Evolution, Localisation and Wronskians.

Lie theoretic aspects of NCDE.

G.H.E. Duchamp

Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
Karol A. Penson, Darij Grinberg, Hoang Ngoc Minh, C. Lavault,
C. Tollu, N. Behr, V. Dinh, C. Bui,
Q.H. Ngô, N. Gargava, S. Goodenough.

CIP seminar,
16 Feb. 2021.

Plan

Useful categories/1

Below a quick list of the categories of use in combinatorics (k is a given field), morphisms are standard.

1. **St**, the category of sets
2. **Mon**, the category of monoids
3. **CMon**, the category of commutative monoids
4. **Gp**, the category of groups
5. **Ring**, the category of rings
6. **CRing**, the category of commutative rings
7. $\mathbf{Vect_k}$, the category of k-vector spaces
8. $\mathbf{Lie_k}$, the category of k-Lie algebras
9. $\mathbf{AAU_k}$, the category of k-Associative Algebras with Unit
10. $\mathbf{CAAU_k}$, the category of k-Associative and Commutative Algebras with Unit

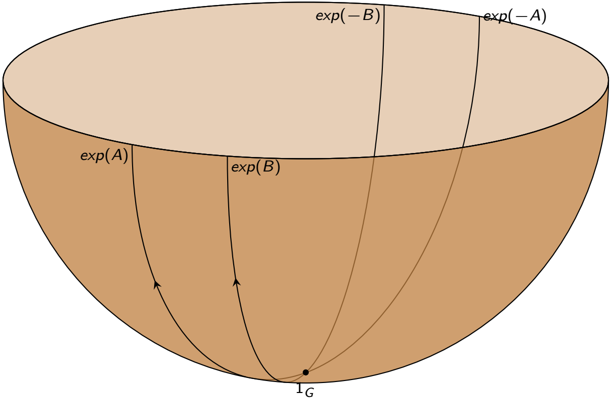
Useful categories/2

11. **Mg**, the category of Magmas i.e. sets with only a binary law (without conditions)
12. $\mathbf{Alg_k}$, the category of k-Algebras (without conditions)
13. $\mathbf{DiffAlg_k}$, the category of k-Associative Differential Algebras with Unit.
14. $\mathbf{CDiffAlg_k}$, the category of k-Associative Commutative Differential Algebras with Unit.
15. **DiffRing**, the category of Differential rings.
16. **CDiffRing**, the category of Commutative Differential Rings.

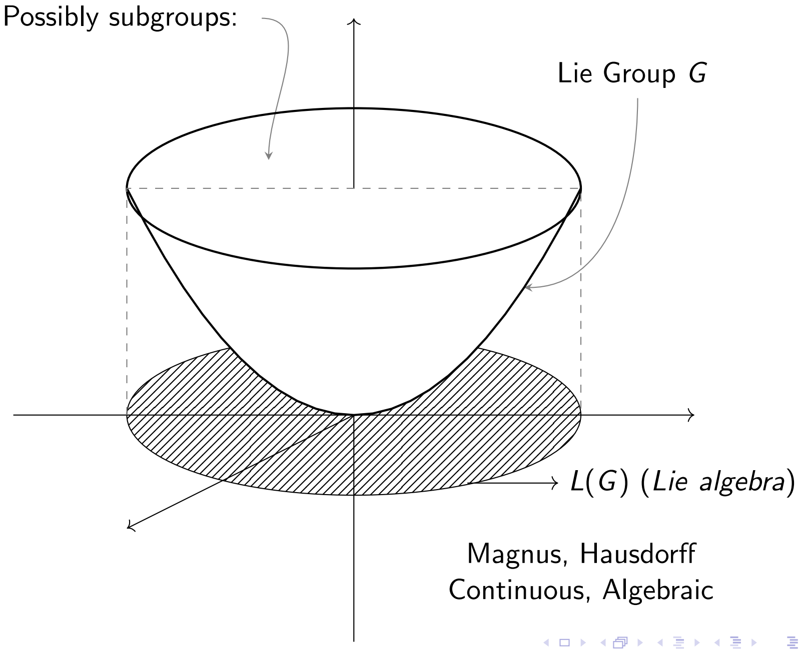
Remarks. –
i) All of these have a standard forgetful functor to **St**. They usually compose and factor nicely. See also [19].
ii) For $\mathbf{k} = \mathbb{Z}$, one has

$$\mathbf{DiffAlg}_{\mathbb{Z}} = \mathbf{DiffRing} \text{ and } \mathbf{CDiffAlg}_{\mathbb{Z}} = \mathbf{CDiffRing}.$$

Magnus and Hausdorff groups



The Magnus group is the set of series with constant term 1_{X^*}, the Hausdorff (sub)-group, is the (conc-)group of ш-characters. When $\mathbf{k}$ is a field, these are the group-like series for $\Delta_{ш}$). When $\mathbf{k} \supset \mathbb{Q}$ these are also Lie exponentials (here A, B are Lie series and $exp(A)exp(B) = exp(H(A, B))$).


Possibly subgroups:
Lie Group G
$L(G)$ (*Lie algebra*)
Magnus, Hausdorff
Continuous, Algebraic

Closed subgroups (Cartan theorem)

of $Gl(n,\mathbf{k})$	Zariski	$U(n,\mathbf{k}), D(n,\mathbf{k})$ $N(n,\mathbf{k})$
of $Gl(n,\mathbf{k})$	Topology ($\mathbf{k}=\mathbb{R},\mathbb{C}$)	$Gl_+(n,\mathbb{R})$
of $Mag(n,\mathbb{C})$	Topology (Formal)	$(Haus_{\text{Ш}_\varphi}(n,\mathbb{C}))$
$U(n,\mathbf{k})$	Equations	$\mathbf{k}=\mathbb{C},\ X^*X=I$
$O(n,\mathbf{k})$	Equations	$\mathbf{k}=\mathbb{R},\ ({}^tX)X=I$
$D(n,\mathbf{k})$	Equations	$a_{i,j}=0,\ i\neq j$
$N(n,\mathbf{k})$	Equations	$a_{i,j}=0,\ i>j$ $a_{ii}=1$

Remarks

i) Here, the formal topology is defined by summability conditions.
ii) For formal aspects of DE, have a look at the chapter “Combinatorial differential equations” in [1].

