Containing many patterns

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A *permutation* π is an ordering of the integers $\{1, 2, \ldots, n\}$.

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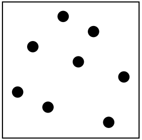

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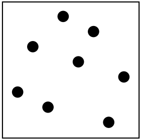
A permutation π *contains* another permutation σ as a *pattern* if π contains a subsequence of entries in the same relative order as those of σ.



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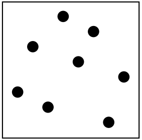

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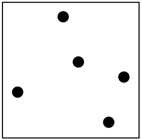

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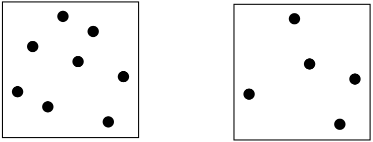

$$\pi = \underline{3}62\underline{8}\underline{5}7\underline{1}\underline{4} \quad \geqslant \quad 2\quad 54\quad 13 = \sigma$$

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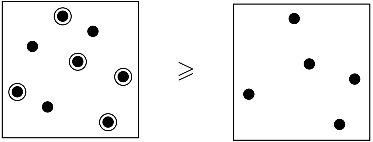

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Definition.
A permutation is n-*universal* if it contains all patterns of length n.

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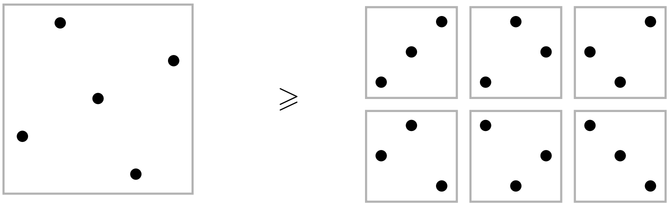

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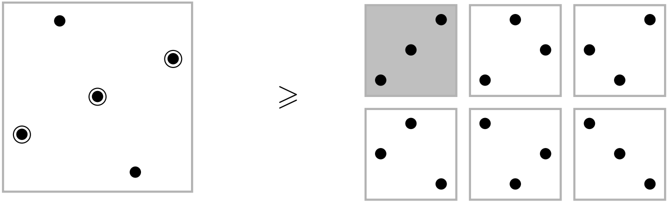

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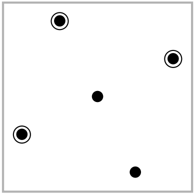

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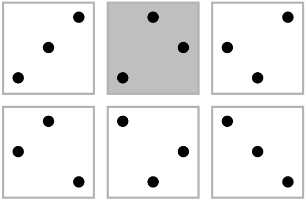

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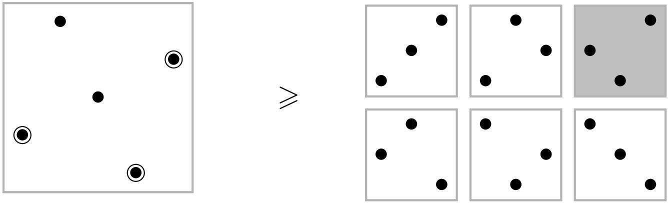

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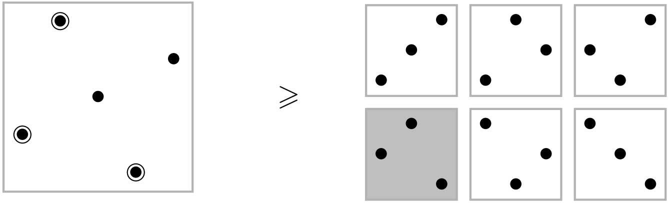

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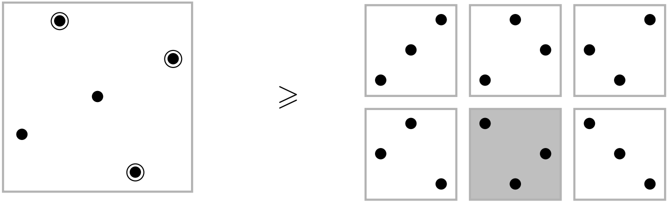

