The Brownian Parabolic Tree

Joint work with J.-F. Marckert

— Motivation: Minimum spanning trees (MST) —

GIVEN
- Connected graph $G = (V, E)$
- Distinct weights $w : E \longrightarrow \mathbb{R}_+$

$\boxed{\text{DEF}}$ T the unique connected subgraph $T = (V, E')$ minimizing $\sum_{e \in E'} w(e)$

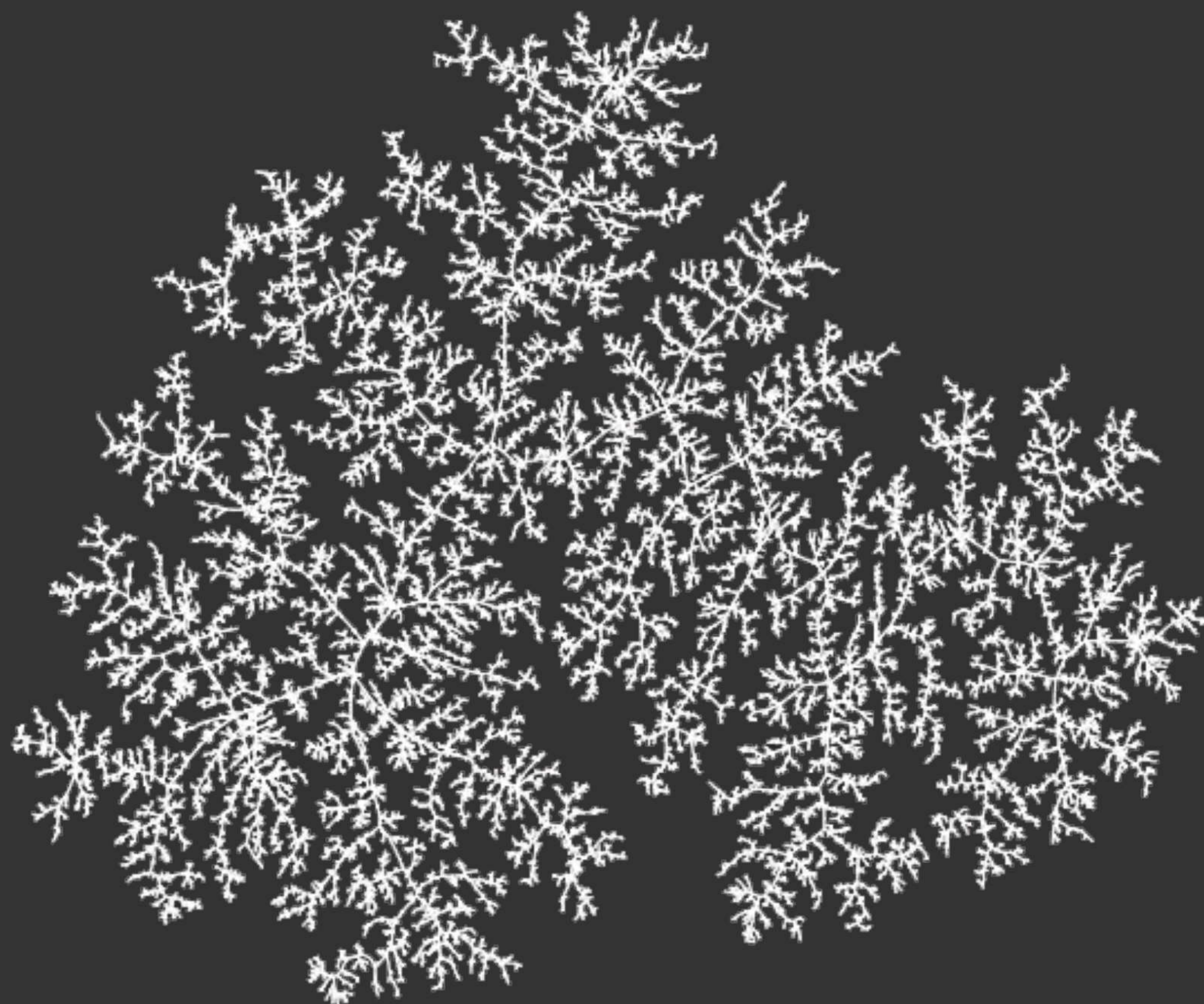
"MST OF THE COMPLETE GRAPH" T_n : K_n the complete graph on $\{1, 2, \ldots, n\}$
weights : $(W_e)_{e \in E}$ iid uniform on $[0, 1]$

- δ_n : graph distance on T_n
- ν_n : empirical measure on $\{1, 2, \ldots, n\}$

$\boxed{\text{THEOREM}}$ (Addario-Berry, B., Goldschmidt, Miermont)
T^n the minimum weight spanning tree of K_n with iid uniform weights
(T_n, δ_n, ν_n) measured metric space

$$(T_n, n^{-1/3}\delta_n, \nu_n) \xrightarrow[\text{GHP}]{d} (\mathcal{M}, \delta, \nu)$$

— A large minimum spanning tree —



— Motivation: Good, and « less good » news —

SOME PROPERTIES OF $(\mathcal{M}, \delta, \nu)$: **Binary $\mathbb{R}$-tree** : all points have degree 1, 2 or 3.

Minkowski dimension : almost surely equal to 3

SOME "ISSUES" : 1) Proof is a Cauchy sequence argument, in distribution

2) No simple / explicit construction of $(\mathcal{M}, \delta, \nu)$

3) Natural / simple questions remain open

→ distribution of the distance between 2 typical points

→ Hausdorff dimension

OBJECTIVE :

- ADDRESS THESE QUESTIONS
- APPROACH ADAPTABLE TO INHOMOGENEOUS SETTINGS

$\Longrightarrow$ THE BROWNIAN PARABOLIC TREE

— The Brownian parabolic tree —

$\boxed{\text{THEOREM}}$ (B.-Marckert) There exists a metric space $\tilde{\mathcal{T}}$ on (a completion of) $\mathbb{R}_+$ such that

i) $(\tilde{\mathcal{T}}, \tilde{d}, \tilde{\mu}, \tilde{\rho}) \overset{d}{=} (\mathcal{T}, \delta, \nu, \rho)$

ii) $\mathcal{T}$ is a compact binary $\mathbb{R}$-tree

iii) $\mathcal{T}$ has box-counting and Hausdorff dimension 3.

INTUITIVE EXPLANATION :

1) Have an explicit construction for $n \in \mathbb{N}$

2) life is beautiful ...

