Lazard's elimination in presented Lie algebras

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Knizhnik-Zamolodchikov equation

- Assume that k is a commutative ring with unit.
- For $n \geq 2$, we denoted by $\mathcal{T}_n = \{t_{i,j}\}_{1 \leq i < j \leq n}$ the set of noncommutative variables.
- The *Knizhnik-Zamolodchikov equation* (see for instance Drinfeld [1], Minh [5])

$$(\mathrm{KZ}_n) \quad \mathrm{d}\,\mathrm{F}(z) = \Omega_n(z)\,\mathrm{F}(z) \tag{1}$$

defined over the complex configuration space

$$\mathbb{C}_*^n = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n | z_i \neq z_j \text{ for } i \neq j\},$$

where the system (so called the *KZ connection*)

$$\Omega_n(z) = \sum_{1 \leq i < j \leq n} \frac{t_{i,j}}{2\mathrm{i}\pi} d\log(z_i - z_j), \tag{2}$$

where the logarithmic function is relative to some section of $\widetilde{\mathbb{C}_*^n}$, for example $\mathbb{C} \backslash]-\infty, 0]$.

Drinfeld-Kohno Lie algebra

- As a consequence of Arnold's theorem, the system (2) is completely integrable i.e. $d\Omega_n - \Omega_n \wedge \Omega_n = 0$, it is equivalent to the fact that $\mathcal{T}_n = \{t_{i,j}\}_{1\leq i<j\leq n}$ satisfy *the infinitesimal pure braid relations*

$$\mathrm{R}[\mathrm{n}] = \begin{cases} \mathrm{R}_1[\mathrm{n}] & [t_{i,j}, t_{i,k} + t_{j,k}] \quad \text{for } 1 \leq i < j < k \leq n, \\ \mathrm{R}_2[\mathrm{n}] & [t_{i,j} + t_{i,k}, t_{j,k}] \quad \text{for } 1 \leq i < j < k \leq n, \\ \mathrm{R}_3[\mathrm{n}] & [t_{i,j}, t_{k,l}] \quad \text{for } \begin{matrix} 1 \leq i < j \leq n, \\ 1 \leq k < l \leq n, \end{matrix} \text{ and } |\{i,j,k,l\}| = \end{cases}$$

- The *Drinfeld-Kohno Lie algebra* $\mathrm{DK}_{\mathrm{k},n}$ is presented as

$$\mathcal{L}_{\mathrm{k}}(\mathcal{T}_n) \Big/ \mathcal{J}_{\mathrm{R}[\mathrm{n}]} \tag{4}$$

where $\mathcal{J}_{\mathrm{R}[\mathrm{n}]}$ is the Lie ideal of $\mathcal{L}_{\mathrm{k}}(\mathcal{T}_n)$ generated by R[n] (3).

- By using the Knizhnik-Zamolodchikov equations, Kohno proved in [2] that $\mathrm{DK}_{\mathrm{k},n}$ can be identified with $\mathfrak{gr}_{\mathrm{k}}(\mathcal{PB}_n)$ the graded Lie algebra of the pure braid group $\mathcal{PB}_n$. Thus, Drinfeld-Kohno Lie algebra $\mathrm{DK}_{\mathrm{k},n}$ is also called the *Lie algebra of infinitesimal braids*.
- By some steps, we can construct a commutative diagram of k-modules with split short exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & gr_{\mathbf{k}}(\mathbb{F}_n) & \longrightarrow & gr_{\mathbf{k}}(\mathcal{PB}_{n+1}) & \longrightarrow & gr_{\mathbf{k}}(\mathcal{PB}_n) & \longrightarrow & 0 \\ & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\ 0 & \longrightarrow & \mathcal{L}_{\mathbf{k}}(x_1,\ldots,x_n) & \longrightarrow & \mathrm{DK}_{\mathbf{k},n+1} & \longrightarrow & \mathrm{DK}_{\mathbf{k},n} & \longrightarrow & 0 \end{array}$$