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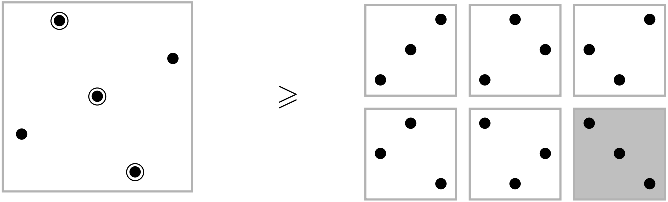

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$$\frac{n^2}{e^2} \leqslant L_n \leqslant n^2.$$

First results

Theorem (Eriksson, Eriksson, Linusson, Wästlund; 2007).

$$L_n \leqslant \frac{2}{3}n^2 + O\left(n^{3/2}(\log n)^{1/2}\right).$$

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Theorem (Miller; 2009).

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Infinite zigzag word

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$$Z = (1\ 3\ 5\ 7\cdots)(\cdots 8\ 6\ 4\ 2)(1\ 3\ 5\ 7\cdots)(\cdots 8\ 6\ 4\ 2)(1\ 3\ 5\ 7\cdots)(\cdots 8\ 6\ 4\ 2)\cdots$$

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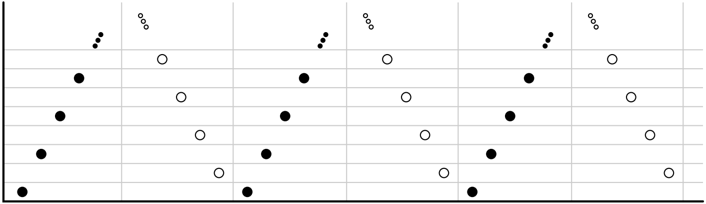

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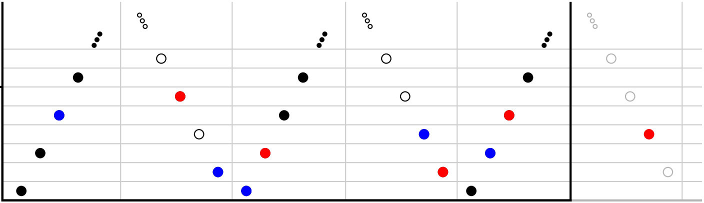

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Theorem (Miller; 2009).

For all $\sigma \in S_n$, either σ or σ^{+1} embed into the first n runs of Z.

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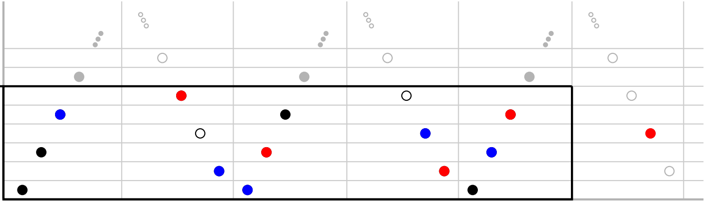

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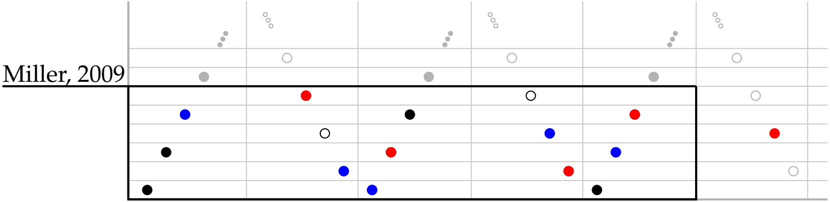

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Let z_n be the restriction of Z to include the first n runs and $n+1$ values,

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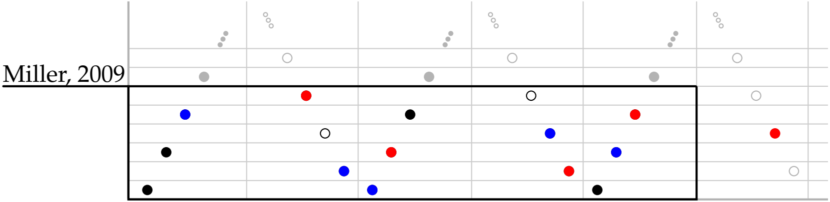

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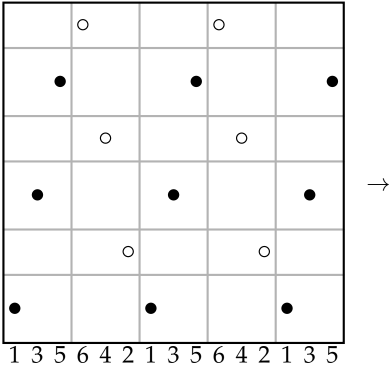
For all $\sigma \in S_n$, either σ or σ^{+1} embed into the first n runs of Z.

