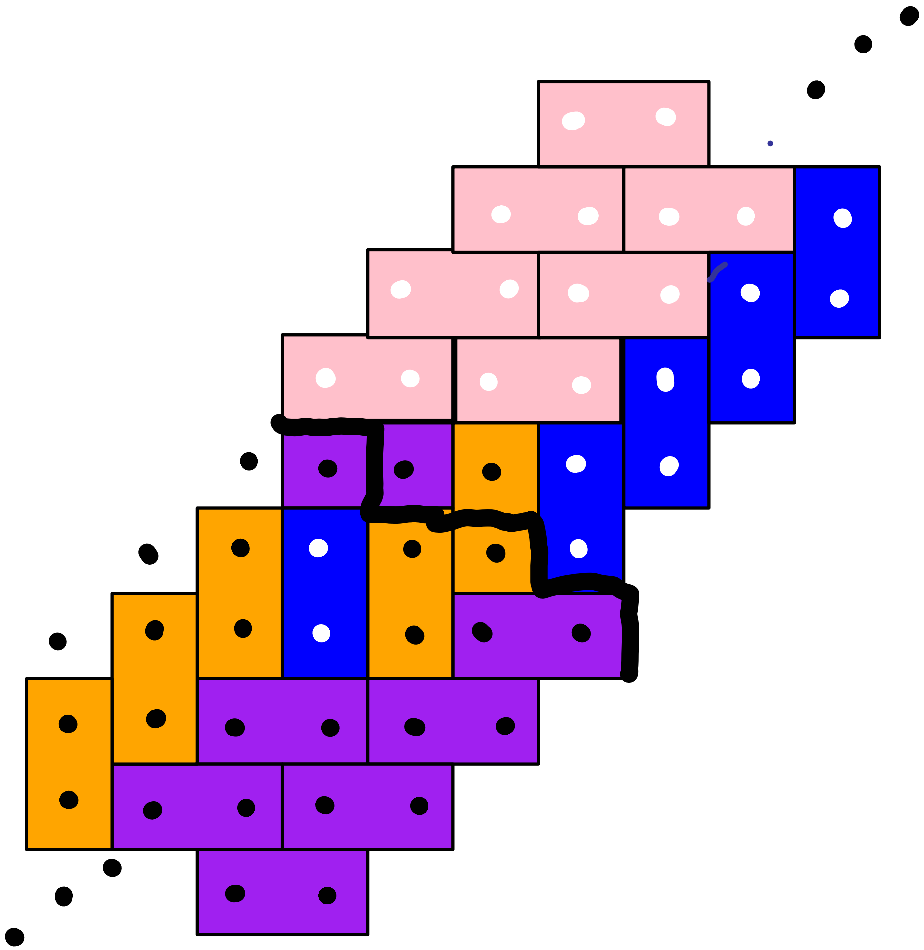
$$\lambda^{(4)} = \dots = \lambda^{(6)} = \emptyset$$

