Combinatorial Diferential Equations

*

Miguel Méndez
IVIC-UCV

Leroux-Viennot combinatorial solution of the differential equation

$$y' = \phi(y)$$
$$y(0) = x$$

$$y(t,x) = \sum_{k,j} a_{k,j} \frac{t^k x^j}{k!j!}$$

Generalization

$$y' = \phi *_s y$$
$$y(0) = x_0$$

$$y(t, x_0, x_1, \dots) = \sum_{k,\alpha} a_{k,\alpha} \frac{t^k \mathbf{x}^\alpha}{k!\alpha_1!\alpha_2!\dots}$$

$$(\phi *_s y)(t, x_0, x_1, \dots) = \phi(y(t, x_0, x_1, \dots), y(t, x_1, x_2, \dots), y(t, x_2, x_3, \dots), \dots)$$

Particular case of the generalization

$$y' = \phi(y(t, x_1, x_2, \ldots))$$
$$y(0) = x_0$$

$$\phi = \phi(x_1)$$

$$(\phi *_s y)(t, x_0, x_1, \ldots) = \phi(y(t, x_1, x_2, \ldots))$$

Formal Power Series

$$F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!}$$

Counts sets of combinatorial objects on 'n vertices'

Derivative

$$F'(x) = \sum_{n=0}^{\infty} f_{n+1} \frac{x^n}{n!}$$

Integral

$$\int_0^t F(x)dx = \sum_{n=0}^{\infty} f_{n-1} \frac{t^n}{n!}$$

Formal Power Series

$$F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!}$$

Counts sets of combinatorial objects on 'n vertices'

Sum

$$(F+G)(x) = \sum_{n=0}^{\infty} (f_n + g_n) \frac{x^n}{n!}$$

Product

$$(F.G)(x) = \sum_{n=0}^{\infty} h_n \frac{t^n}{n!}$$

Formal Power Series

$$F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!}$$

Counts sets of combinatorial objects on ‘n vertices’

$$h_n = \sum_{k=0}^{n} \binom{n}{k} f_k g_{n-k}$$

Product

$$(F.G)(x)dx = \sum_{n=0}^{\infty} h_n \frac{t^n}{n!}$$

Formal Power Series

$$F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!}$$

Counts sets of combinatorial objects on 'n vertices'

Substitution

$$F(G(x)) = \sum_{k=0}^{\infty} f_k \frac{G^k(x)}{k!} = \sum_{n=0}^{\infty} h_n \frac{t^n}{n!}$$

$$h_n = \text{ Given by the Faá di Bruno formula}$$

Shuffle Species

$\mathcal{L}$ **Category of finite totally ordered sets**

$$l = \{l_1, l_2, \ldots, l_n\}$$

$$l = \{a < b < c < d\}$$

$$f \downarrow$$

$$l' = \{1 < 2 < 3 < 4\}$$

Shuffle Species

$\mathcal{L}$ Category of finite totally ordered sets

$\mathcal{F}$ Category of finite sets

$$F : \mathcal{L} \to \mathcal{F}$$

$$F[l] = \{\text{Structures of } F \text{ over the totally ordered set } l\}$$

$$F(x) = \sum_{n=0}^{\infty} |F[n]| \frac{x^n}{n!}$$

Generating Function

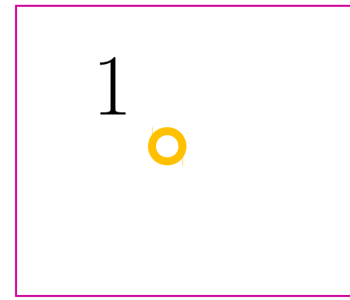
$$F(x) = \sum_{n \geq 0} |F[n]| \frac{x^n}{n!}$$

$$F[n] = F[\{1 < 2 < 3 < \cdots < n\}]$$

The singleton

$$X[l]| = \begin{cases} l & \text{if } |l| = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

X

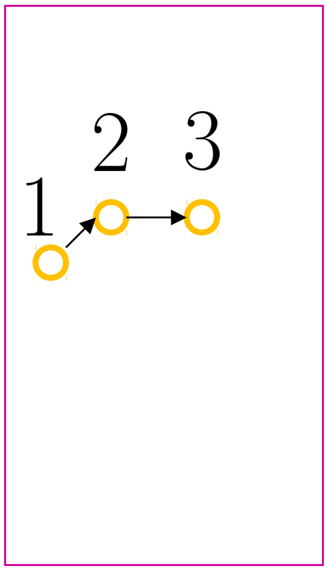


$$|X[1]| = \delta_{1,k}$$

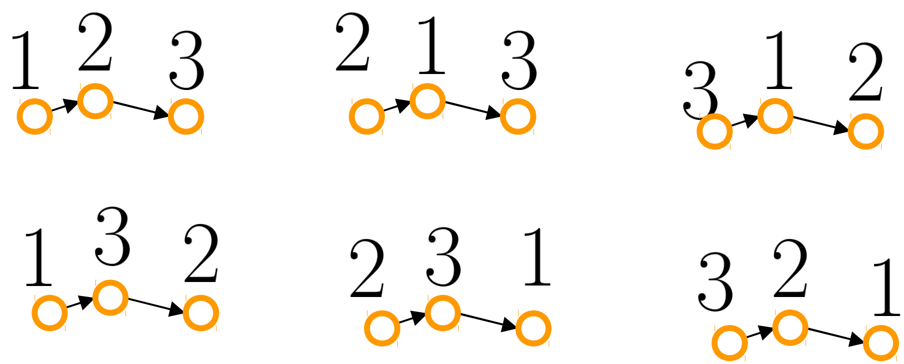
$$X(x) = \sum_{k=0}^{\infty} \delta_{k,1} \frac{x^k}{k!} = x$$