Corollary.

Let z_n be the restriction of Z to include the first n runs and $n+1$ values, and let π be a permutation formed from z_n by breaking ties between values arbitrarily. Then π is n-universal.

Miller's construction



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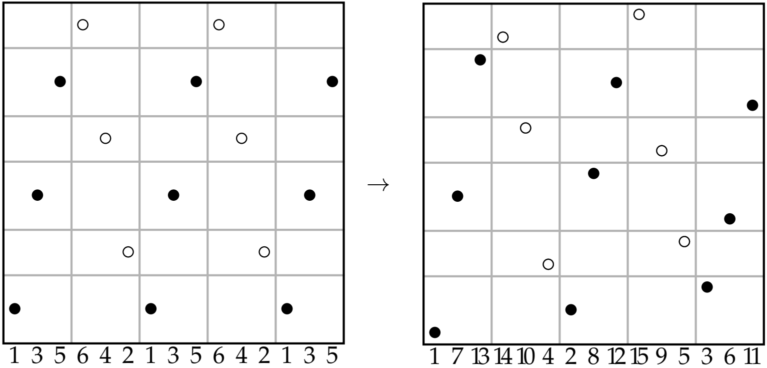

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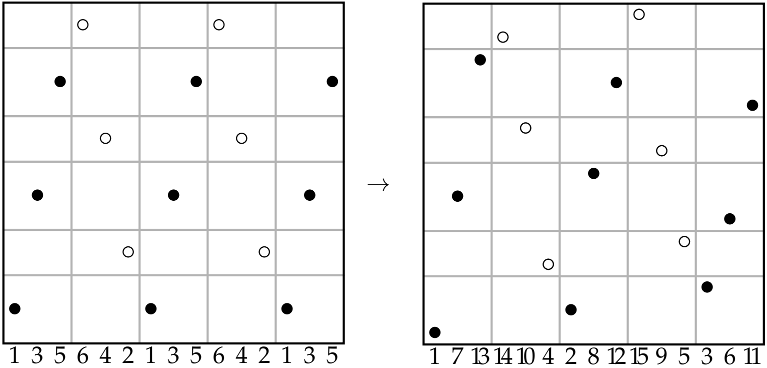
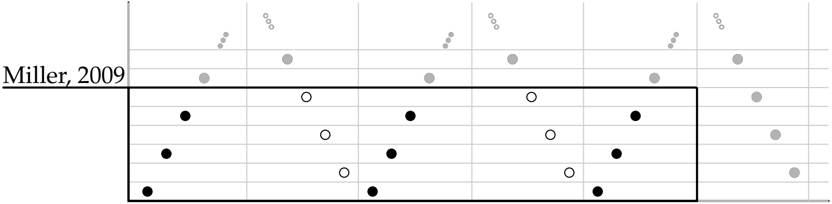


	n	1	2	3	4	5	6
(actual)	L_n	1	3	5	9	13	17
(Miller)	$(n^2+n)/2$	1	3	6	10	15	21

Infinite zigzag word, again



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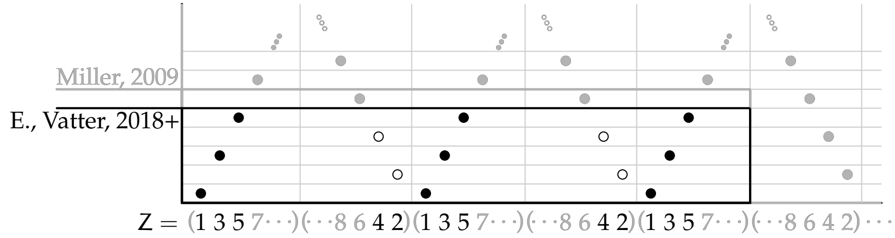

Theorem (E., Vatter; 2018+).