REMAINS :

- FORMAL CONSTRUCTION from
  - * a Brownian motion
  - * countably many $\perp\!\!\!\perp$ uniform r.v.
- IDENTIFICATION OF THE OBJECT
  - $\longrightarrow$ coupling with the discrete
  - $\longrightarrow$ study the dynamics of the construction
- ALLOWS FOR SOME CALCULATIONS
  - $\longrightarrow$ direct proof of compactness
  - $\longrightarrow$ control of mass measure of balls + Hausdorff.

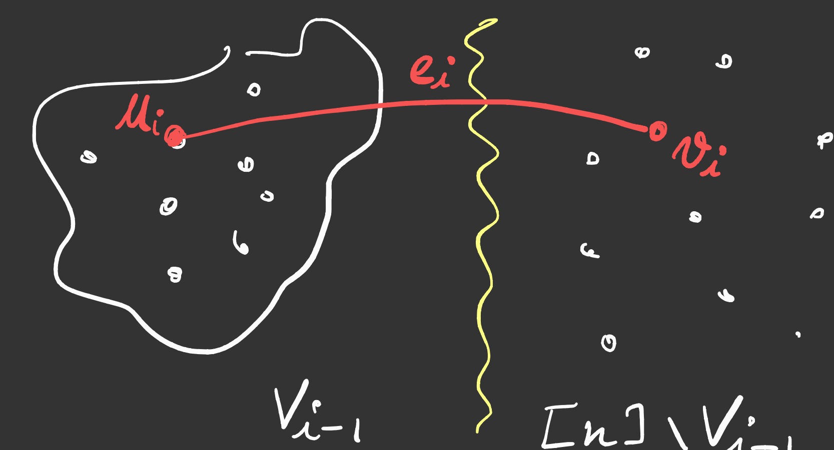
— Prim's algorithm and the MST I —

$G = ([n], E)$ the complete graph on $[n] = \{1, 2, \dots, n\}$

$E = \binom{[n]}{2}$

$(W_e)_{e \in E}$ distinct edge weights

At step i:



PRIM'S ALGORITHM

- $v_1 = 1$, $V_1 = \{v_1\}$
- For $i = 2, 3, \dots, n$:
  - $e_i = \{u_i, v_i\}$ the edge of minimum weight out of $V_{i-1} = \{v_1, v_2, \dots, v_{i-1}\}$

$\Longrightarrow$ $T^n = ([n], \{e_i\}_{i=2}^{n})$ is
- a tree (connected + $(n-1)$ edges)
- minimizes the Σ of the edge weights

$\hookrightarrow$ MINIMUM (WEIGHT) SPANNING TREE

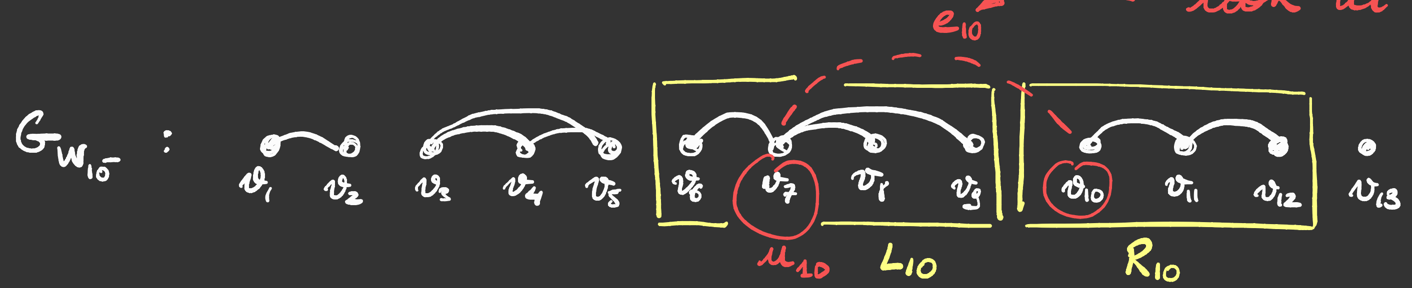
PRIM'S ORDER : $(v_1, v_2, \dots, v_n)$ relates to percolation :

$\forall p \in [0,1]$ $\quad E_p := \{e \in E : W_e \leq p\}$

the connected components of $G_p = ([n], E_p)$ are intervals in the Prim order

— Prim's algorithm and the MST II —

look at e_i with weight $W_i := W_{e_i}$ $(i=10)$



$R_i =$ cc of $G_{W_i^-}$ containing v_i

$L_i =$ ______ ______ v_{i-1}

OBSERVATIONS:

1) By def $e_i = \{u_i, v_i\}$ $\quad u_i \in V_{i-1}$
   ↳ v_i is the first node of a cc of $G_{W_i^-}$, here R_i
2) u_i lies in the cc of $G_{W_i^-}$ which contains v_{i-1}
3) e_i is the min weight edge between V_{i-1} and $[n] \setminus V_{i-1}$
   * conditionally on $G_{W_i^-}$ and $\{v_1, v_2, \dots, v_n\}$
   ↳ each node of L_i has $\# R_i$ available edges to R_i
   so <u>u_i is uniform in L_i</u>

Here: $(W_e)_{e \in E}$ iid

$\boxed{\text{SAMPLING MST}}$ Conditionally on the Prim order + coalescent

- from $p = 0$ ↑ 1
- each time two cc merge, connect
  - min vertex on the right
  - a uniform vertex on the left

— Towards a continuous version —

ONE STEP AT A TIME : decompose this construction in two steps

1) FOCUS ON THE STRUCTURE OF MERGES :

→ How do connected components of G_p merge as $p : 0 \nearrow 1$

2) FOCUS ON THE "LOCATIONS" OF END POINTS

FOR THE COALESCENT : SCALING LIMIT : specific scale at which "what matters" occurs

- which values of p
- what sizes for the connected components

OUTSIDE SPECIFIC TIME RANGE "no more randomness"

- "can be ignored"

WE WILL SEE :

- finite n → INTUITIVE but NASTY
- in the limit → TRACTABLE (≈ nice representation)

— Encoding and the multiplicative coalescent —

RANDOM GRAPHS: $(W_e)_{e\in E}$ iid Exponential(1) $\Rightarrow$ $G_p^n = ([n], E_p) \overset{d}{=} G(n, 1-e^{-p})$