Linear Orders

$$\mathbb{L}(x) = \sum_{k=0}^{\infty} |\mathbb{L}[k]| \frac{x^k}{k!}$$



$\xrightarrow{\mathbb{L}}$



$$|\mathbb{L}[k]| = k!$$

Generating Function

$$\mathbb{L}(x) = \sum_{k=0}^{\infty} |\mathbb{L}[k]| \frac{x^k}{k!}$$

$$\mathbb{L}(x) = \sum_{k=0}^{\infty} k! \frac{x^k}{k!}$$

$$= \sum_{k=0}^{\infty} x^k$$

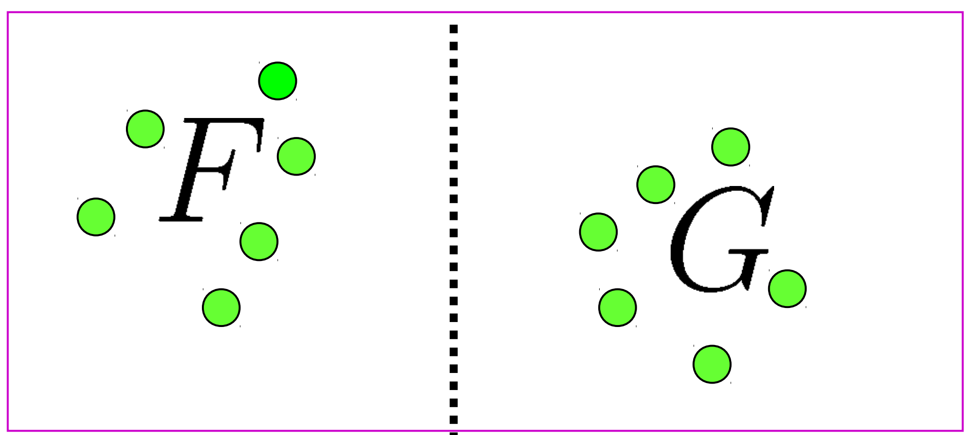
$$= \frac{1}{1-x}$$

Product

$$F(x)G(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \binom{k}{j} |F[j]||G[k-j]| \right) \frac{x^k}{k!}$$

Disjoint Union

$$(F.G)[l] = \sum_{U_1 \uplus U_2 = U} F[l_{U_1}] \times G[l_{U_2}]$$



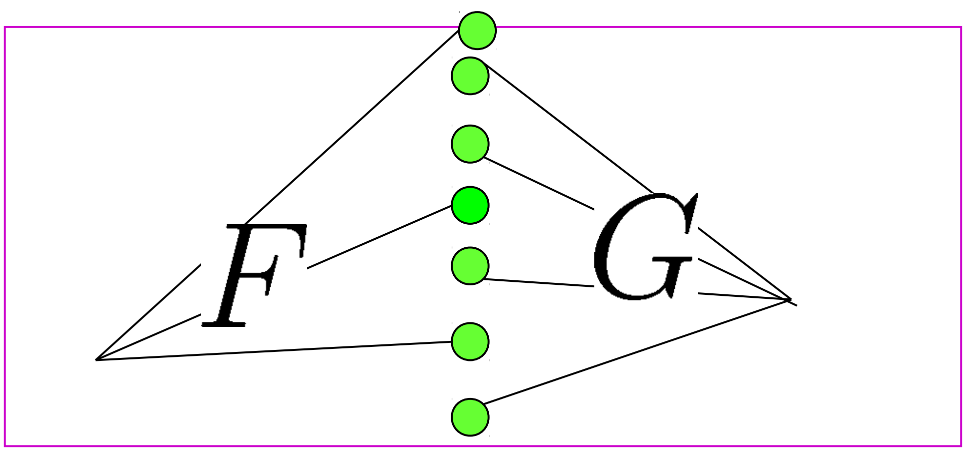
$$(F \cdot G)(x) = F(x)G(x)$$

Product

$$F(x)G(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \binom{k}{j} |F[j]||G[k-j]| \right) \frac{x^k}{k!}$$

Disjoint Union

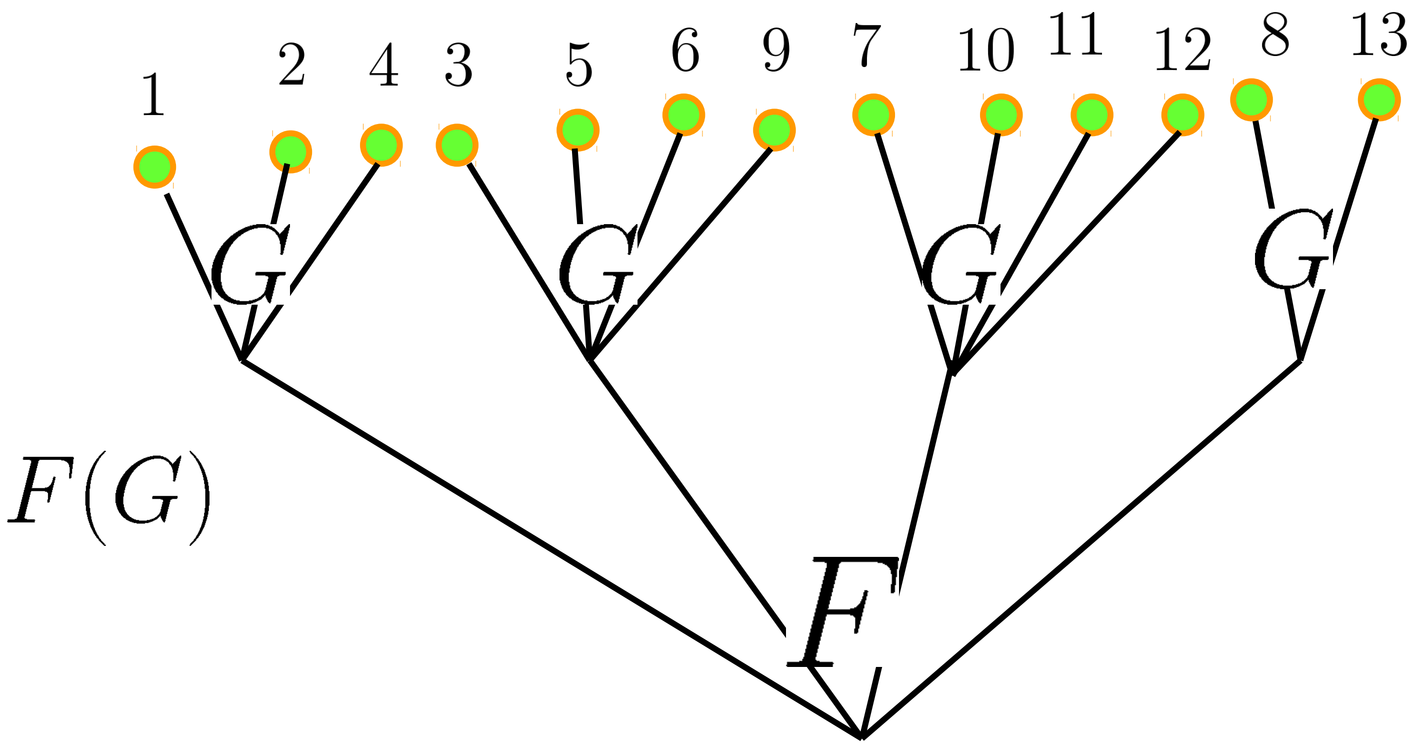
$$(F.G)[l] = \sum_{U_1 \uplus U_2 = U} F[l_{U_1}] \times G[l_{U_2}]$$