In particular, we obtain an isomorphism of k-modules

$$\mathrm{DK}_{\mathrm{k},n+1} \simeq \mathcal{L}_{\mathrm{k}}(x_1,\ldots,x_n) \bigoplus \mathrm{DK}_{\mathrm{k},n}. \tag{5}$$

- A natural question is how to construct a Lie isomorphism from the Drinfeld-Kohno Lie algebra to a semidirect product of Lie algebras

$$\mathrm{DK}_{\mathrm{k},n+1} \xrightarrow{\simeq} \mathcal{L}_{\mathrm{k}}(x_1, \ldots, x_n) \rtimes \mathrm{DK}_{\mathrm{k},n}.$$

- We call the phenomenon by *"the decomposition of Drinfeld-Kohno Lie algebra"*.
- In this talk, we will give a proof for the existence of the decomposition of Drinfeld-Kohno Lie algebra as a corollary of our main theorem and Proposition 2.

Quotients of Lazard's eliminations

Let us recall briefly Lazard's elimination theorem in our setting.

Lazard elimination theorem

Let $X = B \sqcup Z$ be a set partitioned in two blocks. We have an isomorphism of split short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{L}_{\mathbf{k}}(B^*Z) & \xrightarrow{j_{B|Z}} & \mathcal{L}_{\mathbf{k}}(X) & \xrightarrow{p_{B|Z}} & \mathcal{L}_{\mathbf{k}}(B) & \longrightarrow & 0 \\
 & & \downarrow \overline{rn} & & \downarrow Id & & \downarrow \overline{j_B} & & \\
0 & \longrightarrow & \mathcal{L}_{\mathbf{k}}(X)_{BZ} & \xrightarrow{j} & \mathcal{L}_{\mathbf{k}}(X) & \xrightarrow{p} & \mathcal{L}_{\mathbf{k}}(X)_B & \longrightarrow & 0
\end{array}$$

with

- $u = b_1 \ldots b_k \in B^*$ and $z \in Z$ for

$$rn(uz) = \left(\mathrm{ad}_{b_1}^{\mathcal{L}_k(X)} \circ \ldots \circ \mathrm{ad}_{b_k}^{\mathcal{L}_k(X)}\right)(z) =: \mathrm{ad}_{(u)}^{\mathcal{L}_k(X)}(z)$$

Lazard elimination theorem

- bracketing and $\overline{rn}$ is the restriction of rn to its image as in the diagram.
- if $j_B: \ \mathcal{L}_k(B) \to \mathcal{L}_k(X)$ is the subalphabet embedding, (so that the restriction to its image is the isomorphism $\overline{j_B}$) then $\overline{j_B} \circ p_{B|Z}$ is the projector on

$$\mathcal{L}_k(X)_B = \bigoplus_{\substack{\alpha \in \mathbb{N}^{(X)} \\ |\alpha|_Z = 0}} \mathcal{L}_k(X)_\alpha$$

The kernel of $p_{B|Z}$ is

$$\mathcal{L}_k(X)_{BZ} = \bigoplus_{\substack{\alpha \in \mathbb{N}^{(X)} \\ |\alpha|_Z > 0}} \mathcal{L}_k(X)_\alpha$$

- The above diagram is a split SES, its section is given by j_B.