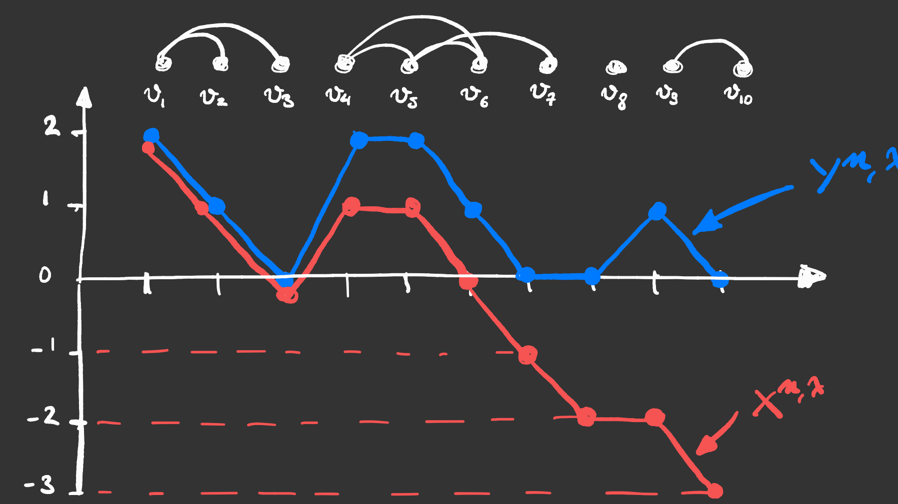
Critical regime: $p = \frac{1}{n} + \frac{\lambda}{n^{4/3}}$ with $\lambda \in \mathbb{R}$ $\longrightarrow$ largest c.c. $\approx n^{2/3}$

ENCODING: $Y^{n,\lambda}(i) = \#\{k > i : v_k \sim_\lambda v_i\}$

$X^{n,\lambda}(i) = Y^{n,\lambda}(i) - \#\{j < i : Y^{n,\lambda}(j) = 0\}$

$\xi(k) = \inf\{i : X^{n,\lambda}(i) = -k\}$

$\{v_i : \xi(k) < i \leqslant \xi(k+1)\}$ c.c.



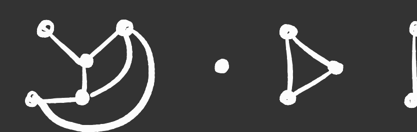
SCALING LIMIT: $\forall k$, $\lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{R}$ one has

$$\left(\frac{X^{n,\lambda_1}(\lfloor t \cdot n^{2/3} \rfloor)}{n^{1/3}}, \dots, \frac{X^{n,\lambda_k}(\lfloor t \cdot n^{2/3} \rfloor)}{n^{1/3}} \right) \xrightarrow[\mathcal{D}]{d} \left(X^{\lambda_1}(\cdot), \dots, X^{\lambda_k}(\cdot) \right)$$

$$\frac{1}{n^{2/3}} \left(\underline{\gamma}^{n,\lambda_1}, \underline{\gamma}^{n,\lambda_2}, \dots, \underline{\gamma}^{n,\lambda_k} \right) \xrightarrow[\ell^2_\downarrow]{d} \left(\underline{\gamma}^{\lambda_1}, \underline{\gamma}^{\lambda_2}, \dots, \underline{\gamma}^{\lambda_k} \right)$$

vector of sizes of c.c.

NEXT MERGE: *multiplicative*



$m_1 = 5$ m_2 m_3 m_4

(i,j) w.p. $\propto m_i \times m_j$

NOTE: $v_1, v_2, \dots, v_n$ is a random order that depends on all the weights.

— A coalescent related to Brownian motion —

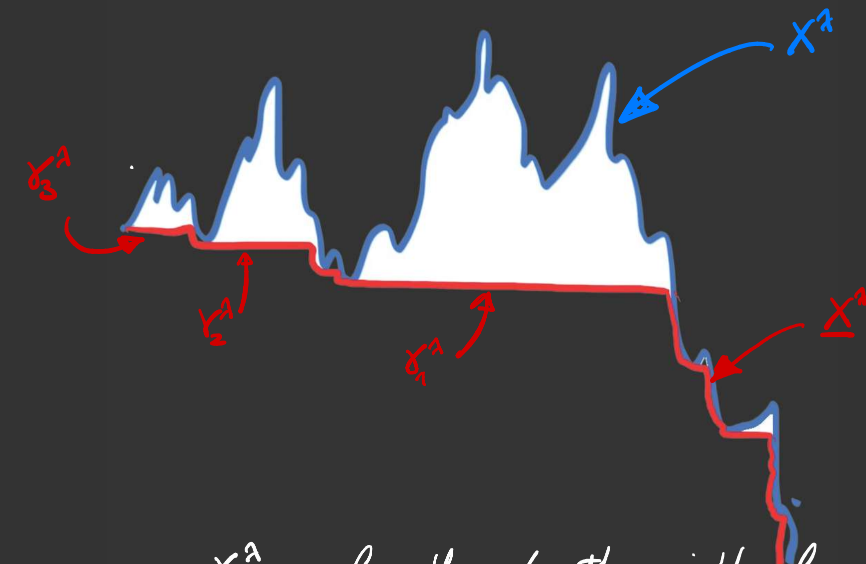
$(W_s)_{s\geq 0}$ Brownian motion

For $\lambda \in \mathbb{R}$ $\quad X^\lambda(s) = W(s) + \lambda s - s^2/2$

$\underline{X}^\lambda(s) = \inf\{X^\lambda(u) : 0 \leq u \leq s\}$

$Z^\lambda = \{s \geq 0 : X^\lambda(s) = \underline{X}^\lambda(s)\}$

For $x < y$ $\quad x \sim_\lambda y$ if $[x,y] \cap Z^\lambda = \phi$



EQUIVALENCE CLASSES: intervals of $\mathbb{R}_+ \setminus Z^\lambda$

- γ_i^λ = length of the i-th largest
- $\forall \lambda$ $\quad \sum_i \gamma_i^\lambda = +\infty$ but $\sum_i (\gamma_i^\lambda)^2 < +\infty$

NESTING PROPERTY: $\lambda \leq \lambda' \Rightarrow Z^\lambda \supseteq Z^{\lambda'}$ so $x \sim_\lambda y \Rightarrow x \sim_{\lambda'} y$

$\lambda \mapsto (\gamma_i^\lambda)_{i\geq 1}$ is a **COALESCENT PROCESS**

$\boxed{\text{THEOREM}}$ (B.-Marckert 2016): (γ^λ) is the (standard) multiplicative coalescent, i.e.