Let z_n^* be the restriction of Z to include the first n runs and n values,

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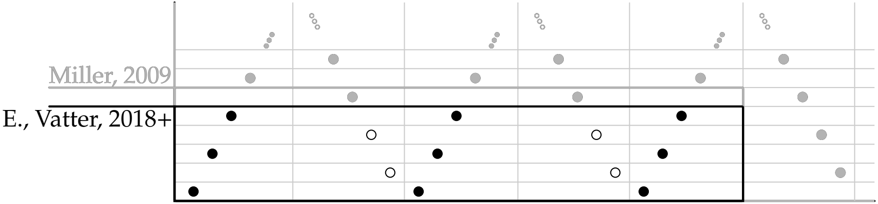

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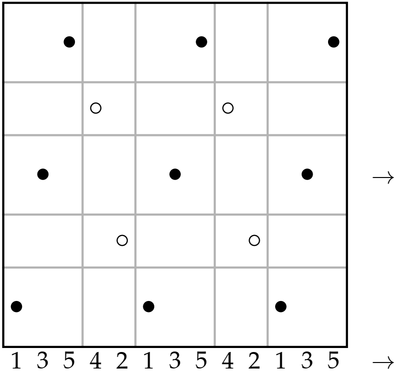
Theorem (E., Vatter; 2018+).

Let z_n^* be the restriction of Z to include the first n runs and n values, and let π be a permutation formed from z_n^* by breaking ties between values in a decreasing fashion. Then π is almost n-universal.

New construction



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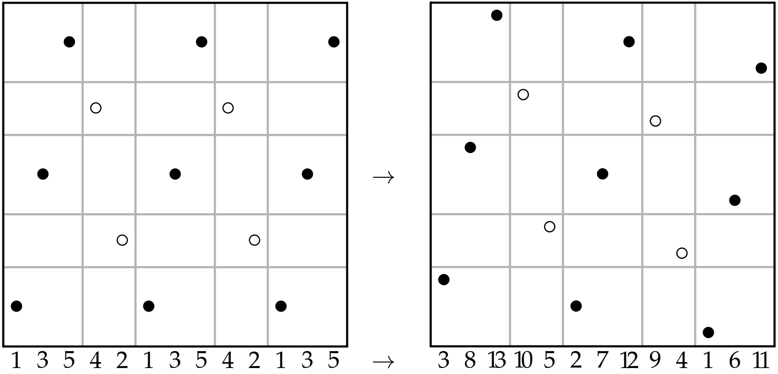
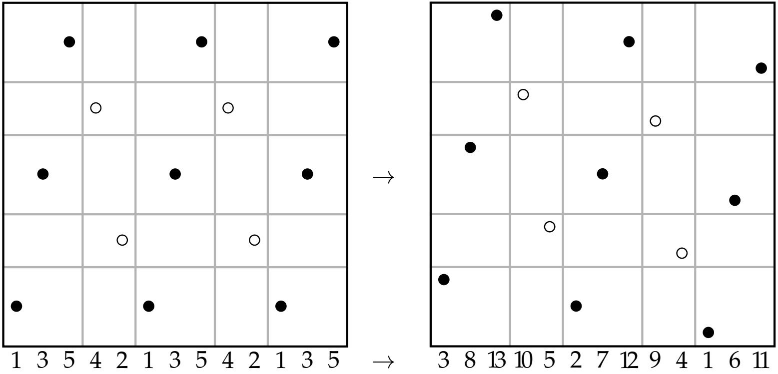


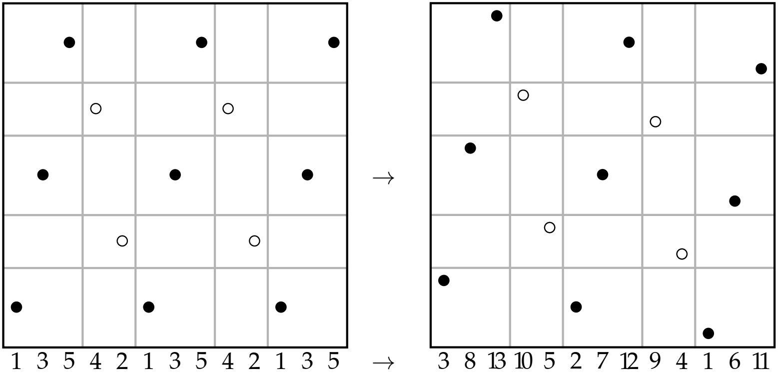
	n	1	2	3	4	5
(actual)	L_n	1	3	5	9	13
(Miller)	$(n^2+n)/2$	1	3	6	10	15