Wei-Norman theorem

mathoverflow

Questions | Tags | Users | Badges | Unanswered | Ask Question

Local coordinates on (infinite dimensional) Lie groups, factorization of Riemann zeta functions

3

3

Given a (finite dimensional) Lie group G (real $k=\mathbb{R}$ or complex $k=\mathbb{C}$) and its Lie algebra $\mathfrak{g}$, one can prove (a basis $B=(b_i)_{1\leq i\leq n}$ of $\mathfrak{g}$ being given) that there exists a neighbourhood W of 1_G (in G) and n *local coordinate* analytic functions

$$W\to k,\ (t_i)_{1\leq i\leq n}$$

such that, for all $g\in W$

$$(*)\quad g=\prod_{1\leq i\leq n}^{\rightarrow} e^{t_i(g)b_i}=e^{t_1(g)b_1}e^{t_2(g)b_2}\dots e^{t_n(g)b_n}$$

to see this, just remark that

$$(t_1,t_2,\cdots t_n)\to exp(t_1b_1)exp(t_2b_2)\cdots exp(t_nb_n)$$

is a *local* diffeomorphism from k^n to G in a neighbourhood of 0 and take the inverse.

This is the local **Wei-Norman's theorem**.

My questions are the following

> Let us loosely call *infinite dimensional* a Lie group whose Lie algebra is not finite dimensional (this includes the example below and infinite dimensional Banach-Lie groups for instance).
>
> Q1) Can you provide examples of infinite dimensional Lie groups where the exponential map

asked **2 years, 6 months ago**
viewed **631 times**
active **11 months ago**

FEATURED ON META

Donations to MathOverflow, Inc

HOT META POSTS

14 How does one cite a MO post with an anonymous author?

3 If I can't get an answer on math.stackexchange.com or dsp.stackexchange.com,...

10 On flagging: what is "research level" *or* does it make sense to flag a...

64 People Chatting

Homotopy Theory

8 hours ago - Aaron Mazel-Gee

Theorem (Wei-Norman theorem)

Let G be a k-Lie group (of finite dimension) ($k = \mathbb{R}$ or $k = \mathbb{C}$) and let $\mathfrak{g}$ be its k-Lie algebra. Let $B = \{b_i\}_{1\leq i\leq n}$ be a (linear) basis of it. Then, there is a neighbourhood W of 1_G (within G) and n analytic functions (local coordinates)

$$W \to k,\ (t_i)_{1\leq i\leq n}$$

such that, for all $g \in W$

$$g = \prod_{1\leq i\leq n}^{\rightarrow} e^{t_i(g)b_i} = e^{t_1(g)b_1} e^{t_2(g)b_2} \ldots e^{t_n(g)b_n}.$$

Example

Example

We take $G = Gl_+(2,\mathbb{R})$ ($Gl_+(2,\mathbb{R})$, connected component of 1 within $Gl(2,\mathbb{R})$),

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{1}$$

We will practically compute the Wei-Norman coefficients through an Iwasawa decomposition

$$M = \text{unitary x diagonal x unitriangular}$$

and compute $MTDU = I_2$ through the following elementary operations

1. (Orthogonalisation)
2. (Normalisation)
3. (Unitarisation)

$$
\begin{aligned}
M &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = (C_1, C_2) = (C_1^{(1)}, C_2^{(1)}) e^{\frac{\langle C_1 | C_2 \rangle}{||C_1||^2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \\
&= \underbrace{e^{arctan(\frac{a_{21}}{a_{11}}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}}_{unitary} \underbrace{e^{log(||C_1^{(1)}||) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} e^{log(||C_2^{(1)}||) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}}_{diagonal\ \ (two\ \ exps)} \underbrace{e^{\frac{\langle C_1 | C_2 \rangle}{||C_1||^2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}}_{triangular}
\end{aligned}
$$

We then get a Wei-Norman decomposition w.r.t. the following basis of

$$
\mathfrak{gl}(2, \mathbb{R}) : \ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
$$

Remark. – Iwasawa decomposiion is general for $Gl(n, \mathbf{k})$, $\mathbf{k}$ being one of the fields $\mathbb{R}, \mathbb{C}, \mathbb{H}$, see [4]. For $\mathbb{R}$ and non-Archimedean fields in the same book, see [2].

Bits and pieces for the BTT

Theorem (DDMS [1])

Let $(\mathcal{A}, d)$ be a k-commutative associative differential algebra with unit and $\mathcal{C}$ be a differential subfield of $\mathcal{A}$ (i.e. $d(\mathcal{C}) \subset \mathcal{C}$). We suppose that $S \in \mathcal{A}\langle\langle X \rangle\rangle$ is a solution of the differential equation

$$\mathbf{d}(S) = MS \ ; \ \langle S | 1_{X^*} \rangle = 1_{\mathcal{A}} \tag{2}$$

where the multiplier M is a homogeneous series (a polynomial in the case of finite X) of degree 1, i.e.

$$M = \sum_{x \in X} u_x \, x \in \mathcal{C}\langle\langle X \rangle\rangle \ . \tag{3}$$

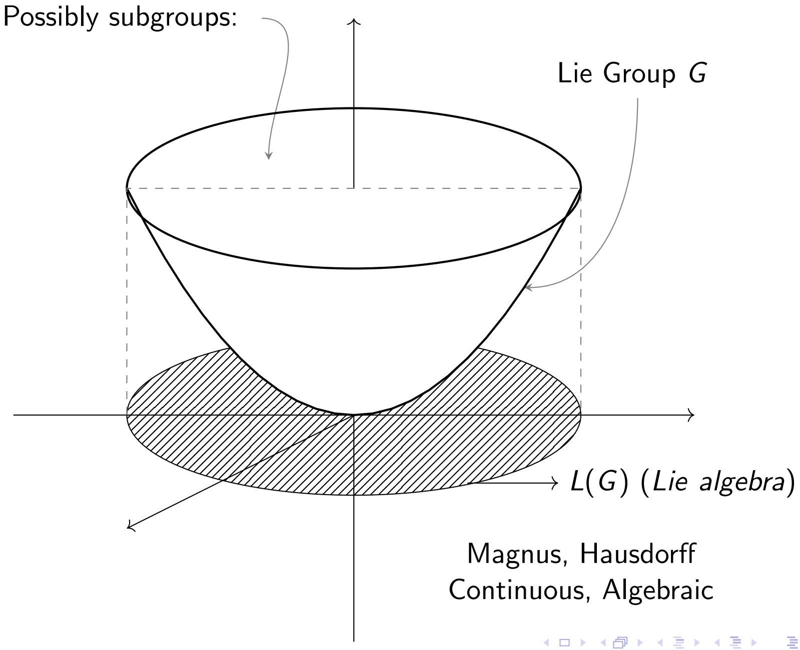
[1] *Independence of Hyperlogarithms over Function Fields via Algebraic Combinatorics*, **M. Deneufchâtel, GHED, V. Hoang Ngoc Minh and A. I. Solomon**, 4th International Conference on Algebraic Informatics, Linz (2011). Proceedings, Lecture Notes in Computer Science, 6742, Springer.