i) a strong Markov pure jump process on $\ell^2_\downarrow$ (homogeneous in time)

ii) any two fragments x and y merge at rate $x \times y$ into a fragment $x+y$

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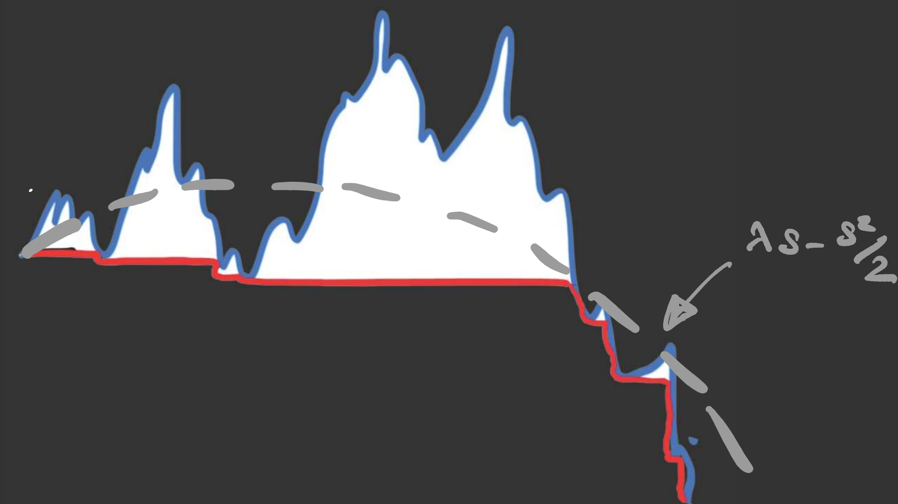

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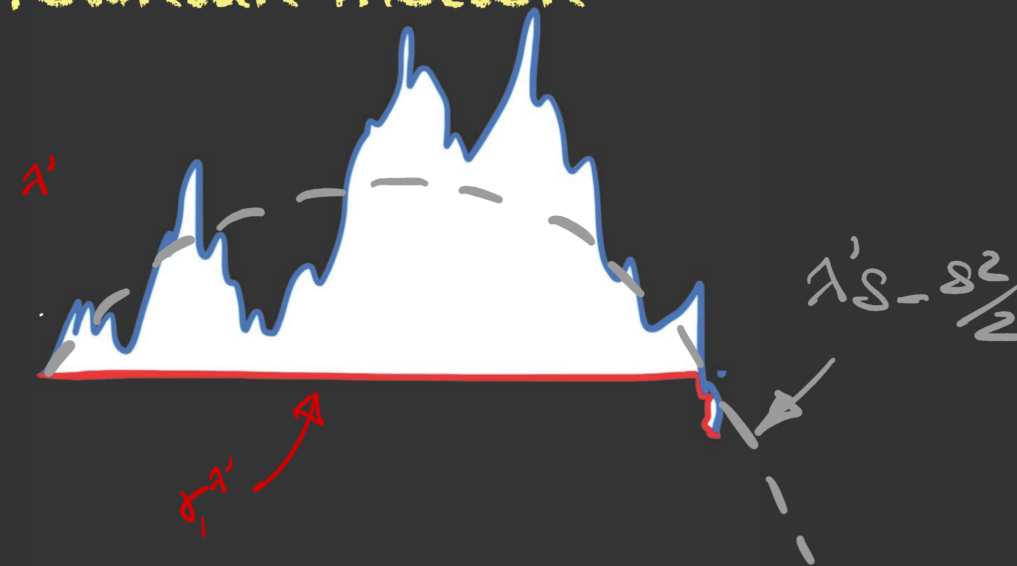

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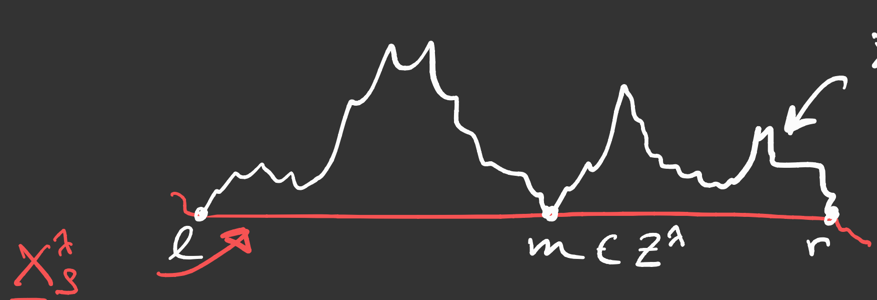
— A coalescent related to Brownian motion II —

<u>BROWNIAN COALESCENT</u> gives a scaling limit for
* masses of fragments
* "<u>identities</u>" of vertex sets

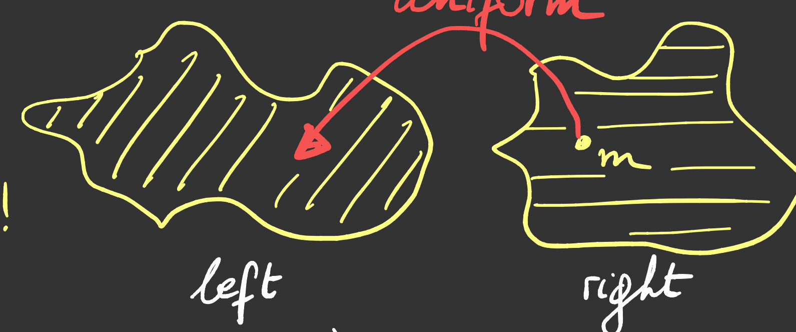
<u>WOULD LIKE TO :</u> ENRICH the structure so that

1) each fragment corresponds to a connected <u>measured metric space</u>
↳ $\forall \lambda \in \mathbb{R}$ countably many metric spaces

2) upon a merge the corresponding spaces connect



$$\mathcal{M}^\lambda_{[\ell,m]} \oplus \mathcal{M}^\lambda_{[m,r]} \longrightarrow \mathcal{M}^{\lambda+}_{[\ell,r]}$$



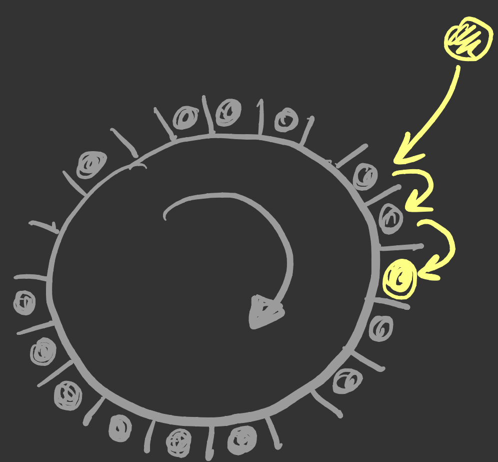
↳ does not create cycles ⇒ FOREST!