New construction



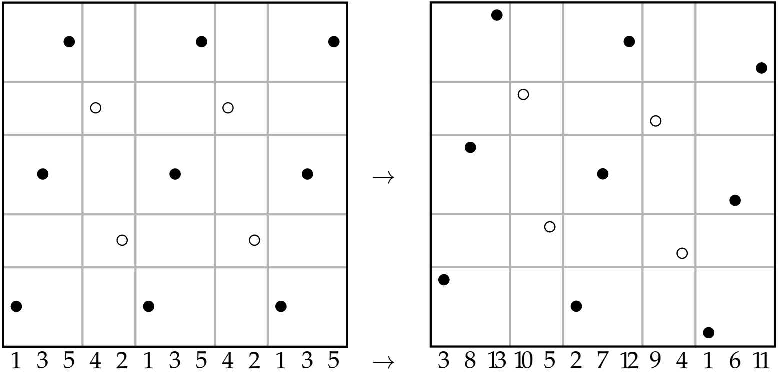
		n	1	2	3	4	5
(actual)	L_n		1	3	5	9	13
(E., Vatter)			1	3	5	9	13
(Miller)	$(n^2+n)/2$		1	3	6	10	15

New construction



	n	1	2	3	4	5	6
(actual)	L_n	1	3	5	9	13	17
(E., Vatter)	$\lceil (n^2+1)/2 \rceil$	1	3	5	9	13	19
(Miller)	$(n^2+n)/2$	1	3	6	10	15	21

New construction



	n	1	2	3	4	5	6	7
(actual)	L_n	1	3	5	9	13	17	$\leqslant 24$
(E., Vatter)	$\lceil (n^2+1)/2 \rceil$	1	3	5	9	13	19	25
(Miller)	$(n^2+n)/2$	1	3	6	10	15	21	28

Pattern counting

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Let

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and

$$C(n, m) = \max_{\pi \in S_n} c(\pi, m).$$

k \ m	1	2	3	4	5	6	7	8	9
9	1	2	6	24	120	670	3107		
8	1	2	6	24	120	618	2412	7064	
7	1	2	6	24	119	526	1724	4214	9037
6	1	2	6	24	112	408	1094	2310	4302
5	1	2	6	24	94	273	614	1127	1856
4	1	2	6	23	71	156	291	477	699
3	1	2	6	19	41	76	114	162	220
2	1	2	6	12	21	28	36	45	55
1	1	2	4	5	6	7	8	9	10

$$C(m+k, m)$$

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$C(m+k, m)$:

- Columns increasing.

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- Entries $\leqslant \binom{m+k}{m}$.

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- Columns are eventually $m!$.

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3	1	2	6	19	41	76	114	162	220	$\longrightarrow \binom{m+3}{3}$
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- Entries $\leqslant \binom{m+k}{m}$.
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- Rows are eventually $\binom{m+k}{m}$.

k-prolific permutations

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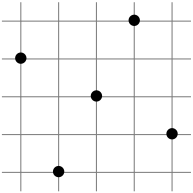

Theorem (Bevan, Homberger, Tenner; 2018)

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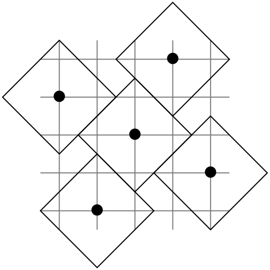

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Permutations of breadth $k+2$ exist for exactly those lengths $n \geqslant \lceil k^2/2 + 2k + 1 \rceil$.