Two flips



$\lambda^{(2)} = \lambda^{(3)} = (1)$

Three flips



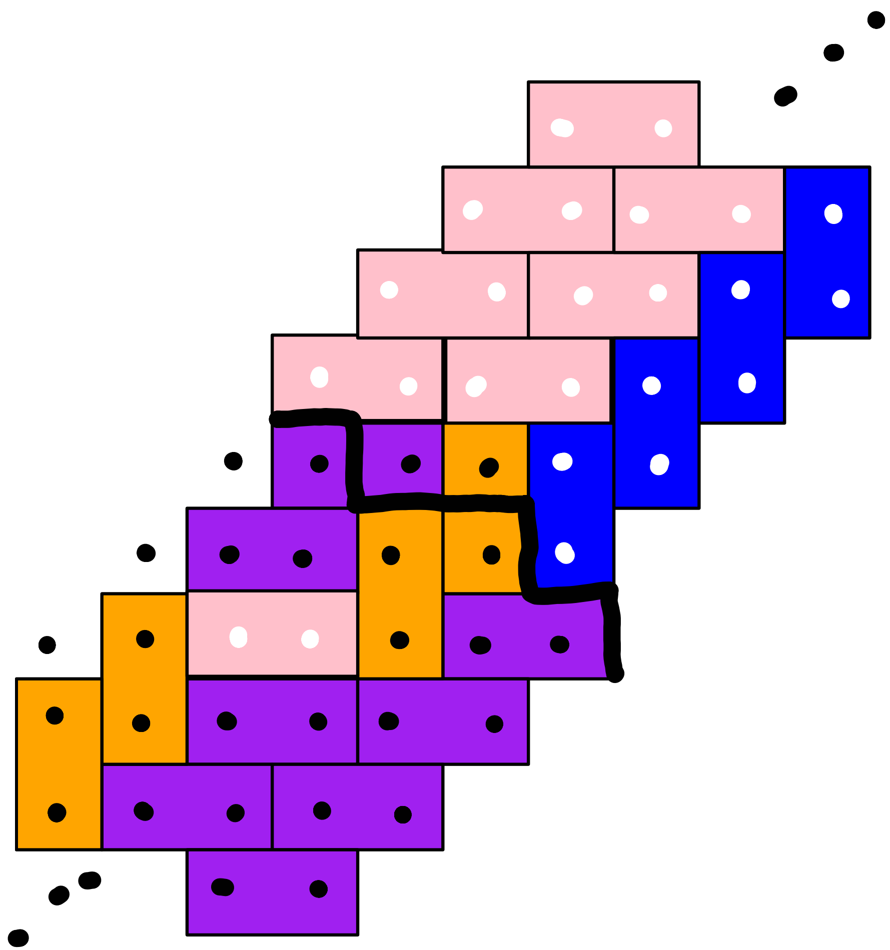
$$\lambda^{(2)} = (1)$$

$$\lambda^{(3)} = (1,1)$$

Four flips

$\lambda^{(2)} = (1,1)$

$\lambda^{(3)} = (1,1)$



After n flips

$\sum |\lambda^{(i)}| = n$

$\lambda^{(0)} \stackrel{i}{=} \lambda^{(6)} = \emptyset$

Proposition

- $\lambda^{(2i)}$ and $\lambda^{(2i-1)}$ differ by a vertical strip
- $\lambda^{(2i+1)}$ and $\lambda^{(2i)}$ differ by a horiz. strip

Moreover

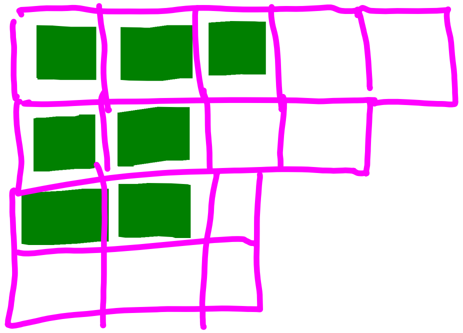
if $w_i = +$ then $\lambda^{(i-1)} \subseteq \lambda^{(i)}$

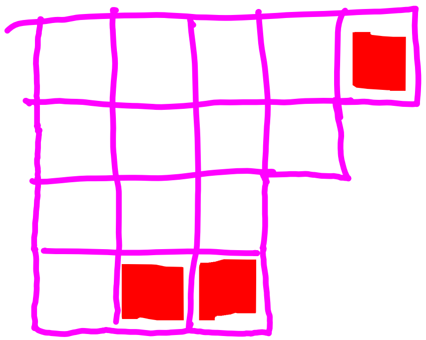
if $w_i = -$ then $\lambda^{(i)} \subseteq \lambda^{(i-1)}$