$$(F \cdot G)(x) = F(x)G(x)$$

substitution $F(G(x)) = \sum_{k=0}^{\infty} a_k \frac{(G(x))^k}{k!}$

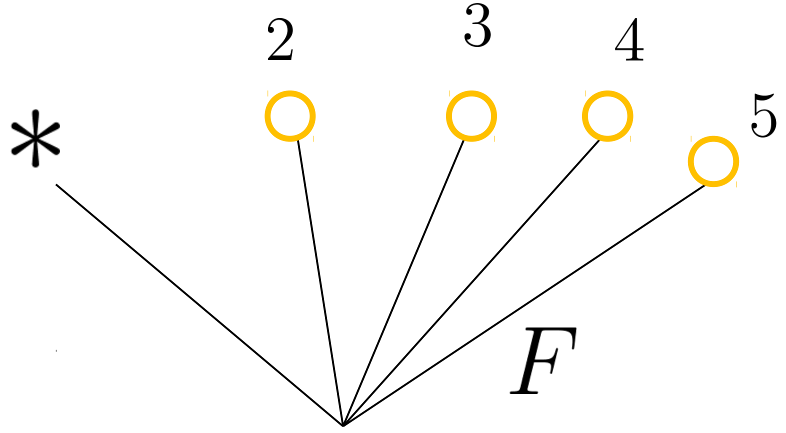
$$F(G)[l] = \sum_{\pi^l \in \Pi[l]} F[\pi^l] \times \left(\prod_{B\in\pi} G[l_B]\right)$$



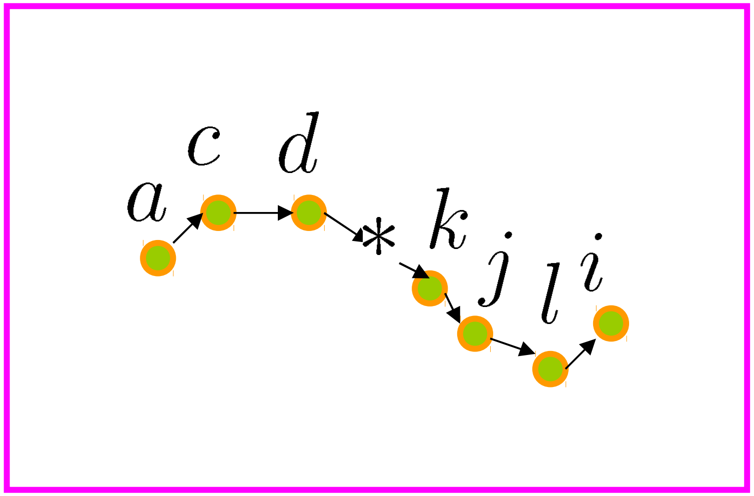
Combinatorial Derivative

$$F'(x) = \sum_{k=0}^{\infty} |F[n+1]| \frac{x^k}{k!}$$

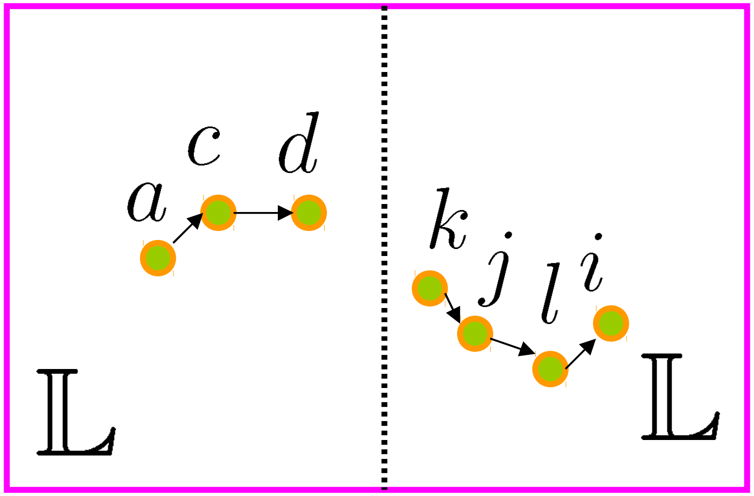
$$F'[l] = F[* + l]$$



$$D\mathbb{L}[a,c,d,k,j,l,i] = \mathbb{L}[*,a,c,d,k,j,l,i]$$



$$D\mathbb{L} = \mathbb{L}^2$$

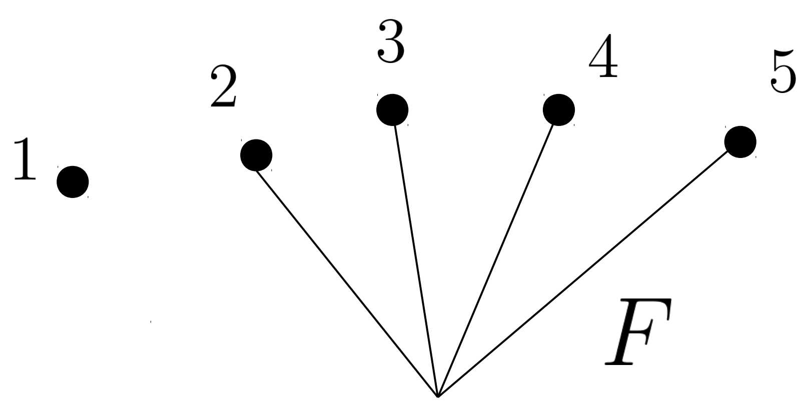


$$D\frac{1}{1-x} = \left(\frac{1}{1-x}\right)^2$$

Combinatorial Integral

$$\int_0^T F(t)dt = \sum_{n=0}^{\infty} |F[n-1]| \frac{T^n}{n!}$$

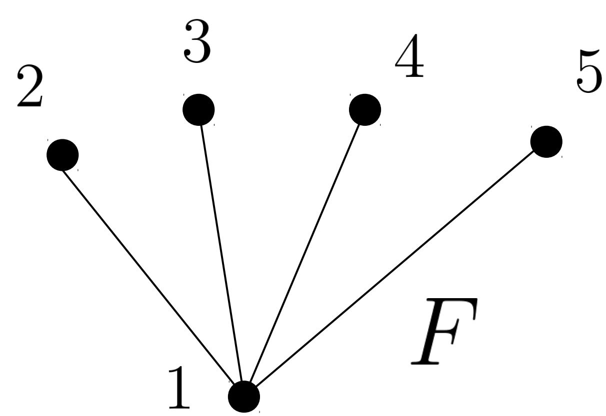
$$\left(\int_0^T F(t)dt\right)[l] = F[l - \min l]$$