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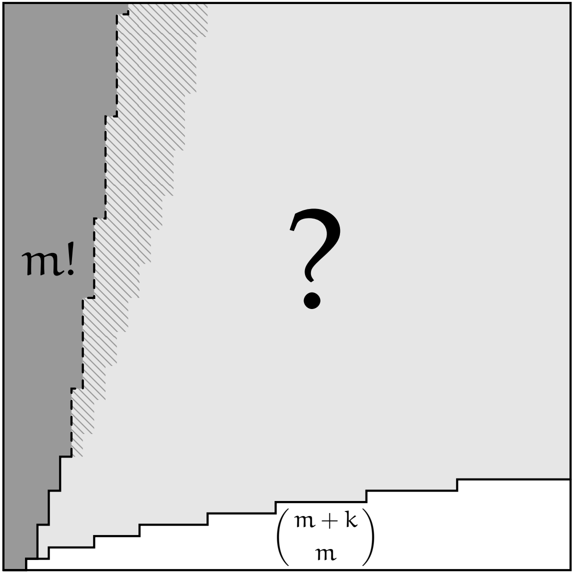
Corollary

For all m, k, $C(m+k, m) = \binom{m+k}{m}$ iff $k \geqslant \sqrt{2m-1} - 1$.

State of Affairs



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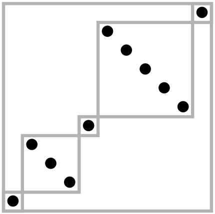

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Corollary (Albert, E., Pantone, and Vatter; 2018).

Among all shortest permutations which are n-universal for $\mathcal{L}$, there is one which itself is layered.

Layerization

Proof of Proposition.

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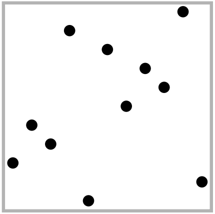

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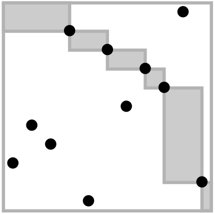

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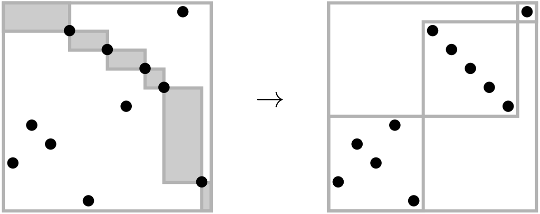

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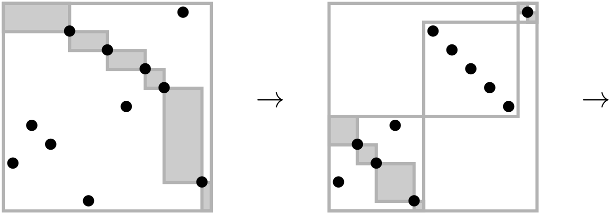

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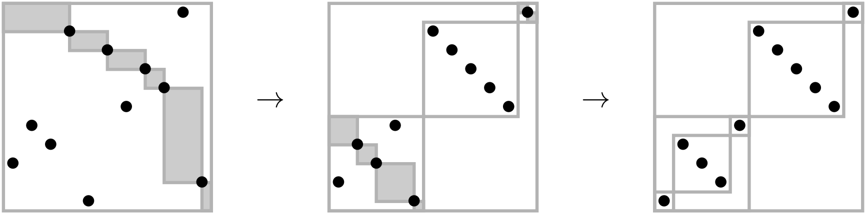

□

Theorem (Albert, E., Pantone, and Vatter; 2018).

For all n, the length of the shortest permutation that is n-universal for layered permutations is given by the sequence defined by

$$a(n) = n + \min\{a(k) + a(n-k-1) \;:\; 0 \leqslant k \leqslant n-1\}$$

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Note: As a consequence, $a(n) \sim n \log_2(n)$.

Containing many patterns in a class

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Definition.
Let

$$c_{\mathcal{L}}(\pi, m) = \#\text{ of } \textit{layered} \text{ patterns of length } m \text{ contained in } \pi$$

and

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Corollary.

$$C_{\mathcal{L}}(n, m) = C_{\mathcal{L}}^{\in}(n, m).$$

$$\mathcal{L} = \text{Av}(231, 312)$$

k \ m	1	2	3	4	5	6	7	8	9
9	1	2	4	8	16	32	63	115	201
8	1	2	4	8	16	32	61	105	181
7	1	2	4	8	16	31	56	95	159
6	1	2	4	8	16	29	51	83	128
5	1	2	4	8	15	26	43	65	99
4	1	2	4	8	13	21	33	49	69
3	1	2	4	7	11	16	22	32	41
2	1	2	4	5	8	10	13	18	20
1	1	2	3	3	4	5	5	6	7