$\lambda = (5,4,3,3)$

$\alpha = (3,2,2)$

$\alpha \subseteq \lambda$

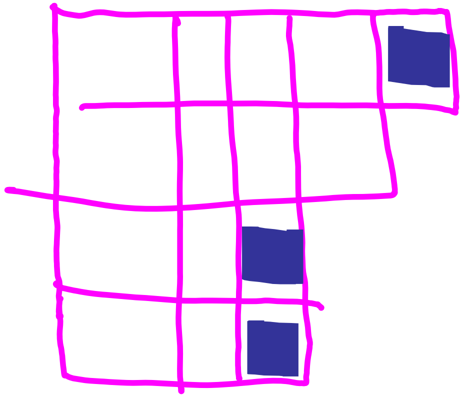




$\mu = (4,4,3,1)$

λ/μ Horizontal strip

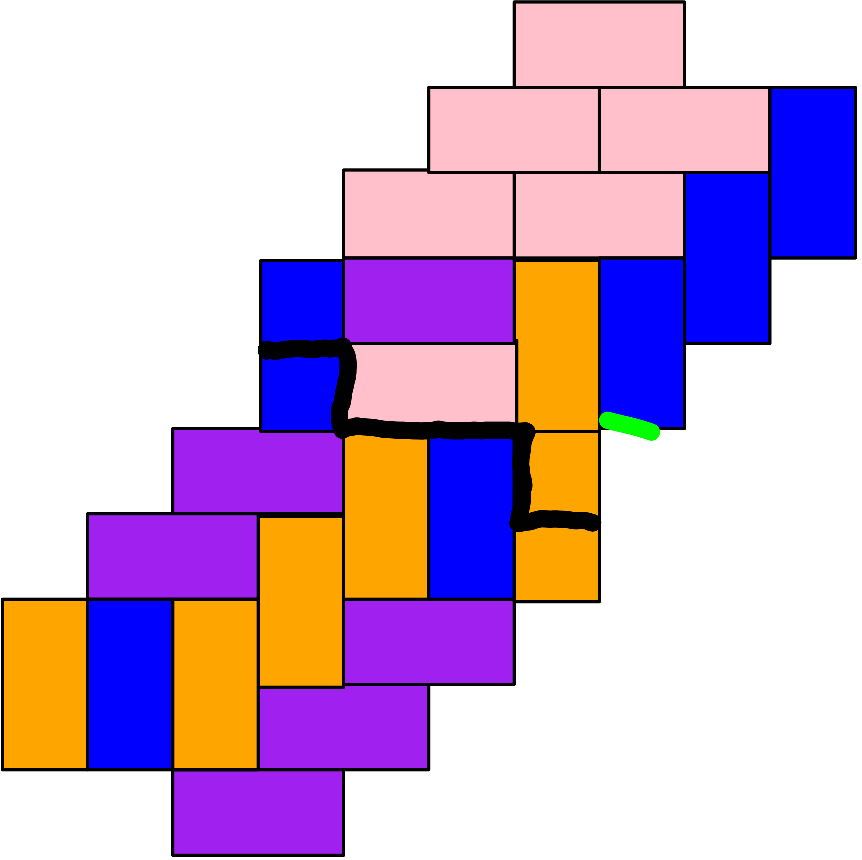
$\lambda \succ \nu$



$\nu = (4,4,2,2)$

λ/ν vertical strip

$\lambda' \succ \nu'$

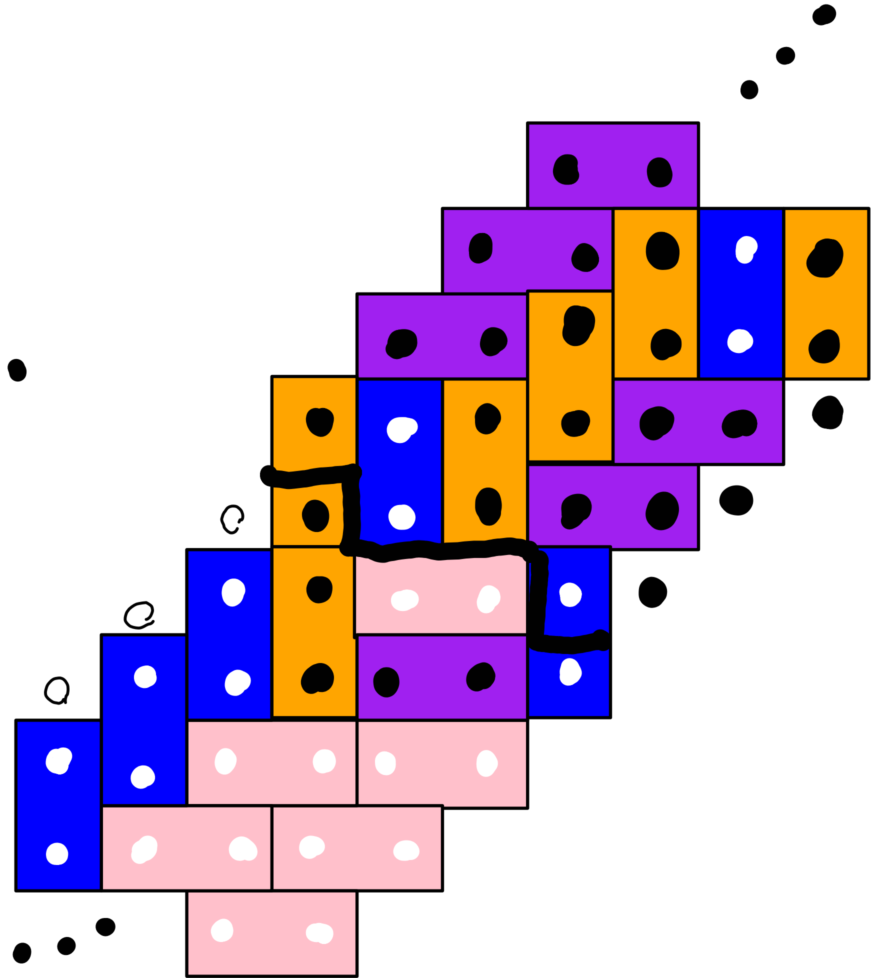


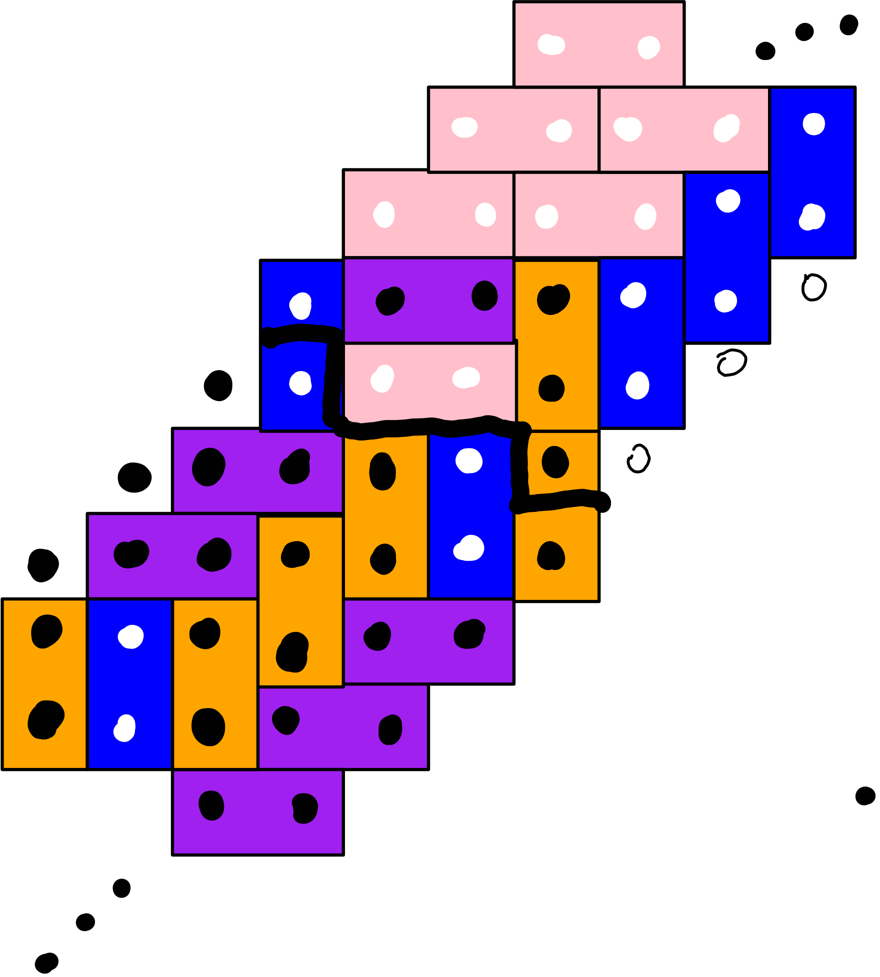
Example

$\phi = {}_{(o)}\gamma$

$\cdots 000 \,|\, \bullet \bullet \bullet \cdots$

${}_{(o)}\gamma$



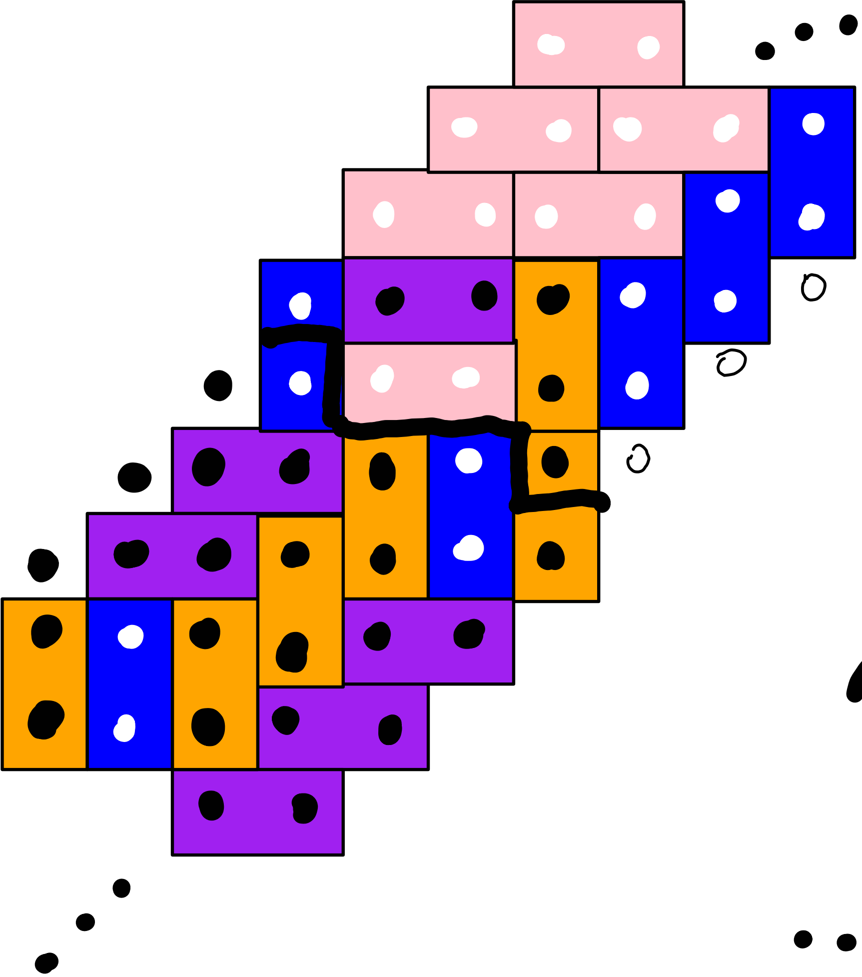


$\lambda^{(0)} = \phi$

$\lambda^{(1)}$