Main results: Quotients of Lazard's eliminations

- Observation and ideas: Put $\mathcal{T}_{n+1} = \mathcal{T}_n \sqcup T_{n+1}$ a set partitioned in two blocks, then the infinitesimal pure braid relator $\mathrm{R}[\mathrm{n}+1] \subset \mathcal{L}_{\mathrm{k}}(\mathcal{T}_{n+1})$ is compatible with the alphabet partition (see Example 1). Thus we deal with a special kind of relators i.e. relators being compatible with an elimination scheme.
- In general, let $X = B \sqcup Z$ be a set partitioned in two blocks. We suppose given a relator $\mathrm{r} = \{r_j\}_{j\in J} \subset \mathcal{L}_{\mathrm{k}}(X)$ which is compatible with the alphabet partition i.e. there exists a partition of the set of indices $J = J_Z \sqcup J_B$ such that $\mathrm{r}_B = \{r_j\}_{j\in J_B} = \mathrm{r} \cap \mathcal{L}_{\mathrm{k}}(X)_B$ and $\mathrm{r}_Z = \{r_j\}_{j\in J_Z} = \mathrm{r} \cap \mathcal{L}_{\mathrm{k}}(X)_{BZ}$. The notations being as above, we construct the following ideals
  1. $\mathcal{J}_B$ is the Lie ideal of $\mathcal{L}_{\mathrm{k}}(X)_B$ generated by $\{r_j\}_{j\in J_B}$
  2. $\mathcal{J}, \mathcal{J}_Z$ and $\mathcal{J}_{BZ}$ are the Lie ideals of $\mathcal{L}_{\mathrm{k}}(X)$ generated respectively by r, r_Z and $\mathrm{r}_{BZ} := \{\mathrm{ad}_Q\, z\}_{Q\in\mathcal{J}_B, z\in Z}$.

Example 1.

A typical example is for the partitioned $X := \mathcal{T}_{n+1} = \mathcal{T}_n \sqcup T_{n+1} := B \sqcup Z$ and the infinitesimal pure braid relator $\mathrm{r} := \mathrm{R}[\mathrm{n}+1] \subset \mathcal{L}_{\mathrm{k}}(\mathcal{T}_{n+1})$. In this case, we observe that the relator $\mathrm{r}_{\mathcal{T}_n} = \mathrm{R}[\mathrm{n}+1] \cap \mathcal{L}_{\mathrm{k}}(\mathcal{T}_{n+1})_{\mathcal{T}_n} = \mathrm{R}[\mathrm{n}]$ and the relator $\mathrm{r}_{T_{n+1}} = \mathrm{R}[\mathrm{n}+1] \cap \mathcal{L}_{\mathrm{k}}(\mathcal{T}_{n+1})_{\mathcal{T}_n T_{n+1}} =$

$$\begin{cases} \mathrm{R}_1^-[\mathrm{n}+1] & [t_{i,j}, t_{i,n+1} + t_{j,n+1}] & \text{for } 1 \leq i < j \leq n, \\ \mathrm{R}_2^-[\mathrm{n}+1] & [t_{i,j} + t_{i,n+1}, t_{j,n+1}] & \text{for } 1 \leq i < j \leq n, \\ \mathrm{R}_3^-[\mathrm{n}+1] & \pm[t_{i,j}, t_{k,n+1}] & \text{for } \begin{matrix} 1 \leq i < j \leq n, \\ 1 \leq k \leq n, \end{matrix} \text{ and } |\{i,j,k\}| = 3. \end{cases}$$

Then we can construct the following Lie ideals

- $\mathcal{J}_{\mathcal{T}_n} = \mathcal{J}_{\mathrm{R}[\mathrm{n}]}$ is the Lie ideal of $\mathcal{L}_{\mathrm{k}}(\mathcal{T}_n)$ generated by the infinitesimal pure braid relator $\mathrm{r}_{\mathcal{T}_n} = \mathrm{R}[\mathrm{n}]$.
- $\mathcal{J}_{T_{n+1}}$ (resp. $\mathcal{J}_{\mathcal{T}_n T_{n+1}}$) is the Lie ideal of $\mathcal{L}_{\mathrm{k}}(\mathcal{T}_{n+1})$ generated by the relator $\mathrm{r}_{T_{n+1}}$ (resp. $\mathrm{r}_{\mathcal{T}_n T_{n+1}} = \{\mathrm{ad}_Q\, z\}_{Q \in \mathcal{J}_{\mathrm{R}[\mathrm{n}]}, z \in T_{n+1}}$).
- $\mathcal{J} = \mathcal{J}_{\mathrm{R}[\mathrm{n}+1]}$ is the Lie ideal of $\mathcal{L}_{\mathrm{k}}(\mathcal{T}_{n+1})$ generated by $\mathrm{R}[\mathrm{n}+1]$.