Bits and pieces for the BTT/2

Theorem (cont'd)

The following conditions are equivalent :

1. *The family* $(\langle S|w\rangle)_{w\in X^*}$ *of coefficients of* S *is free over* $\mathcal{C}$.
2. *The family of coefficients* $(\langle S|y\rangle)_{y\in X\cup\{1_{X^*}\}}$ *is free over* $\mathcal{C}$.
3. *The family* $(u_x)_{x\in X}$ *is such that, for* $f\in\mathcal{C}$ *and* $\alpha_x\in k$
$$d(f)=\sum_{x\in X}\alpha_x u_x \Longrightarrow (\forall x\in X)(\alpha_x=0)\ . \tag{4}$$
4. *The family* $(u_x)_{x\in X}$ *is free over* k *and*
$$d(\mathcal{C})\cap span_k\Big((u_x)_{x\in X}\Big)=\{0\}\ . \tag{5}$$


Possibly subgroups:
Lie Group G
$L(G)$ (*Lie algebra*)
Magnus, Hausdorff
Continuous, Algebraic

Why BTT and NCDE ? : Review of the facts

- $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$ $(\Re(s) > 1)$
- when one multiplies two of these, one gets quantities like

$$\zeta(s_1)\zeta(s_2) = \sum_{n_1,n_2\geq 1} \frac{1}{n_1^{s_1} n_2^{s_2}} = \zeta(s_1,s_2) + \zeta(s_1+s_2) + \zeta(s_2,s_1)$$

- and, with several of them, we are led to the following definition of MultiZeta Values (MZV), converging in
$$\mathcal{H}_r = \{(s_1,\ldots,s_r) \in \mathbb{C}^r \,|\, \forall m = 1,\ldots,r, \Re(s_1) + \ldots + \Re(s_m) > m\}\ .$$

$$\zeta(s_1,\ldots,s_k) := \sum_{n_1>\ldots>n_k\geq 1} \frac{1}{n_1^{s_1}\ldots n_k^{s_k}} \tag{6}$$

- On the other hand, one has the classical polylogarithms defined, for $k \geq 1, |z| < 1$, by

$$-\log(1-z) = \mathrm{Li}_1 = \sum_{n\geq 1} \frac{z^n}{n^1};\ \mathrm{Li}_2 = \sum_{n\geq 1} \frac{z^n}{n^2};\ \ldots;\ \mathrm{Li}_k(z) := \sum_{n\geq 1} \frac{z^n}{n^k}$$

Why BTT and NCDE ? : Review of the facts/2

- The analogue of the classical polylogarithms for MZV reads

$$Li_{y_{s_1}\ldots y_{s_k}}(z) := \sum_{n_1>\ldots>n_k\geq 1} \frac{z^{n_1}}{n_1^{s_1}\ldots n_k^{s_k}} \ ; \ |z|<1$$

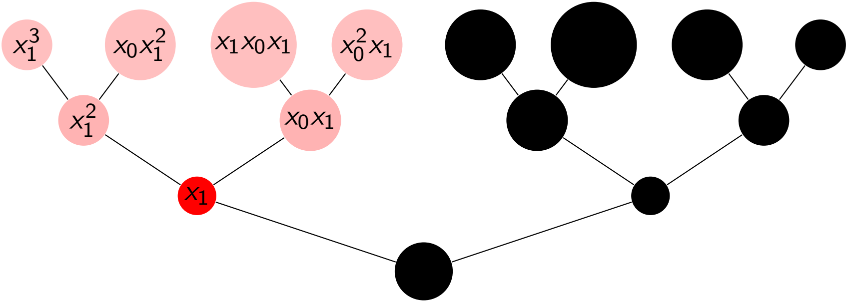
- They satisfy the recursion (ladder stepdown)

$$\begin{aligned} z\frac{d}{dz}Li_{y_{s_1}\ldots y_{s_k}} &= Li_{y_{s_1-1}\ldots y_{s_k}} \text{ if } s_1>1 \\ (1-z)\frac{d}{dz}Li_{y_1 y_{s_2}\ldots y_{s_k}} &= Li_{y_{s_2}\ldots y_{s_k}} \text{ if } k>1 \end{aligned} \tag{7}$$

which, with $s_i\in\mathbb{N}_{\geq 1}$, $k\geq 1$, ends at the "seed"

$$\mathrm{Li}_{y_1}(z)=\mathrm{Li}_1(z)=\log(\frac{1}{1-z}) \tag{8}$$

- For the next step, we code the moves $z\frac{d}{dz}$ (resp. $(1-z)\frac{d}{dz}$) - or more precisely sections $\int_0^z \frac{f(s)}{s}ds$ (resp. $\int_0^z \frac{f(s)}{1-s}ds$) - with x_0 (resp. x_1).



Some coefficients with $X = \{x_0, x_1\}$; $u_0(z) = \frac{1}{z}$; $u_1(z) = \frac{1}{1-z}$, $*_0 = 0$

$$\langle S|x_1^n\rangle = \frac{(-log(1-z))^n}{n!} \quad ; \quad \langle S|x_0x_1\rangle = \underbrace{\mathrm{Li}_2(z)}_{cl.not.} = \mathrm{Li}_{x_0x_1}(z) = \sum_{n\geq 1}\frac{z^n}{n^2}$$

$$\langle S|x_0^2x_1\rangle = \underbrace{\mathrm{Li}_3(z)}_{cl.not.} = \mathrm{Li}_{x_0^2x_1}(z) = \sum_{n\geq 1}\frac{z^n}{n^3} \quad ; \quad \langle S|x_1x_0x_1\rangle = \mathrm{Li}_{x_1x_0x_1}(z) = \mathrm{Li}_{[1,2]}(z) = \sum_{n_1>n_2\geq 1}\frac{z^{n_1}}{n_1n_2^2}$$

$$\langle S|x_0x_1^2\rangle = \mathrm{Li}_{x_0x_1^2}(z) = \mathrm{Li}_{[2,1]}(z) = \sum_{n_1>n_2\geq 1}\frac{z^{n_1}}{n_1^2n_2}$$

; above "cl. not." stands for "classical notation"

Why BTT and NCDE ? : Review of the facts/3

- Calling S the prospective generating series

$$S = \sum_{w \in X^*} \underbrace{\langle S|w\rangle}_{\in \mathcal{H}(\Omega)} w \ ; \ X = \{x_0, x_1\} \tag{9}$$

V. Drinfel'd [1] indirectly proposed a way to complete the tree:

$$\begin{cases} \mathbf{d}(S) = (\frac{x_0}{z} + \frac{x_1}{1-z}).S & (NCDE) \\ \lim_{\substack{z \to 0 \\ z \in \Omega}} S(z) e^{-x_0 log(z)} = 1_{\mathcal{H}(\Omega)\langle\langle X \rangle\rangle} & (Asympt.\ Init.\ Cond.) \end{cases} \tag{10}$$

from the general theory, this system has a unique solution which is precisely Li (called G_0 in [1]) ; $S \mapsto \mathbf{d}(S)$ being the term by term derivation of the coefficients.