$\cdots \bullet \bullet \bullet \circ \mid \bullet \circ \circ \circ \cdots$

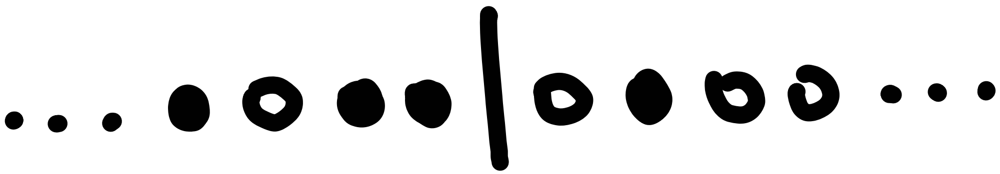
$\lambda^{(1)} = (1)$



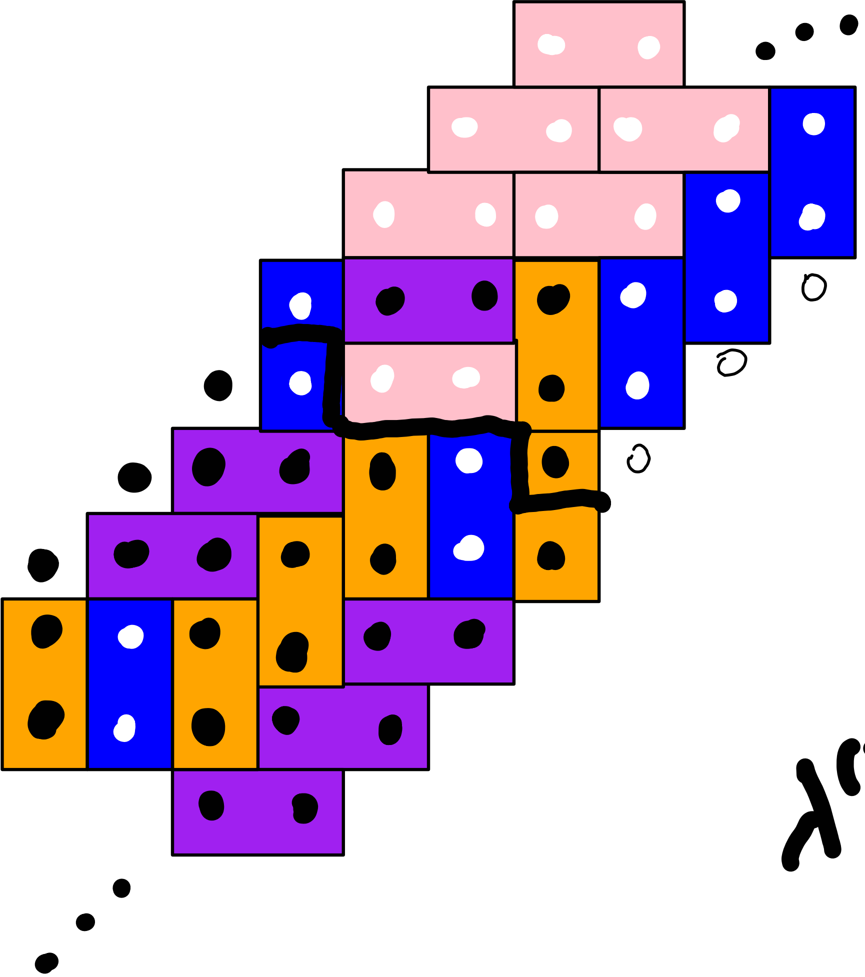
$\lambda^{(0)} = \phi$

$\lambda^{(1)} = (1)$

$\lambda^{(2)}$



$\lambda^{(2)} = (2,1,1)$



$\lambda^{(0)} = \phi$

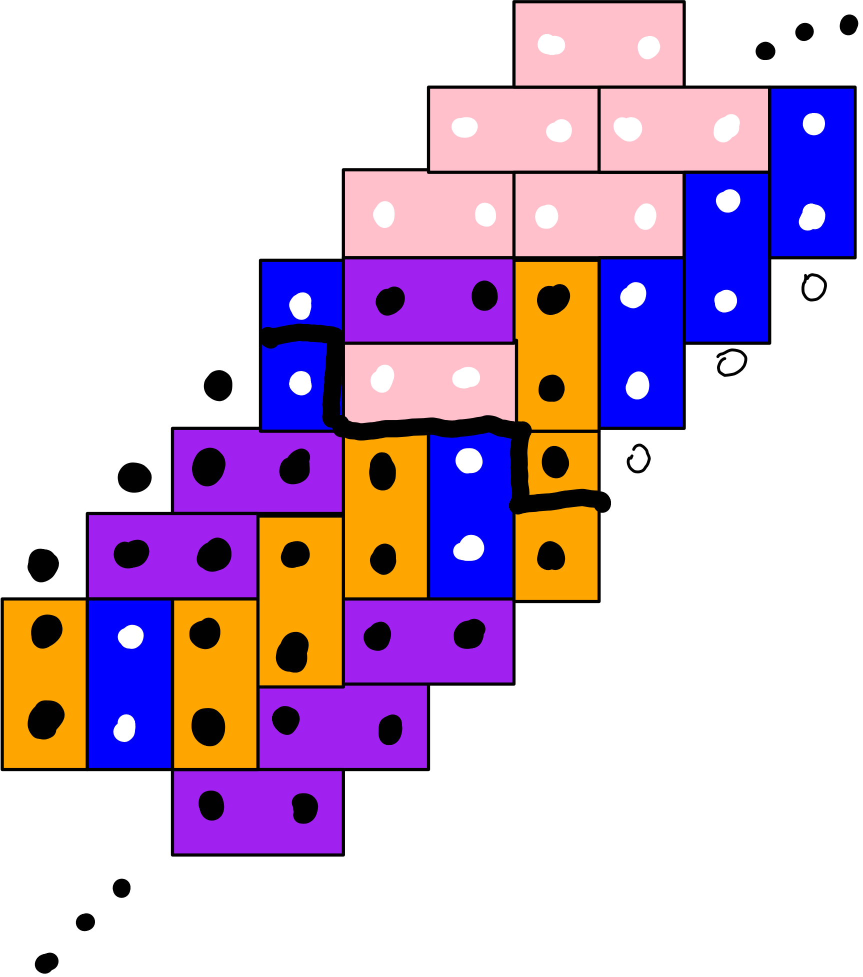
$\lambda^{(1)} = (1)$

$\lambda^{(2)} = (2,1,1)$

$\lambda^{(3)}$

$\cdots \bullet \circ \bullet \bullet \bullet \mid \circ \bullet \circ \circ \cdots$

$\lambda^{(3)} = (2,1,1,1)$



$\lambda^{(0)} = \phi$

$\lambda^{(1)} = (1)$