Combinatorial Integral

$$\int_0^T F(t)dt = \sum_{n=0}^{\infty} |F[n-1]| \frac{T^n}{n!}$$

$$\left(\int_0^T F(t)dt\right)[l] = F[l - \min l]$$



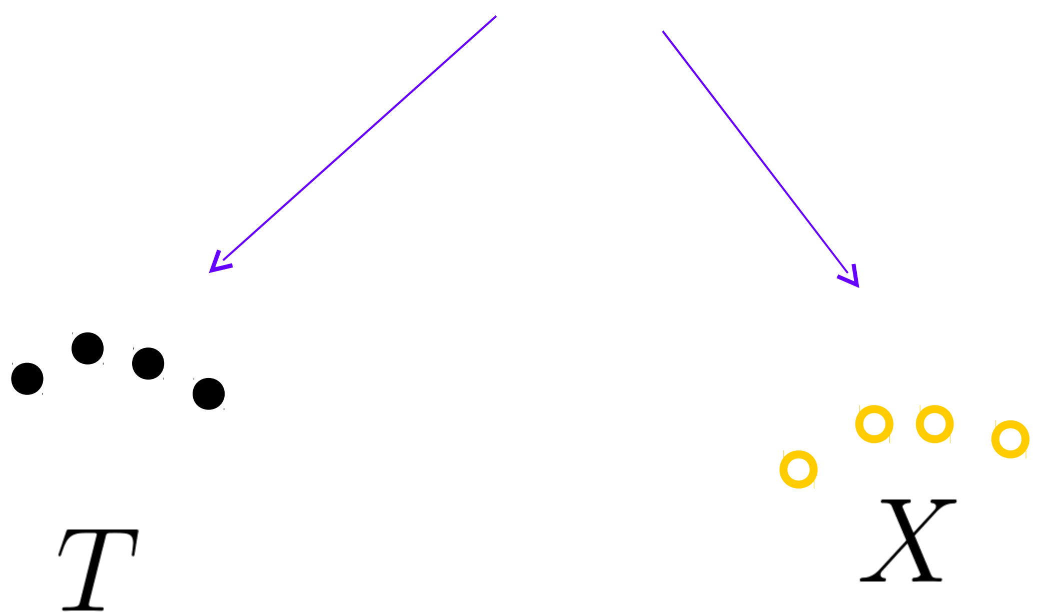
Leroux-Viennot combinatorial solution of the differential equation

$$y' = \phi(y)$$
$$y(0) = x$$

$$y(t, x) = \sum_{k,j} a_{k,j} \frac{t^k x^j}{k!j!}$$

$$F(t,x) = \sum_{k,j} |F[k,j]| \frac{t^k x^j}{k!j!}$$

$$F[l_1, l_2]$$



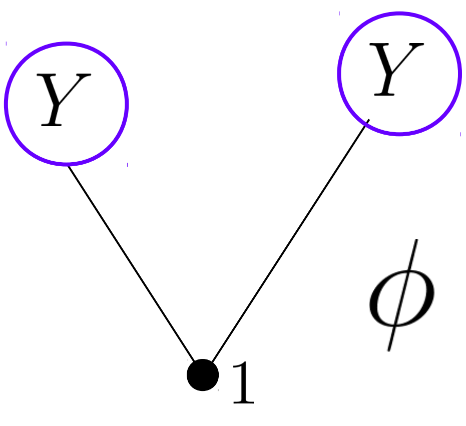
Leroux-Viennot combinatorial solution of the differential equation