3) As $\lambda \uparrow \infty$ we obtain a unique connected space (TREE!)
↳ $\forall x, y \geq 0 \quad \exists \lambda(x,y) < +\infty : \quad x \sim_\lambda y \iff \lambda > \lambda(x,y)$

--- Hashing with linear probing and forests II ---

HASHING INSERTION / PARKING :

* Oriented annulus of slots
* data / cars arrive
  - one by one
  - at an independent uniform location
  - move "right" until find a spot



$\Longrightarrow$ system of coalescing intervals with :

* uniform point in a "left" interval
* slot to the left of the "right" interval

} $\longrightarrow$ coalescence

— Hashing with linear probing and forests II —

BIJECTIVE CORRESPONDANCE :

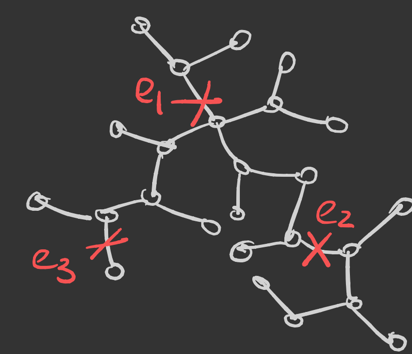
- collection of particles : interval $\longleftrightarrow$ tree
- going backward : removing car / data $\longleftrightarrow$ removing unif. edge

THE "BACKWARDS" FRAGMENTATION

(T_n, d_n) a random Cayley tree

$(e_1, e_2, \dots, e_{n-1})$ a uniform permutation of its edges

$P_k^n = \{e_1, e_2, \dots, e_k\}$



$T_n \setminus P_k^n$ a forest of $k+1$ trees : $T_n^i(k)$ the i-th largest in # nodes

$k \mapsto (\#T_n^1(k), \#T_n^2(k), \dots)$ fragmentation process (discrete)

SCALING LIMIT :

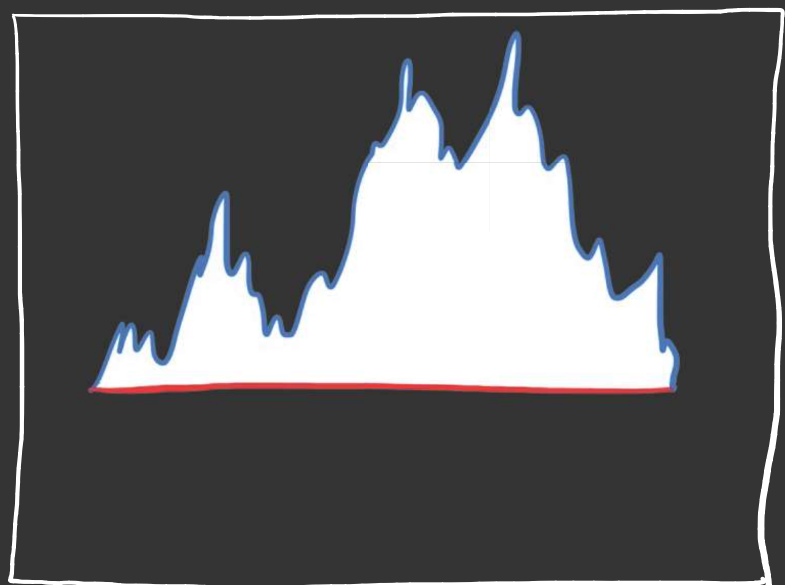
1) when $k \sim t\sqrt{n}$ one has $\#T_n^i(k) \simeq n$

2) $\frac{1}{n}\left(\#T_n^1(\lfloor t\sqrt{n}\rfloor), \#T_n^2(\lfloor t\sqrt{n}\rfloor), \dots\right) \longrightarrow \underline{X}(t) = (X_1(t), X_2(t), \dots)$

3) $t \mapsto \underline{X}(t)$ fragmentation

— A fragmentation of the Brownian excursion —

$(e(s))_{s\in[0,1]}$ a Brownian excursion



For $t \geq 0$ consider $e^{[t]} : s \mapsto e^{[t]}(s) = e(s) - ts$

$$\underline{e}^{[t]}(s) = \inf\{ e^{[t]}(u) : 0 \leq u \leq s\}$$

$$\mathcal{L}^{[t]} := \{ s \in [0,1] : e^{[t]}(s) = \underline{e}^{[t]}(s) \}$$

NESTING PROPERTY: $\mathcal{L} := \bigcup_{t \geq 0} \mathcal{L}^{[t]}$

- $\mathcal{L}^{[t]} \subseteq \mathcal{L}^{[t']}$ for $t \leq t'$.
- the partition $[0,1] \setminus \mathcal{L}^{[t]}$ gets finer as $t \uparrow$.
- $I^{[t]}(x) = \begin{cases} \text{interval of } [0,1]\setminus\mathcal{L}^{[t]} \text{ containing } x \notin \mathcal{L} \\ \text{or } \lim I^{[t]}(x+\varepsilon) \text{ if constant for } \varepsilon \in (0,\varepsilon_0). \end{cases}$
- $F(t) = (F_1(t), F_2(t), \dots)$ reordering of the lengths of intervals

THEOREM (Bertoin)

$\xi \perp\!\!\!\perp$ and uniform $[0,1]$: $F^{\circ}(t) = Leb(I^{[t]}(\xi))$, $F^{*}(t) = Leb(I^{[t]}(0))$

1) $(F(t))_{t \geq 0}$ is a self-similar fragmentation (in $\ell^1_{\downarrow}$)

2) $(F(t))_{t \geq 0}$ is distributed like the Aldous-Pitman frag of the CRT

3) $(F^{*}(t))_{t \geq 0} \overset{d}{=} (F^{\circ}(t))_{t \geq 0}$

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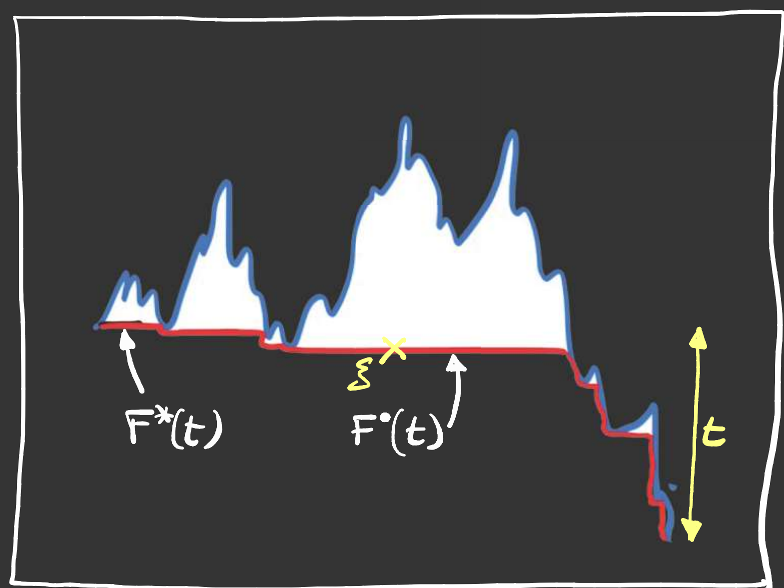