$\lambda^{(2)} = (2,1,1)$

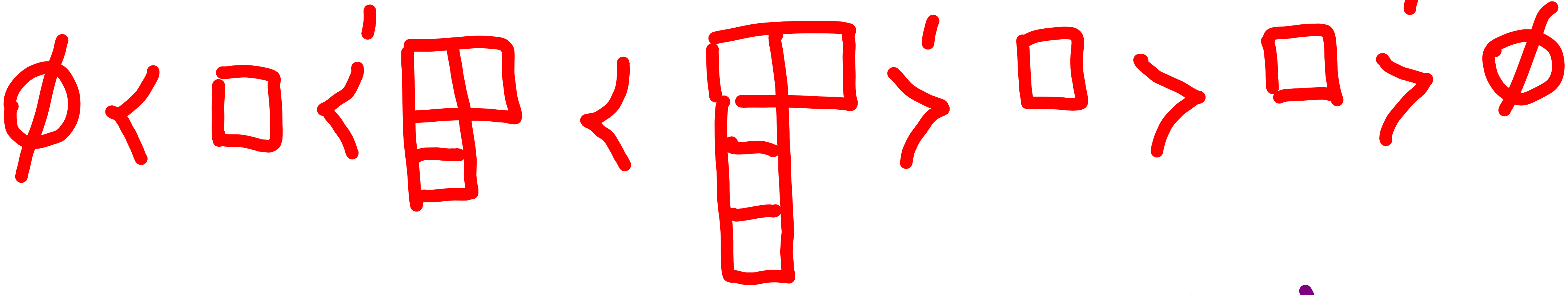
$\lambda^{(3)} = (2,1,1,1)$

$\lambda^{(4)} = (1)$

$\lambda^{(5)} = (1)$

$\lambda^{(6)} = \phi$

$\emptyset, (1), (2,1,1), (2,1,1,1), (1), (1), \emptyset$



$\prec$ add a horizontal strip

$\prec'$ add a vertical strip

GOAL $w = (w_1, \ldots, w_{2e})$

Flip generating function

$$F_w(q) = \prod_{\substack{i<j \\ w_i = + \\ w_j = -}} \left(1 + \varepsilon_{ij}\, q^{j-i}\right)^{\varepsilon_{ij}}$$

$$\varepsilon_{ij} = (-1)^{j-i-1}$$

"Equivalent" proofs of the result

- Vertex operators (O.R.)
- Shuffling algorithms (E et al)
- Growth diagrams (Fomin)
- RSK type algorithms (Sagan, C, Savelief, Vuletic)

Why?

Super - Schur functions $S_\lambda(x,y)$

$$\begin{array}{llllll} 1 & 1 & 1 & \bar{1} & \bar{2} & 3 \\ \bar{1} & 2 & 2 & 2 & \bar{2} & \\ \bar{1} & 3 & 3 & 3 & & \\ 2 & & & & & \\ \bar{2} & & & & & \end{array}$$

$\bar{x}$ vertical strip

x horiz. strip

(Berele Remmel, Krott)

Cauchy identity

$$\sum_\lambda S_\lambda(x,y)\, S_\lambda(w,z) = \prod \frac{(1+x_i z_j)(1+y_j w_i)}{(1-x_i w_j)(1-y_i z_j)}$$

<u>Operators</u> (Okounkov, Borodin, Okada, ...)