$$Y' = \phi(Y)$$
$$Y(0, X) = X$$

$$Y = X + \int_0^T \phi(Y)dt$$

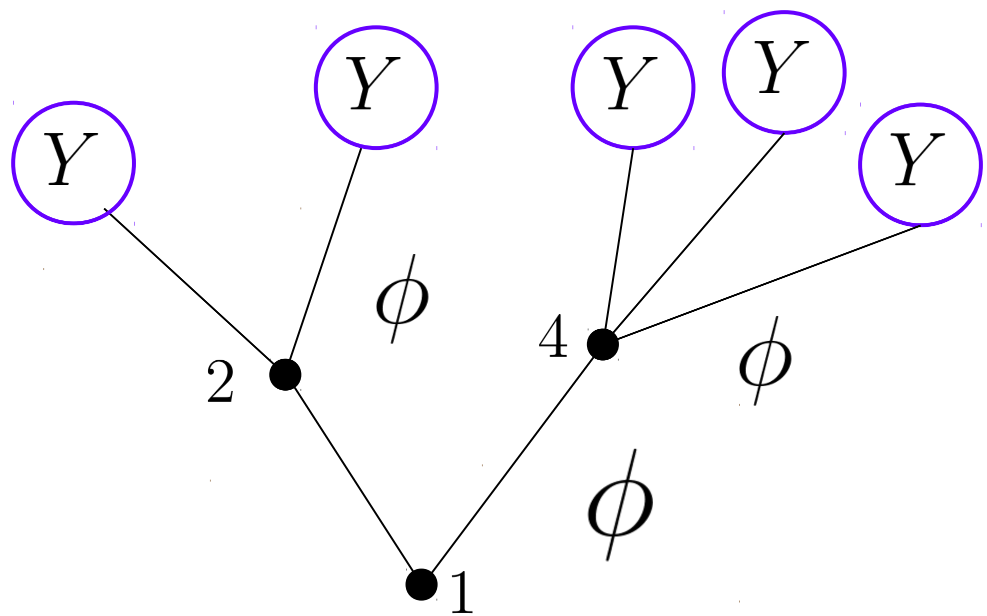
$$Y = X + \int_0^T \phi(Y)dt$$

$$Y[\{1,2,3,4,5\},\{u,v,w,x,y,z,w\}]$$



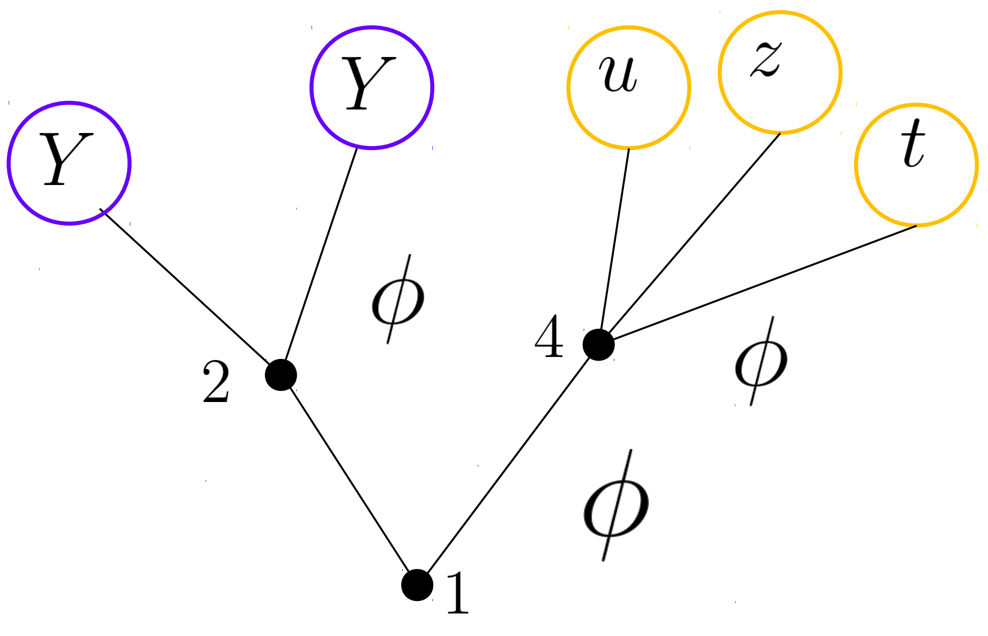
$$Y = X + \int_0^T \phi(Y)dt$$

$$Y[\{1,2,3,4,5\},\{u,v,w,x,y,z,w\}] =$$



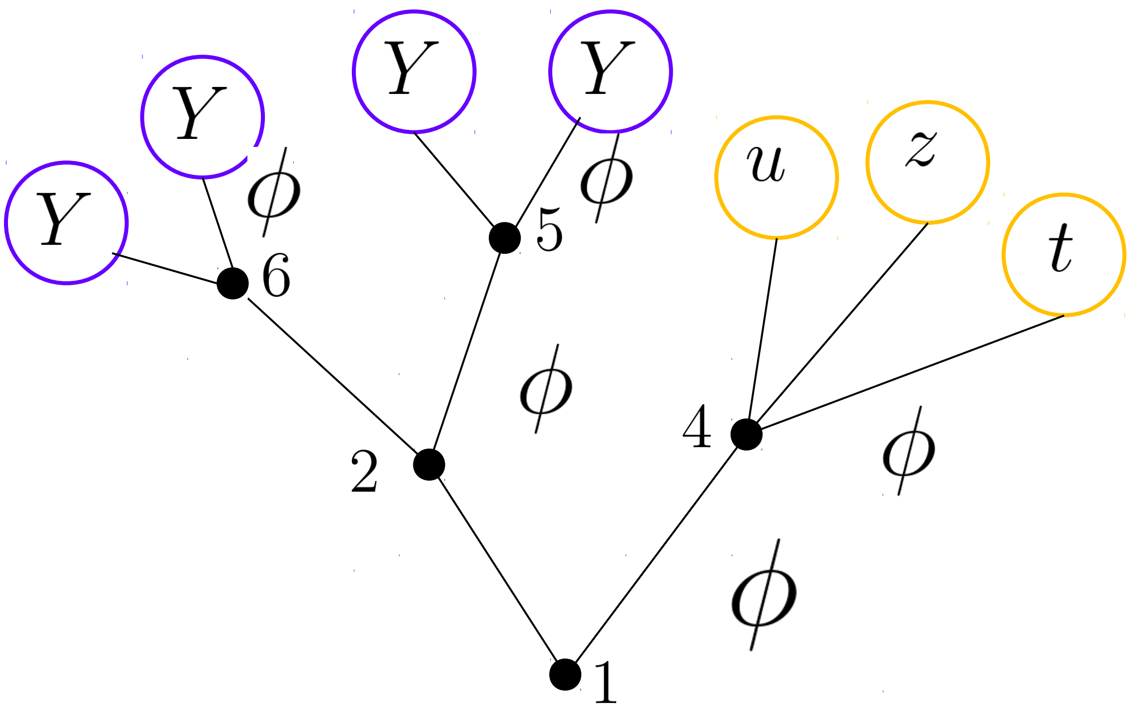
$$Y = X + \int_0^T \phi(Y)dt$$

$$Y[\{1,2,3,4,5,6\},\{u,v,w,x,y,z,w\}] =$$