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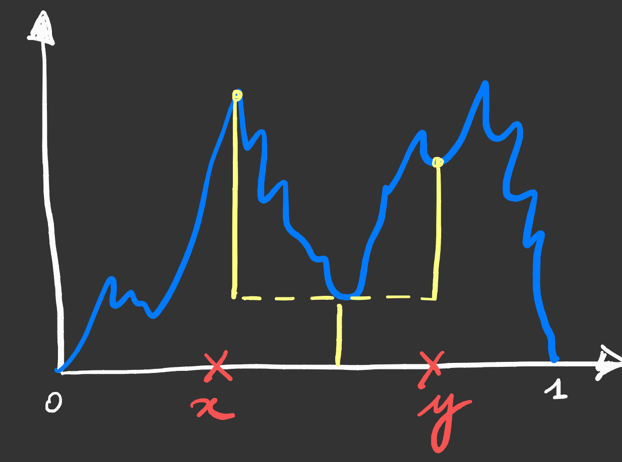
↳ This is the fragmentation of a metric space!

"0 is uniform in $[0,1]$" wrt the fragm.

— The continuum random tree —

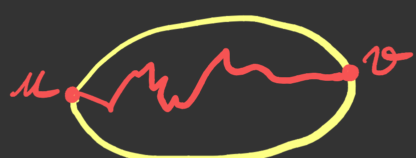
CONSTRUCTION:
- e Brownian excursion on $[0,1]$
- $d(x,y) = e(x) + e(y) - 2 \inf_{x\wedge y \le u \le x \vee y} e(u)$
- $[0,1]/_{\{d=0\}}$ is a compact $\mathbb{R}$-tree $(\mathcal{T}, d)$
- $\pi : [0,1] \to \mathcal{T}$ canonical projection
  $\mu = \mathrm{Leb}(\pi^{-1}(\cdot))$ the mass measure



$(\mathcal{T}, d, \mu)$ measured metric space
= (Brownian) CRT

ℓ = length measure = unique σ-finite measure with $\ell(\llbracket u,v \rrbracket) = d(u,v)$ $\forall\, u,v \in \mathcal{T}$.
path between u and v

FUNDAMENTAL PROPERTIES



1) Two-point distance: u, v ⫫ with distribution μ

$d(u,v) \sim$ Rayleigh $\qquad \mathbb{P}(d(u,v) > x) = e^{-x^2/2}$

2) Brownian scaling: mass $\times\, m \longrightarrow$ distance $\times \sqrt{m}$

REFS: Aldous, Le Gall

— Aldous-Pitman fragmentation of the CRT —

$(\mathcal{T}, d, \mu)$ a Brownian CRT
ℓ the length measure $\big(\ell(\mathcal{T}) = +\infty$ but σ-finite$\big)$

cut at position x_α arriving at time t_α

$\mathcal{P} =$ PPP on $\mathcal{T} \times \mathbb{R}_+$ with intensity $\ell \otimes dt$ $\longrightarrow$ $\mathcal{P} = \{ (x_\alpha, t_\alpha) : \alpha \in \mathcal{A} \}$

$\mathcal{P}_t := \{ x_\alpha : t_\alpha \leq t \}$

For $x, y \in \mathcal{T}$ $x \sim_t y$ if $[\![x, y]\!] \cap \mathcal{P}_t = \emptyset$
$\mathcal{T}_i(t) =$ i-th largest cc of $\mathcal{T} \setminus \mathcal{P}_t$

$\tilde{F}_i(t) = \mu(\mathcal{T}_i(t))$ then with $\tilde{F}(t) = (\tilde{F}_1(t), \tilde{F}_2(t), \dots)$
- $\forall t$ $\sum_{i \geq 0} \tilde{F}_i(t) = 1$
- $(\tilde{F}(t))_{t \geq 0}$ is the Aldous-Pitman fragmentation

$\Longrightarrow$ CAN WE RECONSTRUCT THE TREE $\mathcal{T}$?
$\hookrightarrow$ This is our candidate metric space if the excursion were Brownian

— Aldous-Pitman fragmentation of the CRT —

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NOTE: ALL FRAGMENTS ARE CRT

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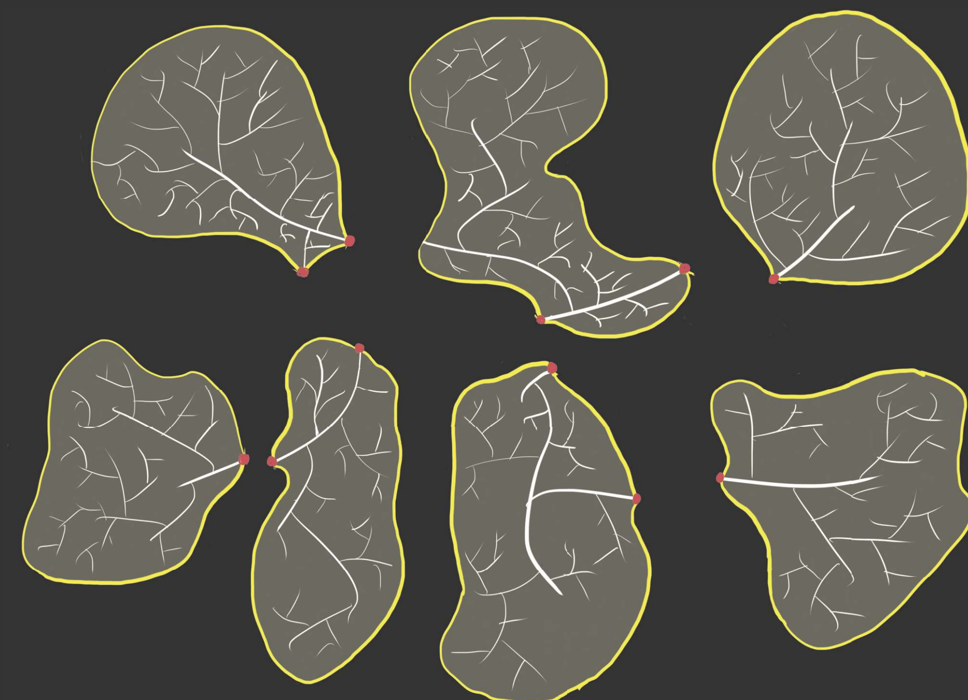
$\tilde{F}_i(t) = \mu(\mathcal{T}_i(t))$ then
- $\forall t \quad \sum_{i \geq 0} \tilde{F}_i(t) = 1$
- $(\tilde{F}_i(t))_{t \geq 0}$ is the Aldous-Pitman fragmentation



$\Longrightarrow$ CAN WE RECONSTRUCT THE TREE $\mathcal{T}$?