- Minh [2] indicated a way to effectively compute this solution through (improper) iterated integrals.

1. V. Drinfel'd, *On quasitriangular quasi-hopf algebra and a group closely connected with Gal($\bar{\mathbb{Q}}/\mathbb{Q}$)*, Leningrad Math. J., 4, 829-860, 1991.
2. H. N. Minh, *Summations of polylogarithms via evaluation transform*, Mathematics and Computers in Simulation, Vol. 42, 4-6, Nov. 1996, pp. 707-728

Explicit construction of Drinfeld's G_0

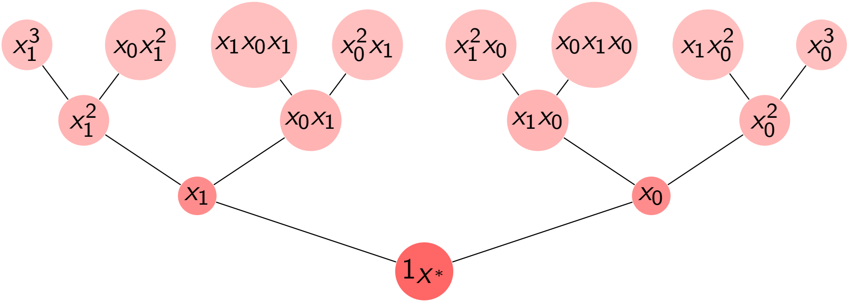
Given a word w, we note $|w|_{x_1}$ the number of occurrences of x_1 within w

$$\alpha_0^z(w) = \begin{cases} 1_\Omega & \text{if} \quad w = 1_{X^*} \\ \int_0^z \alpha_0^s(u)\frac{ds}{1-s} & \text{if} \quad w = x_1 u \\ \int_1^z \alpha_0^s(u)\frac{ds}{s} & \text{if} \quad w = x_0 u \text{ and } |u|_{x_1} = 0 \ (w \in x_0^*) \\ \int_0^z \alpha_0^s(u)\frac{ds}{s} & \text{if} \quad w = x_0 u \text{ and } |u|_{x_1} > 0 \ (w \in x_0 X^* x_1 x_0^*) \end{cases}$$

The third line of this recursion implies

$$\alpha_0^z(x_0^n) = \frac{log(z)^n}{n!}$$

one can check that (a) all the integrals (although improper for the fourth line) are well defined (b) the series $S = \sum_{w \in X^*} \alpha_0^z(w)\, w$ is Li (G_0 in [1]).



Some coefficients with $X = \{x_0, x_1\};\ u_0(z) = \frac{1}{z};\ u_1(z) = \frac{1}{1-z},\ t_0 = 0$

$$\langle S|x_1^n\rangle = \frac{(-log(1-z))^n}{n!} \quad ; \quad \langle S|x_0x_1\rangle = \underbrace{\mathrm{Li}_2(z)}_{cl.not.} = \mathrm{Li}_{x_0x_1}(z) = \sum_{n\geq 1}\frac{z^n}{n^2}$$

$$\langle S|x_0^2x_1\rangle = \underbrace{\mathrm{Li}_3(z)}_{cl.not.} = \mathrm{Li}_{x_0^2x_1}(z) = \sum_{n\geq 1}\frac{z^n}{n^3} \quad ; \quad \langle S|x_1x_0x_1\rangle = \mathrm{Li}_{x_1x_0x_1}(z) = \mathrm{Li}_{[1,2]}(z) = \sum_{n_1>n_2\geq 1}\frac{z^{n_1}}{n_1n_2^2}$$

$$\langle S|x_0x_1^2\rangle = \mathrm{Li}_{x_0x_1^2}(z) = \mathrm{Li}_{[2,1]}(z) = \sum_{n_1>n_2\geq 1}\frac{z^{n_1}}{n_1^2n_2} \quad ; \quad \langle S|x_0^n\rangle = \frac{\log^n(z)}{n!}$$

General solution of NCDE and Picard's process

The series $S_{Pic}^{z_0}$ $(z_0 \in \Omega)$ can be computed by Picard's process

$$S_0 = 1_{X^*} \; ; \; S_{n+1} = 1_{X^*} + \int_{z_0}^{z} M.S_n$$

and its limit is $S_{Pic}^{z_0} := \lim_{n\to\infty} S_n$ $(= \sum_{w\in X^*} \alpha_{z_0}^{z}(w)\, w$ this afternoon).
One has,

Proposition

i) Series $S_{Pic}^{z_0}$ is the unique solution of

$$\left\{ \begin{array}{rcl} \mathbf{d}(S) & = & M.S \textit{ with } M = \sum_{i=1}^{n} \frac{x_i}{z-a_i} \\ S(z_0) & = & 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} \end{array} \right. \tag{11}$$

ii) The complete set of solutions of $\mathbf{d}(S) = M.S$ is $S_{Pic}^{z_0}.\mathbb{C}\langle\langle X\rangle\rangle$.

About solutions of NCDE

1. The set $\mathcal{S}$ of series satisfying $(NCDE)$ has a lot of nice combinatorial properties.
   - Right $\mathbb{C}\langle\langle X\rangle\rangle$ module of rank one ($\mathcal{S} = S_0.\mathbb{C}\langle\langle X\rangle\rangle$, where S_0 is any solution with non-zero constant term, such a solution can be constructed by Picard process).
   - Linear independence of the coefficients (when non-zero).
2. The ones like Li or constructed through Picard's process (Chen series, i.e. limit of $S_0 = 1_{X^*}$; $S_{n+1} = 1_{X^*} + \int_{z_0}^{z} M.S_n$) have moreover
   - Shuffle property
   - Factorisation
   - Extension to rational functions (some of them for Li, all for $S_{Pic}^{z_0}$).

Now, as the lists are coded by words, it is possible to use the rich allowance of notations invented by algebraists, computer scientists, combinatorialists and physicists about NonCommutative Formal Power Series (NCFPS[1]).

[1] This was the initial intent of the series of conferences FPSAC.

What is so special with solutions like Li and $S_{Pic}^{z_0}$.

The following general theorem explains (a) why Li and $S_{Pic}^{z_0}$ have the shuffle property and (b) why Li is unique.