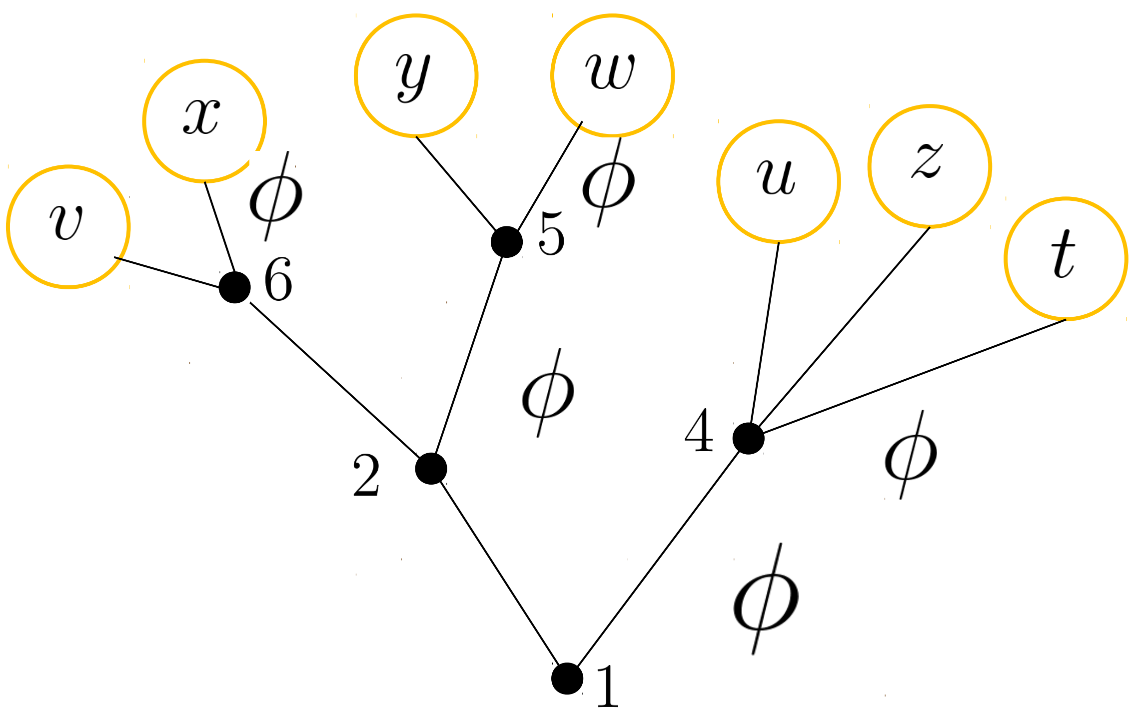
$$Y = X + \int_0^T \phi(Y)dt$$

$$Y[\{\underline{1,2,3,4,5,6}\},\{\underline{u,v,w,x,y,z,w}\}] =$$



$$Y = X + \int_0^T \phi(Y)dt$$

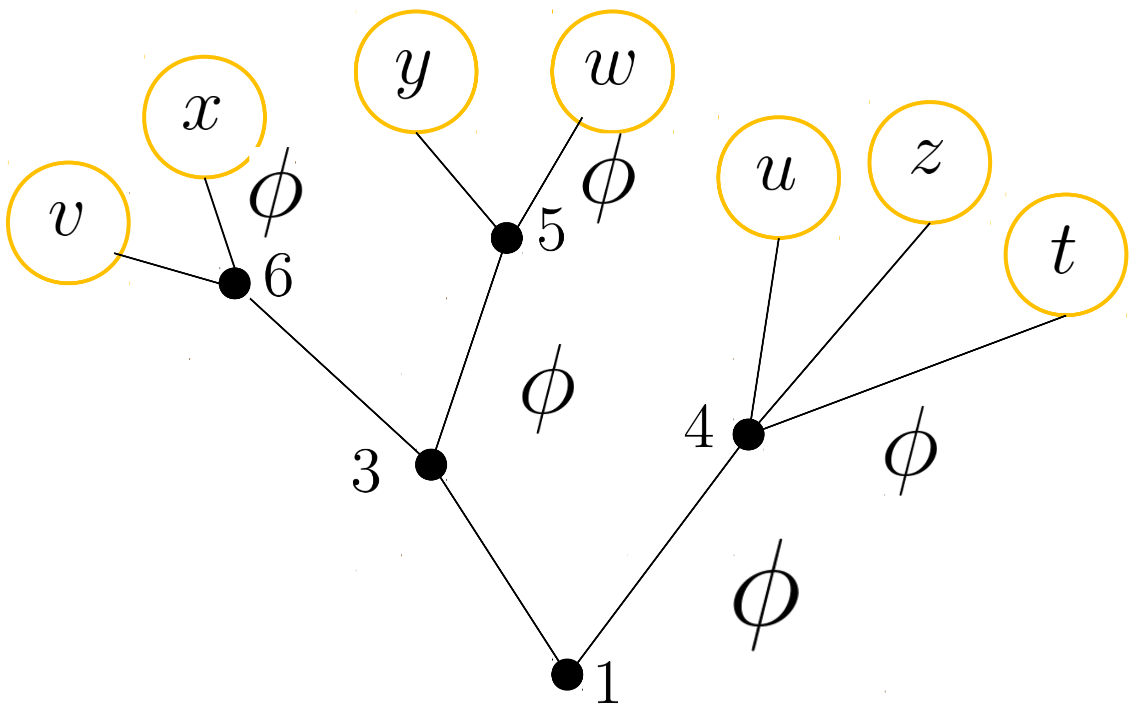
$$Y[\{1, 2, 3, 4, 5, 6\}, \{t, u, v, w, x, y, z\}]$$



$$Y = X + \int_0^T \phi(Y)dt$$

$$Y = \mathcal{A}_\phi^\uparrow$$

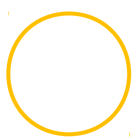
Increasing ϕ -enriched trees.