Main Theorem.

With our constructions above, we get the following properties:

i) we have $(\mathcal{J}_Z + \mathcal{J}_{BZ}) \subset \mathcal{L}_k(X)_{BZ}$ (and then $(\mathcal{J}_Z + \mathcal{J}_{BZ}) \cap \mathcal{J}_B = \{0\}$). Moreover, $(\mathcal{J}_Z + \mathcal{J}_{BZ})$ is a Lie ideal of $\mathcal{L}_k(X)_{BZ}$ (and even, by definition, of $\mathcal{L}_k(X)$).

ii) the action of $\mathcal{L}_k(X)_B$ on $\mathfrak{Der}(\mathcal{L}_k(X)_{BZ})$ (by internal ad) passes to quotients as an action $\alpha : \mathcal{L}_k(X)_B \to \mathfrak{Der}(\mathcal{L}_k(X)_{BZ} \big/ (\mathcal{J}_Z + \mathcal{J}_{BZ}))$ such that $r_B \subset \ker(\alpha)$ and then, we get an action

$$\overline{\alpha} : \mathcal{L}_k(X)_B \big/ \mathcal{J}_B \to \mathfrak{Der}(\mathcal{L}_k(X)_{BZ} \big/ (\mathcal{J}_Z + \mathcal{J}_{BZ})) \tag{7}$$

iii) we can construct an isomorphism from presented Lie algebra $\mathcal{L}_k(X) \big/ \mathcal{J}$ by the set $r = \{r_j\}_{j \in J}$ of relators onto the semidirect product of Lie algebras $\mathcal{L}_k(X)_{BZ} \big/ (\mathcal{J}_Z + \mathcal{J}_{BZ}) \rtimes \mathcal{L}_k(X)_B \big/ \mathcal{J}_B$

Main Theorem.

iii) which will be denoted by

$$\Phi : \mathcal{L}_{\mathsf{k}}(X) \big/ \mathcal{J} \xrightarrow{\simeq} \mathcal{L}_{\mathsf{k}}(X)_{BZ} \Big/ (\mathcal{J}_Z + \mathcal{J}_{BZ}) \rtimes \mathcal{L}_{\mathsf{k}}(X)_B \Big/ \mathcal{J}_B . \tag{8}$$

iv) In fact, one has a commutative diagram of Lie algebras with split short exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{L}_{\mathbf{k}}(X)_{BZ} & \overset{j}{\hookrightarrow} & \mathcal{L}_{\mathbf{k}}(X) & \overset{p}{\longrightarrow} & \mathcal{L}_{\mathbf{k}}(X)_B & \longrightarrow & 0 \\
 & & \big\downarrow s_{\mathcal{J}_Z+\mathcal{J}_{BZ}} & & \big\downarrow s_{\mathcal{J}} & & \big\downarrow s_{\mathcal{J}_B} & & \\
0 & \longrightarrow & \mathcal{L}_{\mathbf{k}}(X)_{BZ} \Big/ (\mathcal{J}_Z + \mathcal{J}_{BZ}) & \longrightarrow & \mathcal{L}_{\mathbf{k}}(X) \big/ \mathcal{J} & \longrightarrow & \mathcal{L}_{\mathbf{k}}(X)_B \Big/ \mathcal{J}_B & \longrightarrow & 0
\end{array}$$