↳ This is our candidate metric space if the excursion were Brownian

NOT UNIQUELY BUT YES!

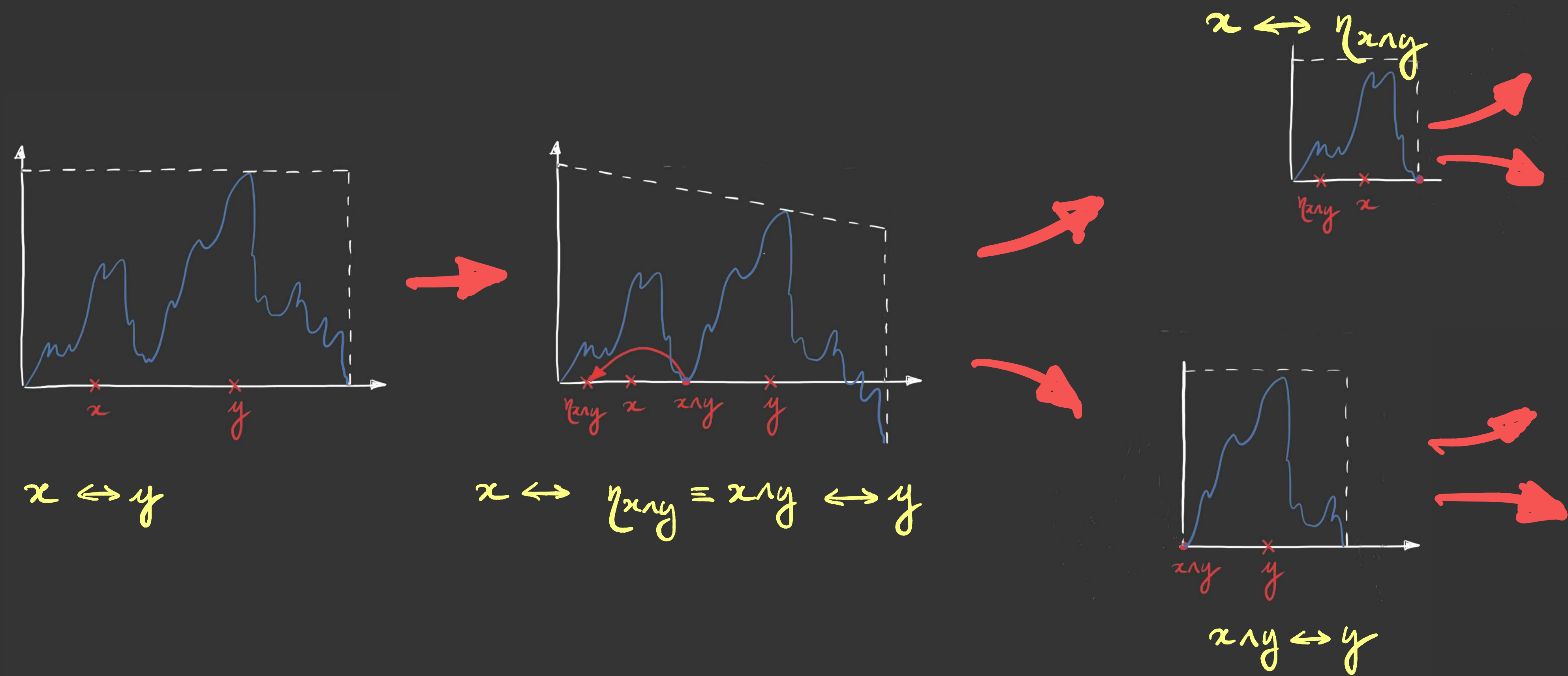
↳ need to "resample the pairs of traces" of the cuts

— Dig in the fragmentation to reconstruct the tree —

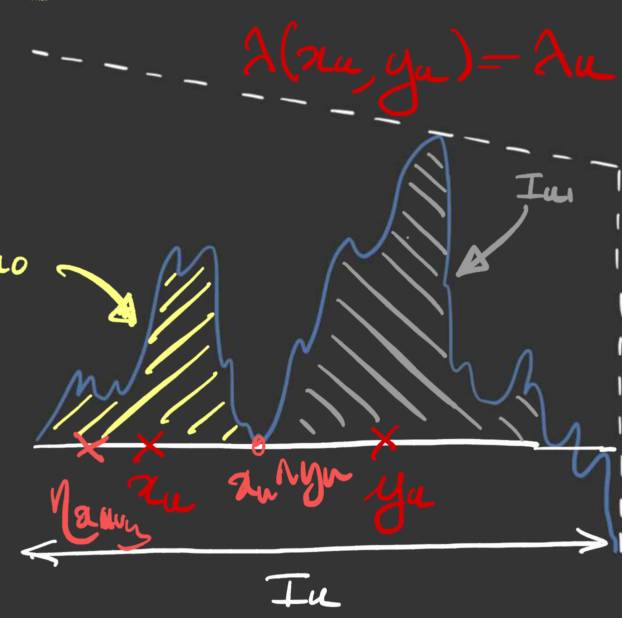
Still in the case of a <u>Brownian excursion</u>

<u>GENERAL IDEA:</u>

1) Focus on 2 points x, y. (then extend to a countable dense set)
2) Recursively construct the set $[\![x, y]\!] \subseteq \mathbb{R}_+$ of points between x and y
3) define $d(x,y)$ as some measure of $[\![x, y]\!]$.