Theorem (Analyse et Géometrie, Cargèse, IESC, 21-24 Nov. 2017)

Let

$$(TSM) \qquad \mathbf{d}S = M_1 S + S M_2 \ . \tag{12}$$

with $S \in \mathcal{H}(\Omega)\langle\langle X\rangle\rangle,\ M_i \in \mathcal{H}(\Omega)_+\langle\langle X\rangle\rangle$

(i) *Solutions of* (TSM) *form a* $\mathbb{C}$*-vector space.*

(ii) *Solutions of* (TSM) *have their constant term (as coefficient of* 1_{X^*}*) which are constant functions (on* Ω*); there exists solutions with constant coefficient* 1_Ω *(hence invertible).*

(iii) *If two solutions coincide at one point* $z_0 \in \Omega$ *(or asymptotically), they coincide everywhere.*

What is so special with solutions like Li and $S_{Pic}^{z_0}$./2

Theorem (cont'd)

(iv) *Let be the following one-sided equations*

$$(LM_1) \quad \mathbf{d}S = M_1 S \qquad (RM_2) \quad \mathbf{d}S = SM_2. \tag{13}$$

and let S_1 (resp. S_2) be a solution of (LM_1) (resp. (LM_2)), then $S_1 S_2$ is a solution of (TSM). Conversely, every solution of (TSM) can be constructed so.

(v) *Let $S_{Pic,LM_1}^{z_0}$ (resp. $S_{Pic,RM_2}^{z_0}$) the unique solution of (LM_1) (resp. (RM_2)) s.t. $S(z_0) = 1_{\mathcal{H}(\Omega)_+\langle\langle X\rangle\rangle}$ then, the space of all solutions of (TSM) is*

$$S_{Pic,LM_1}^{z_0}.\mathbb{C}\langle\langle X\rangle\rangle.S_{Pic,RM_2}^{z_0}$$

(vi) *If $M_i,\ i = 1,2$ are primitive for $\Delta_{\text{ш}}$[a] and if S, a solution of (TSM), is group-like at one point (or asymptotically), it is group-like everywhere (over Ω).*

[a] $\Delta_{\text{ш}}$ is the canonical comultiplcation of $\mathbb{C}\langle X\rangle$ viewed as an enveloping algebra.

The categories **DiffRing**, **CDiffRing**, $\mathbf{DiffAlg_k}$, $\mathbf{CDiffAlg_k}$

1. We begin with $\mathbf{DiffAlg_k}$
   Let $\mathbf{k}$ be a ring $\mathbf{DiffAlg_k}$ is the category of pairs $(\mathcal{A}, \partial)$ where $\mathcal{A} \in \mathbf{AAU_k}$ and $\partial \in Der(\mathcal{A})$. An arrow $f : (\mathcal{A}, \partial_A) \to (\mathcal{B}, \partial_B)$ is an arrow $f \in Hom_{\mathbf{k}}(\mathcal{A}, \mathcal{B})$ such that $f\partial_A = \partial_B f$.
2. For $(\mathcal{A}, \partial_A) \in \mathbf{DiffAlg_k}$, $ker(\partial_A)$ is a $\mathbf{k}$-subalgebra of $\mathcal{A}$ called that of constants of $\mathcal{A}$.

We now describe the free objects

$$\begin{tikzcd}
\mathbf{St} & \mathbf{DiffAlg_k} \arrow[l, dotted, "F"'] \\
X \arrow[r, dashed, "f"] \arrow[dr, dashed, "j_X"'] & \mathcal{A} \\
 & \mathbf{k}\langle\{X\}\rangle \arrow[u, "\hat{f}"']
\end{tikzcd}$$

Figure: A solution of the universal problem w.r.t. the natural forgetful functor from $\mathbf{DiffAlg_k}$ to $\mathbf{St}$.

Construction of $\mathbf{k}\langle\{X\}\rangle$ and $\mathbf{k}\{X\}$

1. We describe the structure. Let X be an alphabet.
   The free object $\mathbf{k}\langle\{X\}\rangle$ is:
   1. a free algebra $\mathbf{k}\langle X \times \mathbb{N}\rangle$ where, for all $x \in X$, is noted $(x, n) = x^{[n]}$ and, for convenience, $x^{[0]} = x$. This algebra is equipped with the derivation ∂ such that $\partial(x^{[k]}) = x^{[k+1]}$
   2. Existence of ∂ as a derivation is standard (see e.g. [5], Ch I, §2.8 *Extension of derivations*).
   3. The construction is similar to what is to be found in [20], but in the noncommutative realm.
2. We now say a word of the construction in [20]

$$\begin{tikzcd}
\mathbf{St} & & \mathbf{CDiffAlg_k} \arrow[ll, dotted, "F"'] \\
X \arrow[rr, dashed, "f"] \arrow[drr, dashed, "j_X"'] & & \mathcal{A} \\
& & \mathbf{k}\{X\} \arrow[u, "\hat{f}"']
\end{tikzcd}$$

Construction of $\mathbf{k}\{X\}$

3. Construction of $\mathbf{k}\{X\}$ is very similar to that of $\mathbf{k}\langle\{X\}\rangle$ but
   1. It is devoted to the category $\mathbf{CDiffAlg_k}$ (commutative differential $\mathbf{k}$-algebras)
   2. It uses commutative polynomials i.e. the basic algebra is $\mathbf{k}[X \times \mathbb{N}]$ (and not $\mathbf{k}\langle X \times \mathbb{N}\rangle$) with the same notations ($(x, n) = x^{[n]}$ and $x^{[0]} = x$).
   3. It is the one used for Proposition 2 in Vu's talk (and, in fact, the construction can be done using $\mathbf{k}\{X\}$ with $Y_i^{[j]} = Y_{ij}$ and a suitable ideal).
   4. We recall Proposition 2.

Proposition 2

Let F be a differential field with algebraically closed field of constants C_F and $\mathcal{L}(Y) = Y^{(n)} + a_{n-1}Y^{(n-1)} + \ldots + a_1 Y' + a_0 Y = 0$ be defined over F. Then there exists a Picard-Vessiot extension L of F for $\mathcal{L}$, that is unique up to differential F-isomorphism.

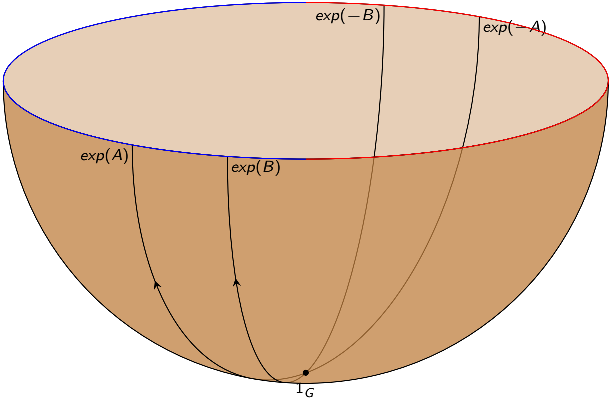
Application: Cartan theorem in Banach algebras (without transversality nor Lipschitz condition)

See `https://mathoverflow.net/questions/356531` for motivation.
Theorem Let $\mathcal{B}$ be a Banach algebra (with unit e) and G be a closed subgroup of $\mathcal{B}^{-1}$ (the group of multiplicative inverses). Let $L(G)$ be the tangent space of G and $m : I \to L(G)$ be a continuous function ($I \subset \mathbb{R}$ is an open interval containing $0_{\mathbb{R}}$), then
i) The following system

$$y'(t) = m(t)y(t) \ ; \ y(0) = e$$

admits a unique solution, say $s(t)$.
ii) The trajectory of s is entirely in G (in other words $t \mapsto s(t)$ is a path drawn on G). My questions are the following:
Q1) Is it known? (I expect so, at least of the specialists)
Q2) If yes, is there a sound reference? (not general, but about this very simple and precise property).