Elimination of the subalphabet Z

- In certain cases (which is that of the Lie algebras $\mathrm{DK}_{\mathrm{k},n}$), it can happen that the left factor of the semidirect product (8) be isomorphic to $\mathcal{L}_{\mathrm{k}}(Z)$. We start from the previous commutative diagram with an additional arrow

$$\begin{array}{ccccccccc}
 & & \mathcal{L}_{\mathbf{k}}(Z) & & & & & & \\
 & & \downarrow j_Z & & & & & & \\
0 & \longrightarrow & \mathcal{L}_{\mathbf{k}}(X)_{BZ} & \xhookrightarrow{j} & \mathcal{L}_{\mathbf{k}}(X) & \xrightarrow{p} & \mathcal{L}_{\mathbf{k}}(X)_B & \longrightarrow & 0 \\
 & & \downarrow s_{\mathcal{J}_Z+\mathcal{J}_{BZ}} & & \downarrow s_{\mathcal{J}} & & \downarrow s_{\mathcal{J}_B} & & \\
0 & \longrightarrow & \mathcal{L}_{\mathbf{k}}(X)_{BZ}\big/(\mathcal{J}_Z+\mathcal{J}_{BZ}) & \longrightarrow & \mathcal{L}_{\mathbf{k}}(X)/\mathcal{J} & \longrightarrow & \mathcal{L}_{\mathbf{k}}(X)_B\big/\mathcal{J}_B & \longrightarrow & 0
\end{array}$$

where j_Z is the subalphabet embedding such that

$$Im(j_Z) = \mathcal{L}_{\mathrm{k}}(X)_Z = \bigoplus_{\substack{\alpha\in\mathbb{N}^{(X)} \\ |\alpha|_B=0}} \mathcal{L}_{\mathrm{k}}(X)_\alpha . \tag{9}$$

We are now in the position to state the following

Proposition 2.

With the notations as in Main Theorem, let us consider the composite map $\beta = s_{\mathcal{J}_Z+\mathcal{J}_{BZ}} \circ j_Z$, then

a. In order that β be injective, it is necessary and sufficient that $(\mathcal{J}_Z + \mathcal{J}_{BZ}) \cap \mathcal{L}_{\mathsf{k}}(X)_Z = \{0\}$.

b. In order that β be surjective, it is necessary and sufficient that, for all $(b, z) \in B \times Z$, we had

$$s_{\mathcal{J}_Z+\mathcal{J}_{BZ}}([b, z]) \in s_{\mathcal{J}_Z+\mathcal{J}_{BZ}}(\mathcal{L}_{\mathsf{k}}(X)_Z). \tag{10}$$

The existence of the decomposition of Drinfeld-Kohno Lie algebra

Recall in Example 1, we denoted by $\mathcal{T}_{n+1} = \mathcal{T}_n \sqcup T_{n+1}$ and the infinitesimal pure braid relator $\mathrm{R}[\mathrm{n}+1] \subset \mathcal{L}_{\mathrm{k}}(\mathcal{T}_{n+1})$. In this case, the existence of the decomposition of Drinfeld-Kohno Lie algebra can be obtained as a consequence of our main theorem and by Proposition 2.

Corollary 3.

There is the decomposition of Drinfeld-Kohno Lie algebra i.e. in the category k-**Lie**,

$$\mathrm{DK}_{\mathrm{k},n+1} \simeq \mathcal{L}_{\mathrm{k}}(X_n) \rtimes \mathrm{DK}_{\mathrm{k},n} \tag{11}$$

where X_n is any alphabet of cardinality n.

About M.-P. Schützenberger's questions on the Partially Commutative Free Lie algebra

- Let $X \in \mathbf{Set}$ be a set viewed as a alphabet. A commutation relation on X is a reflexive and symmetric graph $\theta \subset X^2$ (i.e. $\theta = \theta^{-1}$ and $\{(x, x)\}_{x\in X}$, the diagonal of X, is a subset of θ).
- Firstly, the *free partially commutative monoid* $M(X, \theta)$ is the quotient of X^* by the congruence generated by the family $(xy = yx)_{(x,y)\in\theta}$.
- We will consider the canonical surjection $s_\theta : X^* \to M(X, \theta)$ as well as $j_\theta : M(X, \theta) \to X^*$ an arbitrary set-theoretical section of it.
- The terminal alphabet $\mathrm{TAlph}(t)$ (where $t \in M(X, \theta)$) can be characterized as the set of last letters of preimages of t w.r.t. s_θ, it means that $\mathrm{TAlph}(t) = \{x \in X \mid t \in M(X, \theta).x\}$.
- Secondly, the *free partially commutative Lie algebra* $\mathcal{L}_k(X, \theta)$ is the quotient of $\mathcal{L}_k(X)$ by the ideal generated by the relator $r_\theta = \{[x, y]\}_{(x,y)\in\theta}$.