— The construction of geodesic —



RECURSIVE DESCRIPTION: $\mathcal{U} := \bigcup_{n \geq 0} \{0,1\}^n$

Process $(I_u, x_u, y_u)_{u \in \mathcal{U}}$
with $x_u < y_u \in I_u \quad \forall u$

- $I_\phi = [0,1]$, $x_\phi = x$, $y_\phi = y$
- Given (I_u, x_u, y_u) define $\lambda_u = \lambda(x_u, y_u)$ and

$$\hookrightarrow \left| \begin{array}{l} I_{u0} = \overline{I^{\lambda_u}(x_u)} , \\ x_{u0} = \min\{\eta_{x_u \wedge y_u}, x_u\} \\ y_{u0} = \max\{ \ \cdots \ \} \end{array} \right. \qquad \hookrightarrow \left| \begin{array}{l} I_{u1} = \overline{I^{\lambda_u}(y_u)} \\ x_{u1} = x_u \wedge y_u \\ y_{u1} = y_u \end{array} \right.$$

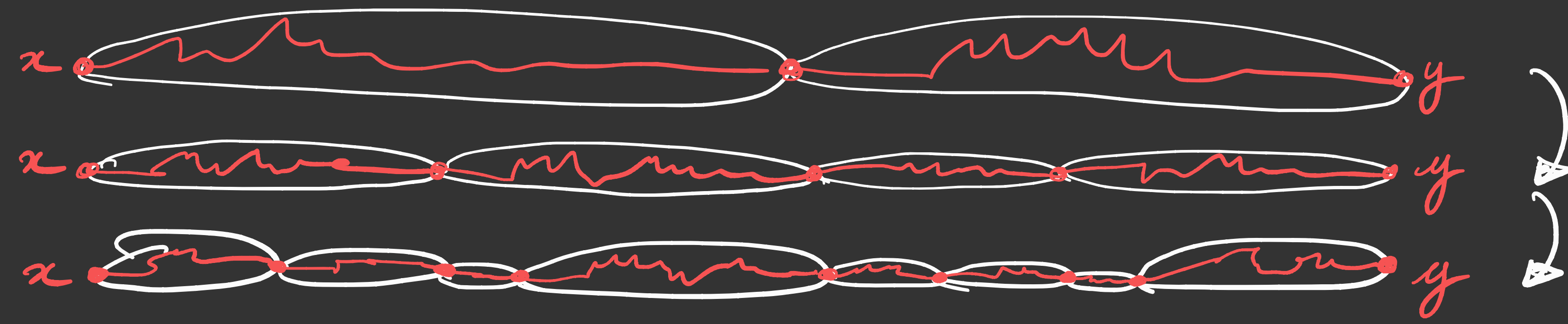
$\forall n \geq 0 \quad C_n = \bigcup_{|u|=n} I_u$ is closed $\Longrightarrow$ $\boxed{[\![x, y]\!] := \bigcap_{n \geq 0} C_n \neq \phi}$

$\boxed{\text{LEMMA}}$ If x, y are $\perp\!\!\!\perp$ uniform then

1) $\dim_H([\![x, y]\!]) = 1/2$ a.s.
2) $H^\psi([\![x, y]\!]) \in (0, \infty)$ for $\psi(r) = \sqrt{r \log|\log r|}$ $\Big\}$ $|I_{u0}| + |I_{u1}| < |I_u|$ a.s.

— The construction of the metric —

INTUITION:



If $x, y \perp\!\!\!\perp$ uniform: $d(x,y) = \sum_{|u|=n} R_u \, |I_u|^{1/2}$ with $(R_u)_{u \in \mathcal{U}}$ iid Rayleigh

↳ $\max\{|I_u| : |u| = n\} \longrightarrow 0 \Rightarrow$ concentration!

A REMARKABLE MARTINGALE:

Set $d_n(x,y) = \sqrt{\frac{2}{\pi}} \sum_{|u|=n} |I_u|^{1/2}$ $\left(\mathbb{E} R_u = \sqrt{\frac{2}{\pi}} \right)$

$\boxed{\text{LEMMA}}$ If x, y are $\perp\!\!\!\perp$ uniform in $[0,1]$

1) $(d_n(x,y))_{n \geq 0}$ is a non-negative martingale

2) $d(x,y) = \lim_{n \to +\infty} d_n(x,y) \sim$ Rayleigh ie $\mathbb{P}(d(x,y) > x) = e^{-x^2/2}$.

$\boxed{\text{PROPOSITION}}$ $(\xi_i)_{i \geq 1}$ iid uniform in $[0,1]$ then

$(d(\xi_i, \xi_j))_{ij} \overset{d}{=} (d_{\mathcal{Z}}(\xi_i, \xi_j))_{ij}$

$\boxed{\text{REFS:}}$

- B.-Wang
- Addario-Berry, Goldschmidt, Dieuleveut.

— But here... this is not Brownian —

$\boxed{\text{EXCURSIONS OF } X^{\lambda}_{\cdot} - \underline{X}^{\lambda}_{\cdot}}$

$e^{(\sigma)}$ Brownian exc on $[0,\sigma]$

Conditionally on duration $= \sigma$: $\tilde{e}^{(\sigma)}(\cdot)$

$$\mathbb{E}\left[f(\tilde{e}^{(\sigma)}) \right] = \frac{\mathbb{E}\left[f(e^{(\sigma)}) \times \exp\left(\int_0^\sigma e^{(\sigma)}(s)\,ds \right) \right]}{\mathbb{E}\left[\exp\left(\int_0^\sigma e^{(\sigma)}(s)\,ds \right) \right]}$$

$\longrightarrow$ ALL OBJECTS ARE A.S. WELL-DEFINED

$\boxed{\text{DISTRIBUTION OF THE SPACE?}}$

1) No limit argument $\longrightarrow$ work directly in the continuum

2) Combination of * dynamics as $\lambda \uparrow$
* analysis as $\lambda \to -\infty$

$$\sup_i |I_i^{\lambda}| \xrightarrow[\lambda \to -\infty]{} 0$$

$$\frac{\tilde{e}^{(\sigma)}(\cdot \times \sigma)}{\sqrt{\sigma}} \xrightarrow[\sigma \to 0]{d} e(\cdot)$$

Conclusions / Recap

"EXPLICIT" CONSTRUCTION OF $(\mathcal{M}, \delta, \nu)$

* from
  - Brownian motion
  - uniform points
* rather than "shearing" $\longrightarrow$ dual uses recursive convex minorant

OPENS POSSIBILITIES OF CALCULATIONS

* monotone construction of distances beween 2 random points
* direct arguments for
  - compactness
  - fractal dimensions (Hausdorff included)
* replace BM by thinned Levy
  $\longrightarrow$ inhomogeneous MST.