Magnus and Hausdorff groups



The Magnus group is the set of series with constant term 1_{X^*}, the Hausdorff (sub)-group, is the group of group-like series for $\Delta_{Ш}$. These are also Lie exponentials (here A, B are Lie series and $exp(A)exp(B) = exp(H(A, B))$).

About Magnus expansion and Poincaré-Hausdorff formula/1

Let $(\mathbb{C}\langle\{X\}\rangle, \partial)$ be the differential algebra freely generated by X (a single formal variable). We define a comultiplication Δ by asking that all $X^{[k]}$ be primitive note that Δ commutes with the derivation. Setting, in $\widehat{\mathbb{C}\langle\{X\}\rangle}$, $D = \partial(e^X)e^{-X}$, direct computation shows that D is primitive and hence a Lie series[2], which can therefore be written as a sum of (evaluations of) Dynkin trees.
On the other hand, the formula

$$D = \sum_{k\geq 1} \frac{1}{k!} \sum_{l=0}^{k-1} X^l (\partial X) X^{k-1-l} \cdot \sum_{n\geq 0} \frac{(-X)^n}{n!} \tag{14}$$

suggests that all bidegrees, in $(X, \partial X)$, are of the form $[n, 1]$ and thus, there exists an univariate series $\Phi(Y) = \sum_{n\geq 0} a_n Y^n$ such that $D = \Phi(ad_X)[\partial X]$.

---

[2]Which would be trivial,if we were in $\mathbb{C}\{X\}$ (i.e. X commutes with ∂X, as there $D = \partial(X)$, but this is not the case within $\mathbb{C}\langle\{X\}\rangle$ as shows the computation (14).

About Magnus expansion and Poincaré-Hausdorff formula/2

Using left and right multiplications by X (resp. noted g, d), we can rewrite (14) as

$$D = \Big(\sum_{k \geq 1} \frac{1}{k!} \sum_{l=0}^{k-1} g^l d^{k-1-l} [\partial X] \Big) e^{-X} \tag{15}$$

but, from the fact that g, d commute, the inner sum $\sum_{l=0}^{k-1} g^l d^{k-1-l}$ is ruled out by the the following identity (in $\mathbb{C}[Y, Z]$, but computed within $\mathbb{C}(Y, Z)$) and

$$\sum_{l=0}^{k-1} Y^l Z^{k-1-l} = \frac{Y^k - Z^k}{Y - Z} = \frac{\big((Y - Z) + Z\big)^k - Z^k}{Y - Z} = \sum_{j=1}^{k} \binom{k}{j} (Y - Z)^j Z^{k-j}$$

$$\sum_{l=0}^{k-1} Y^l Z^{k-1-l} = \frac{Y^k - Z^k}{Y - Z} = \frac{\big((Y-Z)+Z\big)^k - Z^k}{Y - Z} = \sum_{j=1}^{k} \binom{k}{j} (Y-Z)^j Z^{k-j} \tag{16}$$

Taking notice that $(g-d) = ad_X$ and pluging (16) into (14), one gets

$$D = \Big(\sum_{k\geq 1} \frac{1}{k!} \sum_{j=1}^{k} \binom{k}{j} (ad_X)^{j-1} d^{k-j}[\partial X]\Big) e^{-X} =$$

$$\frac{1}{ad_X}\Big(\sum_{k\geq 1}\sum_{j=1}^{k} \frac{1}{j!(r-j)!}(ad_X)^j d^{k-j}[\partial X]\Big) e^{-X} = \frac{e^{ad_X}-1}{ad_X}[X'] \tag{17}$$

which is Poincaré-Hausdorff formula (of course $\dfrac{e^{ad_X}-1}{ad_X}$ stands for the substitution of ad_X in the formal series corresponding to the entire function $\dfrac{e^z-1}{z}$).

Abstract BTT theorem towards localisation

Theorem (DDMS.[1] "Linz")

Let $(\mathcal{A}, d)$ be a k-commutative associative differential algebra with unit ($\ker(d) = k$ is a field) and $\mathcal{C}$ be a differential subfield of $\mathcal{A}$ (i.e. $d(\mathcal{C}) \subset \mathcal{C}$). We suppose that $S \in \mathcal{A}\langle\langle X\rangle\rangle$ is a solution of the differential equation

$$\mathbf{d}(S) = MS \ ; \ \langle S|1_{X^*}\rangle = 1_{\mathcal{A}} \tag{18}$$

where the multiplier M is a homogeneous series (a polynomial in the case of finite X) of degree 1, i.e.

$$M = \sum_{x\in X} u_x x \in \mathcal{C}\langle\langle X\rangle\rangle \ . \tag{19}$$

The following conditions are equivalent :

[1] *Independence of Hyperlogarithms over Function Fields via Algebraic Combinatorics*, **M. Deneufchâtel, GHED, V. Hoang Ngoc Minh and A. I. Solomon**, 4th International Conference on Algebraic Informatics, Linz (2011). Proceedings, Lecture Notes in Computer Science, 6742, Springer.

Abstract BTT theorem towards localisation/2

Theorem (cont'd)

1. *The family* $(\langle S|w\rangle)_{w\in X^*}$ *of coefficients of* S *is free over* $\mathcal{C}$.
2. *The family of coefficients* $(\langle S|y\rangle)_{y\in X\cup\{1_{X^*}\}}$ *is free over* $\mathcal{C}$.
3. *The family* $(u_x)_{x\in X}$ *is such that, for* $f\in\mathcal{C}$ *and* $\alpha_x\in k$

$$d(f)=\sum_{x\in X}\alpha_x u_x \Longrightarrow (\forall x\in X)(\alpha_x=0)\ . \tag{20}$$

4. *The family* $(u_x)_{x\in X}$ *is free over* k *and*

$$d(\mathcal{C})\cap span_k\Big((u_x)_{x\in X}\Big)=\{0\}\ . \tag{21}$$

Need for localization

In practical cases, we only have a differential subalgebra of $\mathcal{C}_0 \subset \mathcal{H}(\Omega)$ (as image, through Li, of a shuffle subalgebra of $Dom(\mathrm{Li})$).