$C^{\in}_{\mathcal{L}}(m+k, m)$

=

k \ m	1	2	3	4	5	6	7	8	9
9	1	2	4	8	16	32	63	115	201
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4	1	2	4	8	13	21	33	49	69
3	1	2	4	7	11	16	22	32	41
2	1	2	4	5	8	10	13	18	20
1	1	2	3	3	4	5	5	6	7

$C_{\mathcal{L}}(m+k, m)$

$$\mathcal{A} = \text{Av}(321)$$

$C^{\in}_{\mathcal{A}}(m+k, m)$

k \ m	1	2	3	4	5	6	7	8	9
9	1	2	5	14	42	132	429		
8	1	2	5	14	42	132	419		
7	1	2	5	14	42	132	397	1030	
6	1	2	5	14	42	128	338	790	1624
5	1	2	5	14	42	116	261	522	949
4	1	2	5	14	40	91	169	289	457
3	1	2	5	14	30	53	86	126	176
2	1	2	5	11	17	24	32	41	51
1	1	2	4	5	6	7	8	9	10

$\stackrel{?}{=}$

$C_{\mathcal{A}}(m+k, m)$

k \ m	1	2	3	4	5	6	7	8	9
9	1	2	5	14	42	132	429		
8	1	2	5	14	42	132			
7	1	2	5	14	42	132			
6	1	2	5	14	42	128	338		
5	1	2	5	14	42	116	261	522	
4	1	2	5	14	40	91	169	289	457
3	1	2	5	14	30	53	86	126	176
2	1	2	5	11	17	24	32	41	51
1	1	2	4	5	6	7	8	9	10

$$C^{\in}_{\mathcal{A}}(m+k, m) \stackrel{?}{=} C_{\mathcal{A}}(m+k, m)$$

$$\mathcal{S} = \text{Av}(2413, 3142)$$

k \ m	1	2	3	4	5	6	7	8	9
9	1	2	6	22					
8	1	2	6	22	80				
7	1	2	6	22	75	200			
6	1	2	6	22	64	164	375		
5	1	2	6	20	54	123	257	479	
4	1	2	6	18	41	83	149	249	390
3	1	2	6	14	27	46	72	106	149
2	1	2	5	9	14	20	27	35	44
1	1	2	3	4	5	6	7	8	9

$C^{\in}_{\mathcal{S}}(m+k, m)$

k \ m	1	2	3	4	5	6	7	8	9
9	1	2	6	22	90				
8	1	2	6	22	90				
7	1	2	6	22	89				
6	1	2	6	22	82	188			
5	1	2	6	22	68	130	257		
4	1	2	6	21	48	83	149	249	
3	1	2	6	17	28	46	72	106	149
2	1	2	6	10	14	20	27	35	44
1	1	2	4	4	5	6	7	8	9

$C_{\mathcal{S}}(m+k, m)$

$$\mathcal{S} = \mathrm{Av}(2413, 3142)$$

k \ m	1	2	3	4	5	6	7	8	9
9	1	2	6	22					
8	1	2	6	22	80				
7	1	2	6	22	75	200			
6	1	2	6	22	64	164	375		
5	1	2	6	20	54	123	257	479	
4	1	2	6	18	41	83	149	249	390
3	1	2	6	14	27	46	72	106	149
2	1	2	5	9	14	20	27	35	44
1	1	2	3	4	5	6	7	8	9

$C^{\in}_{\mathcal{S}}(m+k, m)$

$\neq$

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9	1	2	6	22	90				
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$C_{\mathcal{S}}(m+k, m)$

$$\mathcal{A} = \mathrm{Av}(231)$$

k \ m	1	2	3	4	5	6	7	8	9
9	1	2	5	14	42	132			
8	1	2	5	14	42	126	323		
7	1	2	5	14	42	116	273	609	
6	1	2	5	14	40	101	216	435	823
5	1	2	5	14	38	81	164	295	504
4	1	2	5	14	31	60	106	175	282
3	1	2	5	12	22	38	58	88	122
2	1	2	5	8	13	18	25	32	41
1	1	2	3	4	5	6	7	8	9

$C^{\in}_{\mathcal{A}}(m+k, m)$

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$C_{\mathcal{A}}(m+k, m)$

$$\mathcal{A} = \mathrm{Av}(231)$$

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$C_{\mathcal{A}}(m+k, m)$

Merci beaucoup!