Theorem 4.

Let (X, θ) be an alphabet with commutations. We consider a partition of X, $X = B \sqcup Z$ such that Z is totally non-commutative i.e. no two letters of Z commute between themselves ($\theta \cap Z^2 = \Delta_Z$) and the code

$$C_B(Z) = \{s_\theta(uz) | u \in B^*, z \in Z, \mathrm{TAlph}(s_\theta(uz)) = \{z\}\} \tag{12}$$

Let $C = j_\theta(C_B(Z))$ and j_C be the subalphabet embedding, we have the diagram

Theorem 4.

$$
\begin{array}{ccccccccc}
 & & \mathcal{L}_{\mathbf{k}}(C) & & & & & & \\
 & & \big\downarrow j_C & & & & & & \\
0 & \longrightarrow & \mathcal{L}_{\mathbf{k}}(X)_{BZ} & \xrightarrow{\ j\ } & \mathcal{L}_{\mathbf{k}}(X) & \xrightarrow{\ p\ } & \mathcal{L}_{\mathbf{k}}(X)_B & \longrightarrow & 0 \\
 & & \big\downarrow s_{\mathcal{J}_Z+\mathcal{J}_{BZ}} & & \big\downarrow s_{\mathcal{J}} & & \big\downarrow s_{\mathcal{J}_B} & & \\
0 & \longrightarrow & \mathcal{L}_{\mathbf{k}}(X)_{BZ} \Big/ (\mathcal{J}_Z + \mathcal{J}_{BZ}) & \longrightarrow & \mathcal{L}_{\mathbf{k}}(X) / \mathcal{J} & \longrightarrow & \mathcal{L}_{\mathbf{k}}(X)_B \Big/ \mathcal{J}_B & \longrightarrow & 0
\end{array}
$$

Then, with the above hypotheses (Z totally non-commutative and $C = j_\theta(C_B(Z))$), $s_{\mathcal{J}_Z+\mathcal{J}_{BZ}} \circ j_C$ is an isomorphism. In particular, the left factor of the semi-direct product (8), here $\mathcal{L}_k(X)_{BZ} \Big/ (\mathcal{J}_Z + \mathcal{J}_{BZ})$ is a free Lie algebra.

It would be interesting to have alternative proofs for answers to Schützenberger's questions about the Partially Commutative Free Lie algebra (cf. Duchamp and Krob [4], Thm. III.3) as a consequence of our main theorem.

Corollary 5. (Lazard's Partially Commutative Elimination)

Let X be a set equipped with a commutation relation θ and B be a subset of X such that $Z = X - B$ is totally non-commutative. Then there is an isomorphism from the free partially commutative Lie algebra $\mathcal{L}_{\mathrm{k}}(X, \theta)$ to the semidirect of product of Lie algebras, namely

$$\mathcal{L}_{\mathrm{k}}(X, \theta) \simeq_{\mathrm{k}\text{-}\mathbf{Lie}} \mathcal{L}_{\mathrm{k}}(C) \rtimes \mathcal{L}_{\mathrm{k}}(B, \theta_B). \tag{13}$$

Upcoming works

- Suppose that a commutative ring k of characteristic zero (hence $\mathbb{Q} \hookrightarrow$ k) and $X = B \sqcup Z$ is a set partitioned in two blocks, where $B = \{b_1, \ldots, b_n\}$ and $Z = \{z_1, z_2, z_3, \ldots\}$. Let us consider the polynomial algebra

$$\mathrm{k}\langle X\rangle = \mathrm{k}\langle b_1, \ldots, b_n, z_1, z_2, z_3, \ldots\rangle.$$

The collection (called by Magnus polynomials (cf. Nakamura [3]))

$$u.\,\mathrm{ad}_{(w_1)}\, z_{i_1} \ldots \mathrm{ad}_{(w_k)}\, z_{i_k}, \tag{14}$$

where $k \geq 0, w_1, \ldots, w_k \in B^*, i_1, \ldots, i_k \geq 1$ and $u \in B^*$, are k-linear basis of $\mathrm{k}\langle X\rangle$.