- $\mathbb{C}[z]$
- $\mathbb{C}[z, z^{-1}, (1-z)^{-1}]$
- $\mathbb{C}[z^{\alpha}(1-z)^{-\beta}]_{\alpha,\beta\in\mathbb{C}} = \mathcal{C}_{\mathbb{C}}$

Realizing the fraction field $Fr(\mathcal{C}_0)$ as (differential) field of germs makes the computation difficult to handle. It is easier to check the freeness of the " basic triangle" directly with the algebra. For instance, for the polylogarithms, we just have to show that, given $P_i \in \mathcal{C}_{\mathbb{C}}$,

$$P_1(z) + P_2(z)\log(z) + P_3(z)(\log(\frac{1}{1-z})) = 0_{\Omega} \Longrightarrow P_i \equiv 0 \qquad (22)$$

which can be done using deck transformations (see below).

Localization

Theorem (Thm1 in "Linz", Localized form)

Let $(\mathcal{A}, d)$ be a commutative associative differential ring ($\ker(d) = k$ being a field) and $\mathcal{C}$ be a differential subring (i.e. $d(\mathcal{C}) \subset \mathcal{C}$) of $\mathcal{A}$ which is an integral domain containing the field of constants.
We suppose that, for all $x \in X$, $u_x \in \mathcal{C}$ and that $S \in \mathcal{A}\langle\langle X \rangle\rangle$ is a solution of the differential equation (18) and that $(u_x)_{x \in X} \in \mathcal{C}^X$.
The following conditions are equivalent :

- i) *The family $(\langle S|w \rangle)_{w \in X^*}$ of coefficients of S is free over $\mathcal{C}$.*
- ii) *The family of coefficients $(\langle S|y \rangle)_{y \in X \cup \{1_{X^*}\}}$ is free over $\mathcal{C}$.*
- iii') *For all $f_1, f_2 \in \mathcal{C}$, $f_2 \neq 0$ and $\alpha \in k^{(X)}$, we have the property*

$$W(f_1, f_2) = f_2^2 \Big(\sum_{x \in X} \alpha_x u_x\Big) \Longrightarrow (\forall x \in X)(\alpha_x = 0) \ . \tag{23}$$

where $W(f_1, f_2)$, the wronskian, stands for $d(f_1) f_2 - f_1 d(f_2)$.

Discussion

In fact, in the localized form and with $\mathcal{C}$ not a differential field, (iii) is strictly weaker than (iii'), as shows the following family of counterexamples

1. $\Omega = \mathbb{C} \setminus (]-\infty, 0])$
2. $X = \{x_0\}$, $u_0 = z^\beta$, $\beta \notin \mathbb{Q}$
3. $\mathcal{C}_0 = \mathbb{C}\{\{z^\beta\}\} = \mathbb{C}.1_\Omega \oplus span_\mathbb{C}\{z^{(k+1)\beta - l}\}_{k,l\geq 0}$
4. $S = 1_\Omega + (\sum_{n\geq 1} \frac{z^{n(\beta+1)}}{(\beta+1)^n n!})$

Let us show that, for these data (iii) holds but not (i).
Firstly, we show that $\mathcal{C}_0 = \mathbb{C}\{\{z^\beta\}\}$ corresponds to the given direct sum.
We remark that the family $(z^\alpha)_{\alpha\in\mathbb{C}}$ is $\mathbb{C}$-linearly free (within $\mathcal{H}(\Omega)$), which is a consequence of the fact that they are eigenfunctions, for different eigenvalues, of the Euler operator $z\frac{d}{dz}$.

Then

$$\mathbb{C}\{\{z^\beta\}\} = \mathbb{C}1_\Omega \oplus span_{\mathbb{C}}\{z^{(k+1)\beta-l}\}_{k,l\geq 0} = span_{\mathbb{C}}\{z^{(k')\beta-l}\}_{k',l\geq 0}$$

comes from the fact that the RHS is a subset of the LHS as, for all, $k, l \geq 0$, $z^{(k+1)\beta-l} \in \mathbb{C}\{\{z^\beta\}\}$. Finally $1_\Omega \in \mathbb{C}\{\{z^\beta\}\}$ by definition ($\mathbb{C}\{\{X\}\}$ is a $\mathbb{C}$-AAU).

<u>(iii) is fulfilled</u>. Here

$u_0(z) = z^\beta$ is such that, for any $f \in \mathcal{C}_0$ and c_0 in $\mathbb{C}$, we have

$$c_0 u_0 = \partial_z(f) \Longrightarrow (c_0 = 0) \tag{24}$$

<u>But (i) is not</u> Because we have the following relation

$$(\beta+1)z^{\beta-1}\langle S|x_0\rangle - z^{2\beta}.1_\Omega = 0$$

Sketch of the proof

After some technicalities, we show that (18) can be transported in $\mathcal{A}[(\mathcal{C}^\times)^{-1}]$ by means of the following commutative diagram and back.

$$
\begin{tikzcd}
\mathcal{C} \arrow[rr, hook, "\varphi_{\mathcal{C}}"] \arrow[dr, hook, "j"'] \arrow[dd, "d"'] & & Fr(\mathcal{C}) \arrow[dr, hook, "j_{frac}"] \arrow[dd, "d_{frac}"] & \\
& \mathcal{A} \arrow[rr, "\varphi_{\mathcal{A}}"] \arrow[dd, "d"] & & \mathcal{A}[(\mathcal{C}^\times)^{-1}] \arrow[dd, "d_{frac}"] \\
\mathcal{C} \arrow[rr, hook, "\varphi_{\mathcal{C}}"] \arrow[dr, hook, "j"'] & & Fr(\mathcal{C}) \arrow[dr, hook, "j_{frac}"'] & \\
& \mathcal{A} \arrow[rr, "\varphi_{\mathcal{A}}"] & & \mathcal{A}[(\mathcal{C}^\times)^{-1}]
\end{tikzcd}
\tag{25}
$$

Proof that $[1_\Omega, \log(z), \log(\frac{1}{1-z})]$ is $\mathcal{C}_\mathbb{C}$-free.

Let us suppose $P_i, i = 1\ldots 3$ such that

$$P_1(z) + P_2(z)\log(z) + P_3(z)(\log(\frac{1}{1-z})) = 0_\Omega$$

We first prove that $P_2 = \sum_{i\in F} c_i z^{\alpha_i}(1-z)^{\beta_i}$ is zero using the deck transformation D_0 of index one around zero.
One has $D_0^n(\sum_{i\in F} c_i z^{\alpha_i}(1-z)^{\beta_i}) = \sum_{i\in F} c_i z^{\alpha_i}(1-z)^{\beta_i} e^{2i\pi.n\alpha_i}$, the same calculation holds for all P_i which proves that all $D_0^n(P_i)$ are bounded. But one has $D_0^n(\log(z)) = \log(z) + 2i\pi.n$ and then

$$D_0^n(P_1(z) + P_2(z)\log(z) + P_3(z)(\log(\frac{1}{1-z}))) =$$

$$D_0^n(P_1(z)) + D_0^n(P_2(z))(\log(z) + 2i\pi.n) + D_0^n(P_3(z))\log(\frac{1}{1-z}) = 0$$

It suffices to build a sequence of integers $n_j \to +\infty$ such that $\lim_{j\to\infty} D_0^{n_j}(P_2(z)) = P_2(z)$ which is a consequence of the following lemma.