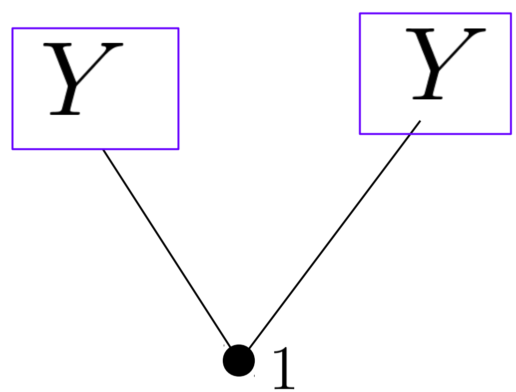
Example

$$Y' = Y^2$$
$$Y(0, X) = 1$$

$$Y = 1 + \int_0^T Y^2 dt$$

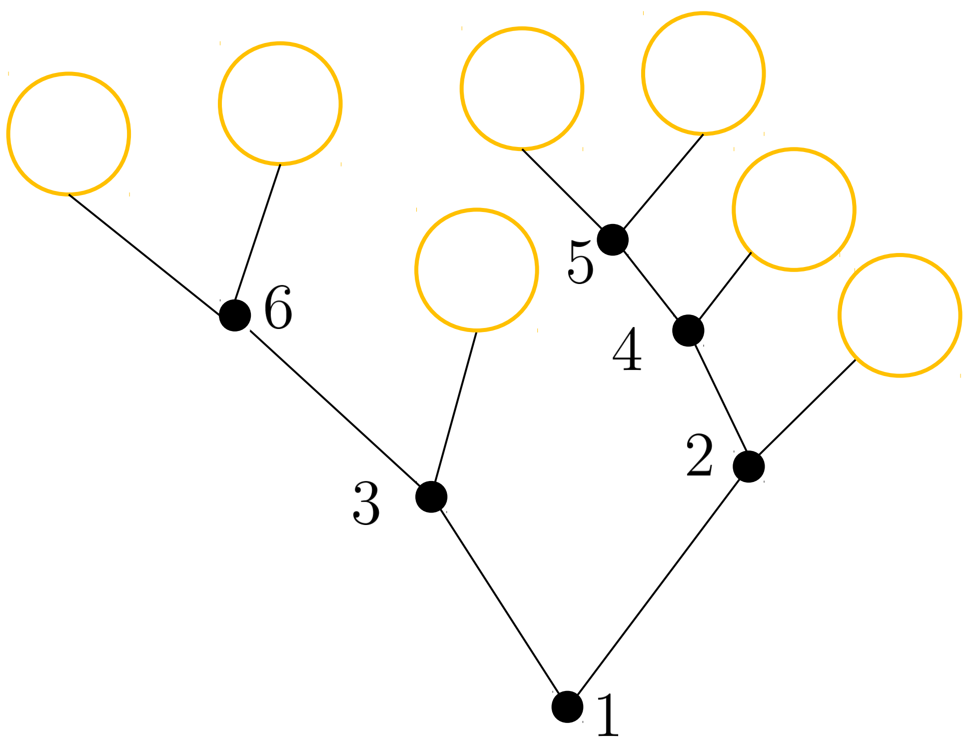


or



Example

$$Y' = Y^2$$
$$Y(0, X) = 1$$

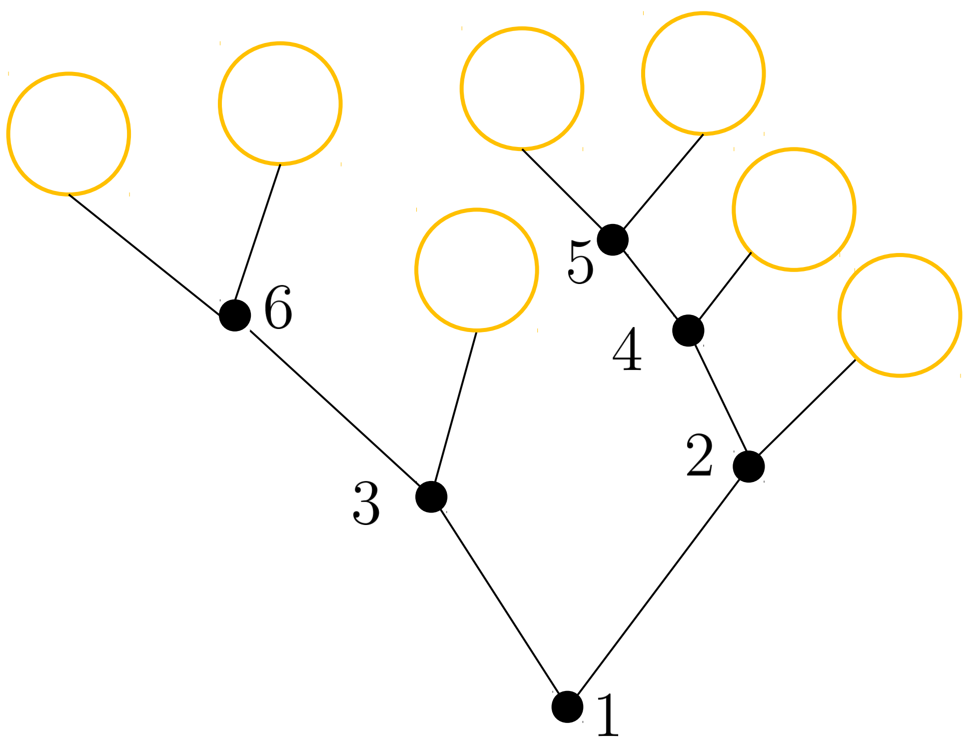


$$Y = \mathcal{A}^{\uparrow}_{X^2} = \mathbb{L}$$

631542

Example

$$Y' = Y^2$$
$$Y(0,X) = 1$$



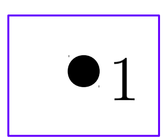
$$Y = \mathcal{A}^{\uparrow}_{X^2} = \mathbb{L}$$

631542

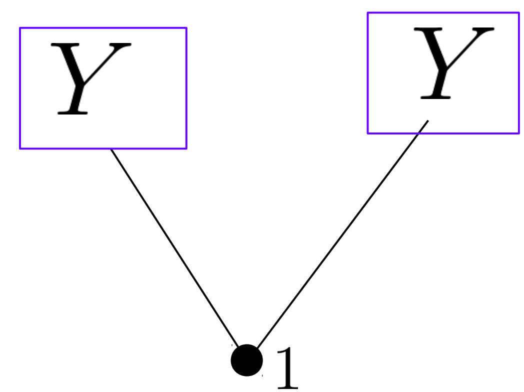
Example

$$Y' = Y^2 + 1$$
$$Y(0, X) = 0$$

$$Y = T + \int_0^T Y^2 dt$$



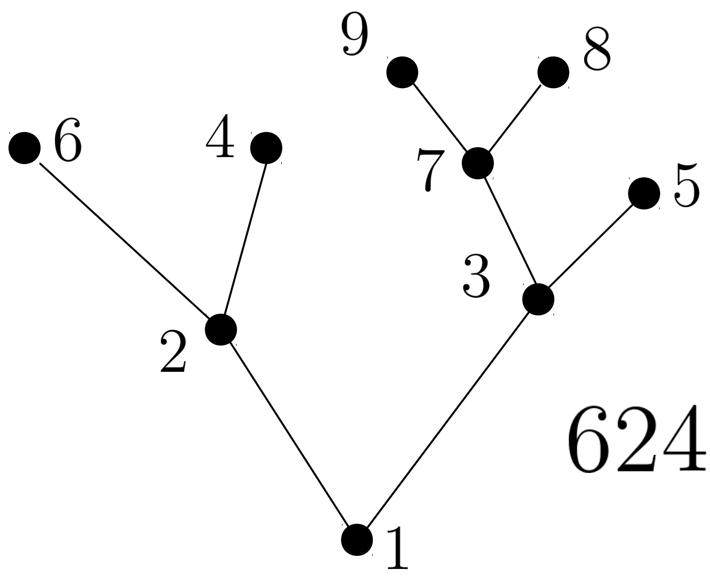
or



Example

$$Y' = Y^2 + 1$$
$$Y(0, X) = 0$$

$$Y = \mathcal{A}^{\uparrow}_{X^2+1}(T, 0) = Alt_{odd}$$



$$Y(t) = \tan(t)$$

624197835

The plethystic case

$$y' = \phi(y(t, x_1, x_2, \dots))$$
$$y(0) = x_0$$

$$\phi = \phi(x_1)$$

$$(\phi *_s y)(t, x_0, x_1, \dots) = \phi(y(t, x_1, x_2, \dots))$$

$$y'(t, x_0, x_1, \ldots) = \phi(y(t, x_1, x_2, \ldots))$$
$$y(0) = x_0$$

solution

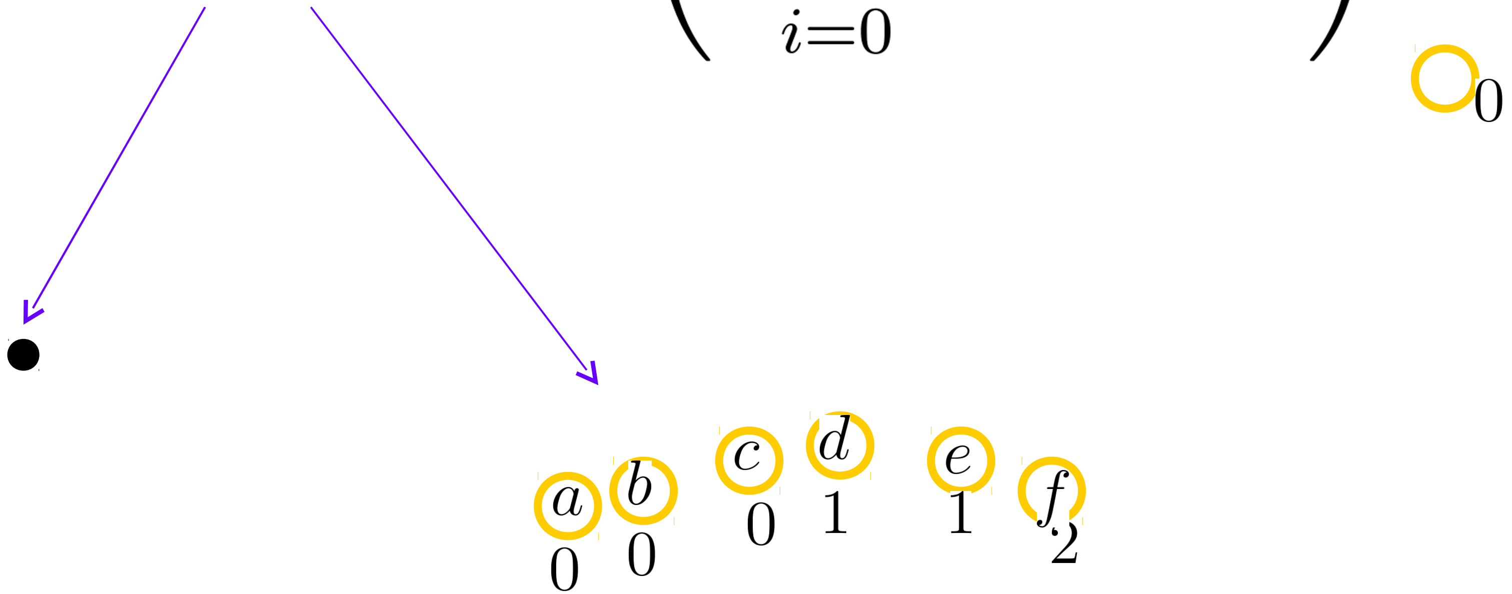
$$Y(t, \mathbf{x}) = \mathcal{A}^{\uparrow}_{\phi}(t, \mathbf{x})$$