- We introduce the *half-shuffle* in the polynomial algebra $k\langle X\rangle$ as the linear extension of the binary product on words given by

$$(x_1\ldots x_p)\frac{\sqcup\!\sqcup}{2}(x_{p+1}\ldots x_n) = x_1(x_2\ldots x_p \sqcup\!\sqcup\, x_{p+1}\ldots x_n),$$
$$1_{X^*}\frac{\sqcup\!\sqcup}{2}(x_{p+1}\ldots x_n) = 0,$$
$$(x_1\ldots x_p)\frac{\sqcup\!\sqcup}{2}1_{X^*} = x_1\ldots x_p$$

and then elements arising by the half-shuffle of ZB^*:

$$z_{i_1}w_1\frac{\sqcup\!\sqcup}{2}(z_{i_2}w_2\frac{\sqcup\!\sqcup}{2}(\ldots\frac{\sqcup\!\sqcup}{2}(z_{i_{k-1}}w_{k-1}\frac{\sqcup\!\sqcup}{2}z_{i_k}w_k)\ldots)), \quad (15)$$

where $k \geq 0, w_1,\ldots,w_k \in B^*, i_1,\ldots,i_k \geq 1$ (if $k=0$ then (15) will be denoted by 1_{X^*}). Henceforth we write simply $z_{i_1}w_1\frac{\sqcup\!\sqcup}{2}\ldots\frac{\sqcup\!\sqcup}{2}z_{i_k}w_k$ instead of (15).

The purpose of the following theorem is to describe the dual of Magnus basis under the standard pairing

$$\langle \bullet | \bullet \rangle : \ \mathrm{k}\langle X\rangle^{\vee} \otimes \mathrm{k}\langle X\rangle = \mathrm{k}\langle\langle X\rangle\rangle \otimes \mathrm{k}\langle X\rangle \to \mathrm{k}$$

with respect to the monomials of $\mathrm{k}\langle X\rangle$ (here for all $S \in \mathrm{k}\langle\langle X\rangle\rangle$ and $P \in \mathrm{k}\langle X\rangle$ then the pairing $\langle S|P\rangle = \sum_{w\in X^*} \langle S|w\rangle\langle P|w\rangle$).

Theorem 6.

The collections

$$\{u.(-1)^{|w_1|}\,\mathrm{ad}_{(w_1)}\, z_{i_1} \ldots (-1)^{|w_k|}\,\mathrm{ad}_{(w_k)}\, z_{i_k}\}^{k\geq 0, w_1,\ldots,w_k\in B^*}_{i_1,\ldots,i_k\geq 1, u\in B^*}$$

and

$$\{u \sqcup\!\sqcup (z_{i_1}\widetilde{w_1} \frac{\sqcup\!\sqcup}{2} \ldots \frac{\sqcup\!\sqcup}{2} z_{i_k}\widetilde{w_k})\}^{k\geq 0, w_1,\ldots,w_k\in B^*}_{i_1,\ldots,i_k\geq 1, u\in B^*}$$

are dual bases of, respectively $\mathrm{k}\langle X\rangle$ and $\mathrm{k}\langle X\rangle^{\vee}$, where $\widetilde{w} = b_{i_k} b_{i_{k-1}} \ldots b_{i_1}$ reverses the order of letters in the word $w = b_{i_1} b_{i_2} \ldots b_{i_k} \in B^*$.

- Describe the dual basis in a suitable algebraic framework.
- Applying the dual basis to provide finally solutions for Knizhnik-Zamolodchikov equations given in (1) with asymptotic conditions by *dévissage*.

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Thank you very much for your attention!