$$\Gamma_+'(z)\,|\lambda\rangle = \sum_{\mu' \succ \lambda'} z^{|\mu/\lambda|}\,|\mu\rangle$$

$$\Gamma_+(z)\,|\lambda\rangle = \sum_{\mu \succ \lambda} z^{|\mu/\lambda|}\,|\mu\rangle$$

$$\Gamma_-'(z)\ ,\quad \Gamma_-(z)$$

$$D(q)\,\Gamma_+^*(z) = \Gamma_+^*(zq)\,D(q)$$

$$D(q)\,\Gamma_-^*(z) = \Gamma_-^*(z/q)\,D(q)$$

Proposition (BCC)

$$w = (w_1, \dots, w_{2\ell})$$

$$F(q) = \langle \emptyset | \prod_{i=1}^{\ell} D(q)\, \Gamma_{w_{2i-1}}(1)\, D(q)\, \Gamma'_{w_{2i}}(1) | \emptyset \rangle$$

Commutation relations (MacDonald)

$$\Gamma_+(u)\, \Gamma_-(v) = \frac{1}{1-uv}\, \Gamma_-(v)\, \Gamma_+(u)$$

$$\Gamma_+(u)\, \Gamma'_-(v) = (1+uv)\, \Gamma'_-(v)\, \Gamma_+(u)$$

$$D(q)\,\Gamma_+(u) = \Gamma_+(uq)\,D(q)$$

$$D(q)\,\Gamma_-(u) = \Gamma_-(u/q)\,D(q)$$

$$\Gamma_-(u)\,|\phi\rangle = |\phi\rangle$$

$$\langle\phi|\,\Gamma_+(u) = \langle\phi|$$

FLIP GF

$\square$

__Example__ · Aztec Diamond

$$\langle \emptyset | \left(\Gamma_+ D(q) \Gamma_-' D(q) \right)^{\ell} | \emptyset \rangle = \prod_{i=1}^{\ell} \left(1 + q^{2k-1} \right)^{\ell - k}$$

· Pyramid partitions

$$\langle \emptyset | \left(\Gamma_+ D(q) \Gamma_+' D(q) \right)^n \left(\Gamma_- D(q) \Gamma_-' D(q) \right)^n | \emptyset \rangle$$

Generalizations

- Change the region $\lambda^{(0)} = \lambda \quad \lambda^{(e)} = \mu$
- Follow the flips in each diagonal
- Mix vertical and horizontal strips → plane partitions