$$\mathcal{A}^{\uparrow}_{\phi}(t, \mathbf{x}) = \sum_{k,\alpha} |\mathcal{A}^{\uparrow}_{\phi}[k, \alpha]| \frac{t^k x_0^{\alpha_0} x_1^{\alpha_1} \ldots}{k! \alpha_0! \alpha_1! \ldots}$$

number of ϕ-enriched increasing trees with k internal vertices and α_j leaves of height j

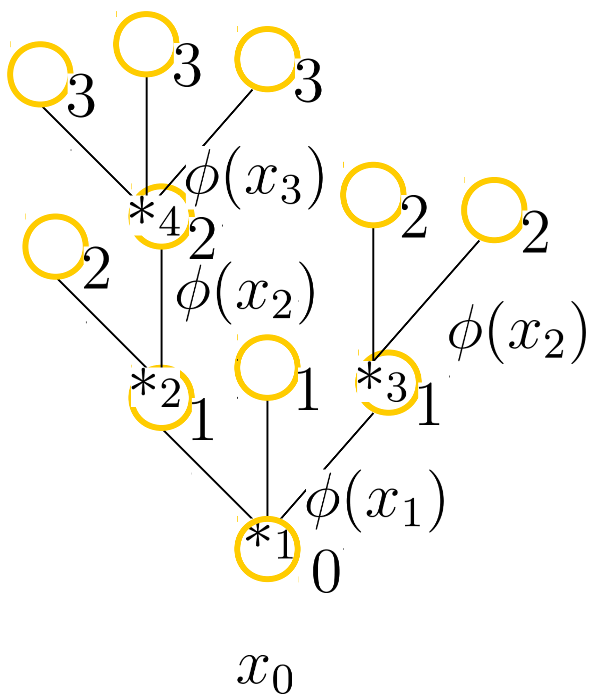
Lie-Gröbner-Taylor Formula

$$\mathcal{A}_{\phi}^{\uparrow}(t,\mathbf{x}) = \exp\left(t \sum_{i=0}^{\infty} \phi(x_{i+1}) \partial_i \right) x_0$$



Lie-Gröbner-Taylor Formula

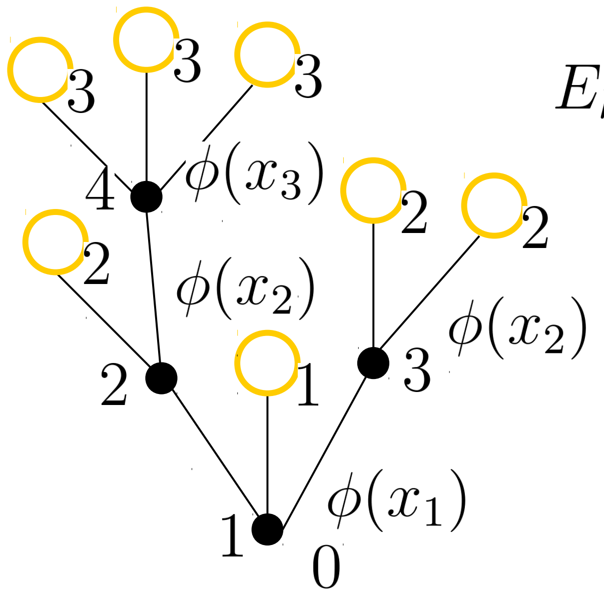
$$\exp\left(t\sum_{i=0}^{\infty}\phi(x_{i+1})\partial_i\right)x_0 = \sum_{k=0}^{\infty}\frac{t^k}{k!}\left(\sum_{i=0}^{\infty}\phi(x_{i+1})\partial_i\right)^k x_0$$



$$\left(\sum_{i=0}^{\infty}\phi(x_{i+1})\partial_i\right)^4 x_0$$

Lie-Gröbner-Taylor Formula

$$\exp\left(t\sum_{i=0}^{\infty}\phi(x_{i+1})\partial_i\right)x_0=\sum_{k=0}^{\infty}\frac{t^k}{k!}\left(\sum_{i=0}^{\infty}\phi(x_{i+1})\partial_i\right)^k x_0$$



$$E_k[1,2,3,4]=\{\{1,2,3,4\}\}$$

$$\boxed{\frac{t^4}{4!}}\left(\sum_{i=0}^{\infty}\phi(x_{i+1})\partial_i\right)^4 x_0$$

Lie-Gröbner-Taylor Formula

$$\exp\left(t\sum_{i=0}^{\infty}\phi(x_{i+1})\partial_i\right)F(x_0,x_1,x_2,\ldots) = (F *_s \mathcal{A}_\phi^{\uparrow})(t,\mathbf{x})$$

$$(F *_s \mathcal{A}_\phi^{\uparrow})(t,x_0,x_1,\ldots) = F(\mathcal{A}_\phi^{\uparrow}(t,x_0,x_1,\ldots),\mathcal{A}_\phi^{\uparrow}(t,x_1,x_2,\ldots),\ldots)$$

Creation and anhilation bosonic operators

$$a_i a_i^+ - a_i^+ a_i = 1$$

$$\exp\left(t\sum_{i=0}^{\infty}\phi(x_{i+1})\partial_i\right) F(x_0, x_1, x_2, \ldots) = F *_s \mathcal{A}_\phi^{\uparrow}(t, \mathbf{x})$$

$$\exp\left(t\sum_{i=0}^{\infty}\phi(a_{i+1}^+)a_i\right) =: \exp(\sum_{i=0}^{\infty}\mathcal{A}_\phi^{\uparrow}(t, a_{i+1}^+, a_{i+2}^+, \ldots)a_i :$$