Lemma

Let us consider a homomorphism $\varphi : \mathbb{N} \to G$ where G is a compact (Hausdorff) group, then it exists $u_j \to +\infty$ such that

$$\lim_{j\to\infty} \varphi(u_j) = e$$

Proof.

First of all, due to the compactness of G, the sequence $\varphi(n)$ admits a subsequence $\varphi(n_k)$ convergent to some $\ell \in G$. Now one can refine the sequence as n_{k_j} such that

$$0 < n_{k_1} - n_{k_0} < \ldots < n_{k_{j+1}} - n_{k_j} < n_{k_{j+2}} - n_{k_{j+1}} < \ldots$$

With $u_j = n_{k_{j+1}} - n_{k_j}$ one has $\lim_{j\to\infty} \varphi(u_j) = e$.
End of the proof One applies the lemma to the morphism

$$n \mapsto (e^{2i\pi . n\alpha_i})_{i\in F} \in \mathbb{U}^F$$

Conclusion

- For Series with variable coefficients, we have a theory of Noncommutative Evolution Equation sufficiently powerful to cover iterated integrals and multiplicative renormalisation
- Use of combinatorics on words gives a necessary and sufficient condition on the "inputs" to have linear independance of the solutions over higher function fields.
- Picard (Chen) solutions admit enlarged indexing w.r.t. compact convergence on Ω (polylogarithmic case) but Drinfeld's G_0 has a domain which includes only some rational series.
- Localization is possible (under certain conditions).
- Local BTT theorem allows to explore linear and algebraic independences w.r.t. subalgebras of $Dom(\mathrm{Li})$.

Thank you for your attention.

Links

1. Categorical framework(s)

   `https://ncatlab.org/nlab/show/category`
   `https://en.wikipedia.org/wiki/Category_(mathematics)`

2. Universal problems

   `https://ncatlab.org/nlab/show/universal+construction`
   `https://en.wikipedia.org/wiki/Universal_property`

3. Paolo Perrone, *Notes on Category Theory with examples from basic mathematics*, 181p (2020)
   arXiv:1912.10642 [math.CT]

   `https://en.wikipedia.org/wiki/Abstract_nonsense`

4. Heteromorphisms

   - `https://ncatlab.org/nlab/show/heteromorphism`
   - D. Ellerman, *MacLane, Bourbaki, and Adjoints: A Heteromorphic Retrospective*, David EllermanPhilosophy Department, University of California at Riverside.

Links/2

5. https://en.wikipedia.org/wiki/Category_of_modules
6. https://ncatlab.org/nlab/show/Grothendieck+group
7. Traces and hilbertian operators

   https://hal.archives-ouvertes.fr/hal-01015295/document
8. State on a star-algebra

   https://ncatlab.org/nlab/show/state+on+a+star-algebra
9. Hilbert module

   https://ncatlab.org/nlab/show/Hilbert+module

[1] F. Bergeron, G. Labelle, P. Leroux, *Combinatorial Species and Tree-like Structures*, Cambridge University Press, 1998.

[2] D. Bump, *Automorphic Forms and Representations*, Cambridge University Press (1998).

[3] N. Bourbaki, *Algèbre, Chapitre 8*, Springer, 2012.

[4] N. Bourbaki.– *Integration II Ch 7-9* (Translated by Sterling K. Berberian), Springer-Verlag (2004).

[5] N. Bourbaki.– *Lie Groups and Lie Algebras, Ch 1-3*, Addison-Wesley, ISBN 0-201-00643-X

[6] P. Cartier, *Jacobiennes généralisées, monodromie unipotente et intégrales itérées*, Séminaire Bourbaki, Volume 30 (1987-1988) , Talk no. 687 , p. 31-52

[7] M. Deneufchâtel, GD, V. Hoang Ngoc Minh and A. I. Solomon, *Independence of Hyperlogarithms over Function Fields via Algebraic Combinatorics*, 4th International Conference on Algebraic Informatics, Linz (2011). Proceedings, Lecture Notes in Computer Science, 6742, Springer.

[8] Jean Dieudonné, *Foundations of Modern Analysis*, Volume 2, Academic Press; 2nd rev edition (January 1, 1969)

[9] GD, Quoc Huan Ngô and Vincel Hoang Ngoc Minh, *Kleene stars of the plane, polylogarithms and symmetries*, (pp 52-72) TCS 800, 2019, pp 52-72.

[10] GD, Darij Grinberg, Vincel Hoang Ngoc Minh, *Three variations on the linear independence of grouplikes in a coalgebra*, ArXiv:2009.10970 [math.QA] (Wed, 23 Sep 2020)

[11] Gérard H. E. Duchamp, Christophe Tollu, Karol A. Penson and Gleb A. Koshevoy, *Deformations of Algebras: Twisting and Perturbations* , Séminaire Lotharingien de Combinatoire, B62e (2010)

[12] GD, Nguyen Hoang-Nghia, Thomas Krajewski, Adrian Tanasa, *Recipe theorem for the Tutte polynomial for matroids, renormalization group-like approach*, Advances in Applied Mathematics 51 (2013) 345–358.

[13] K.T. Chen, R.H. Fox, R.C. Lyndon, *Free differential calculus, IV. The quotient groups of the lower central series*, Ann. of Math. , 68 (1958) pp. 81–95

[14] M.E. Hoffman, *Quasi-shuffle algebras and applications*, arXiv preprint arXiv:1805.12464, 2018

[15] M. Lothaire, *Combinatorics on Words*, 2nd Edition, Cambridge Mathematical Library (1997).

[16] Szymon Charzynski and Marek Kus, *Wei-Norman equations for a unitary evolution*, Classical Analysis and ODEs, J. Phys. A: Math. Theor. 46 265208

[17] Rimhac Ree, *Lie Elements and an Algebra Associated With Shuffles*, Annals of Mathematics Second Series, Vol. 68, No. 2 (Sep., 1958)

[18] J. Voight, *Quaternion algebras*, `https://math.dartmouth.edu/~jvoight/quat-book.pdf`

[19] Adjuncts in nlab. `https://ncatlab.org/nlab/show/adjunct`

[20] M. van der Put, M. F. Singer.– *Galois Theory of Linear Differential Equations*, Springer (2003)