$$\prod_{i<j} \left(1 + \varepsilon_{ij}\, q^{j-i}\right)^{\varepsilon_{ij}}$$

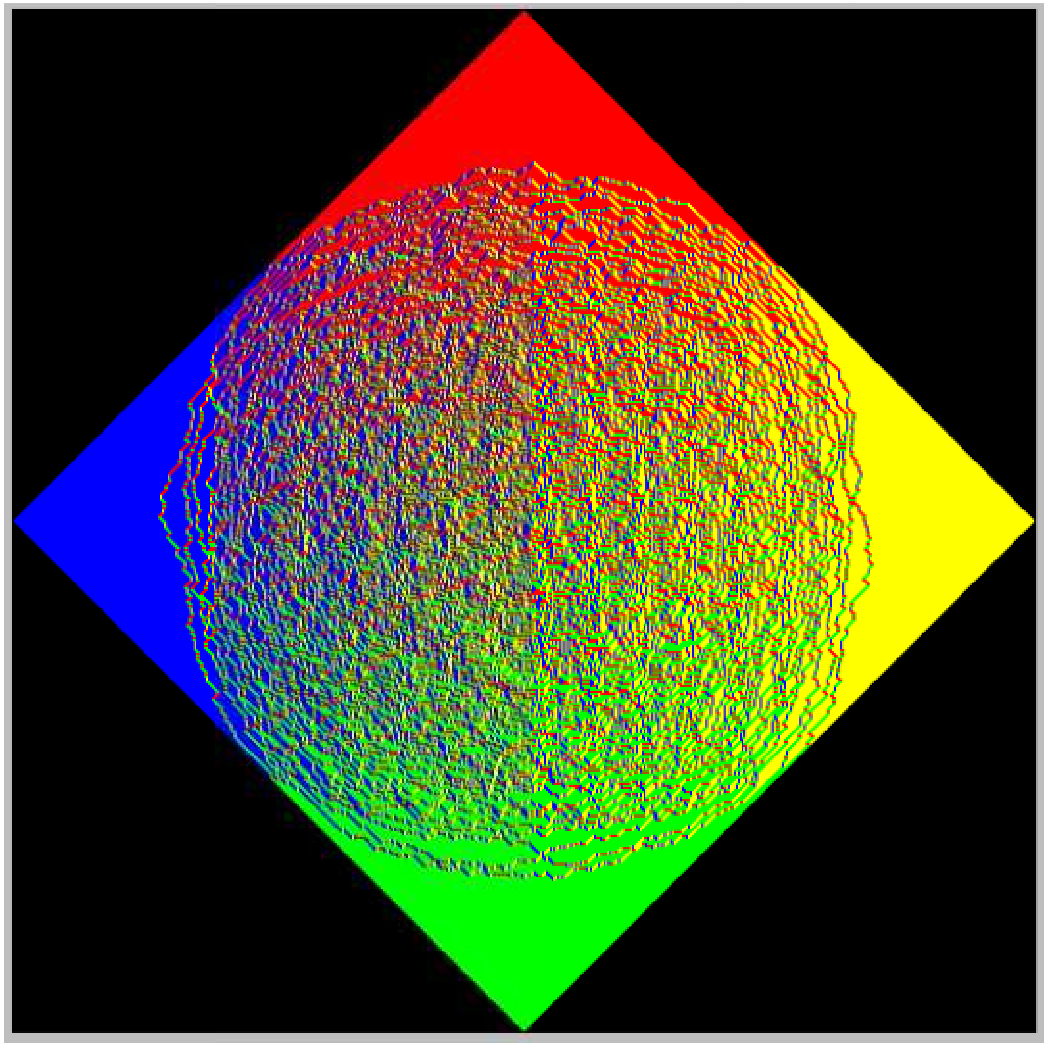
$w_i = +, w_j = -i$

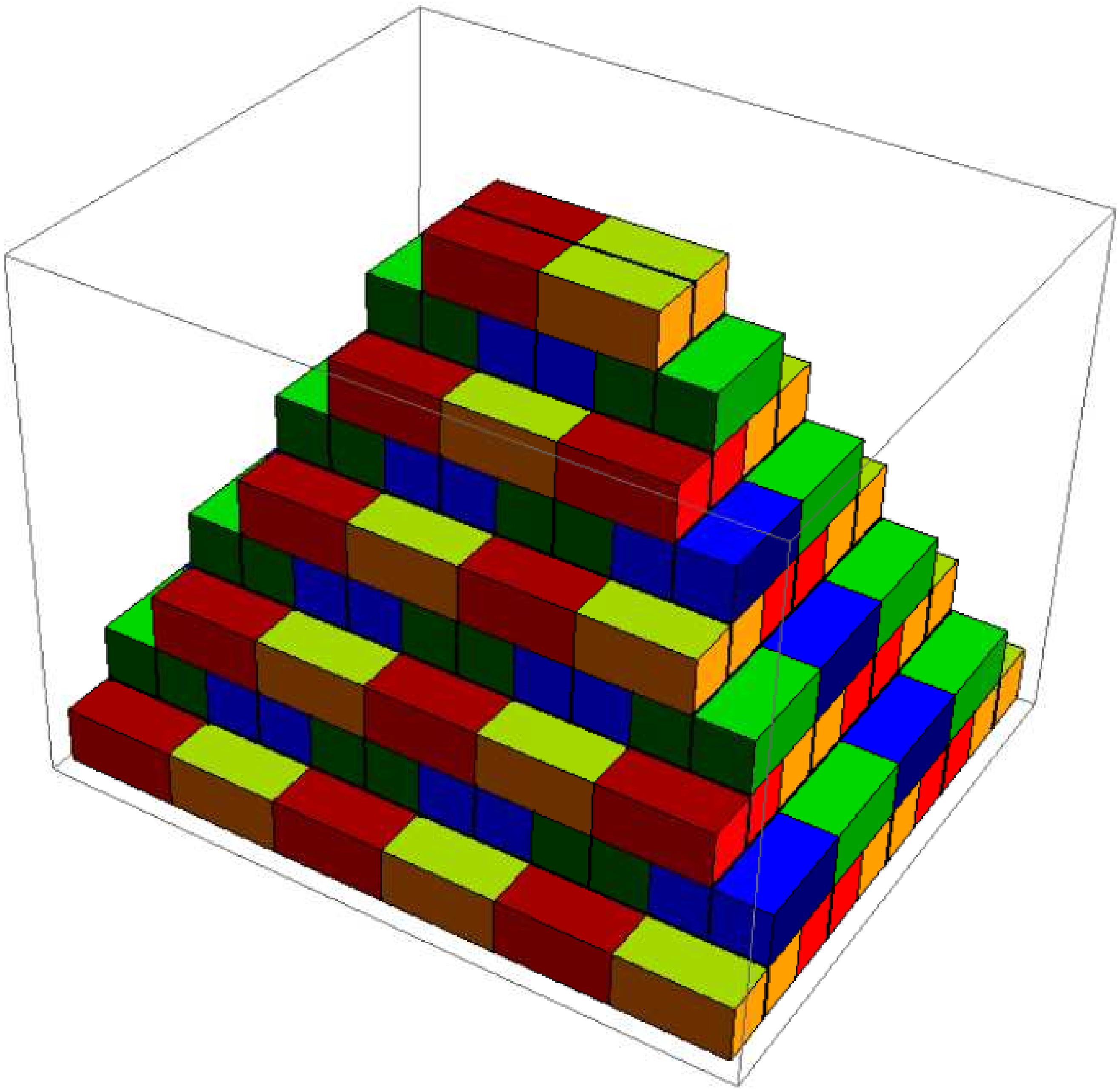
$$\varepsilon_{ij} = \begin{cases} -1 & VV \quad HH \\ 1 & HV \quad VH \end{cases}$$

Other questions

- Random generation (Shuffling)
- Correlation functions (Borodin and Shlosman) Super Schur process
- Bijective proofs (Hillmann Grassl, Krattenthaler)

- Limit shape (BCC Boutillier, Ramassamy)





Merci!