Applications of Lazard's elimination and of MRS[1] factorizations

V. Hoang Ngoc Minh
Université Lille, 1 Place Déliot, 59024 Lille, France.

Séminaire Combinatoire, Informatique et Physique,
12 et 20 Janvier 2021, Villetaneuse

[1] Mélançon-Reutenauer-Schützenberger

Bibliography I

- J. Berstel & C. Reutenauer.– *Rational series and their languages*, Spr.-Ver., 1988.
- V.C. Bui, G.H.E. Duchamp, V. Hoang Ngoc Minh, H. Nguyen, C. Tollu.– *Combinatorics of φ-deformed stuffle Hopf algebras*, https://hal.archives-ouvertes.fr/hal-00793118 (2014).
- V.C. Bui, G.H.E. Duchamp, V. Hoang Ngoc Minh, Q.H. Ngo, C. Tollu.– *(Pure) transcendence bases in φ-deformed shuffle bialgebras*, Journal électronique du Séminaire Lotharingien de Combinatoire B74f (2018).
- P. Cartier.– *Jacobiennes généralisées, monodromie unipotente et intégrales itérées*, Séminaire Bourbaki, 687 (1987), 31–52.
- P. Cartier.– *Fonctions polylogarithmes, nombres polyzetas et groupes pro-unipotents*– Séminaire Bourbaki, 53$^{\text{ème}}$, n°885, 2000-2001.
- K.-T. Chen.– *Iterated integrals and exponential homomorphisms*, Proc. Lond. Mathem. Soc. (3) **4** (1954) 502–512. bibitemAofAC. Costermans, J.Y. Enjalbert and V. Hoang Ngoc Minh.– *Algorithmic and combinatoric aspects of multiple harmonic sums*, Discrete Mathematics & Theoretical Computer Science Proceedings, 2005.
- M. Deneufchâtel, G.H.E. Duchamp, V. Hoang Ngoc Minh, A.I. Solomon.– *Independence of hyperlogarithms over function fields via algebraic combinatorics*, in LNCS (2011), 6742.
- G.H.E. Duchamp, V. Hoang Ngoc Minh, K.A. Penson.– *About Some Drinfel'd Associators*, International Workshop on Computer Algebra in Scientific Computing CASC 2018 - Lille, 17-21 September 2018.
- V. Drinfel'd– On quasitriangular quasi-hopf algebra and a group closely connected with gal$(\bar{q}/q)$, *Leningrad Math. J.*, 4, 829-860, 1991.
- J. Ecalle.– ARI/GARI, la dimorphie et l'arithmétique des multizêtas : un premier bilan, J. Th. des nombres de Bordeaux, 15, (2003), pp. 411-478.

Bibliography II

M. Lothaire.– *Combinatorics on Words*, Encyclopedia of Mathematics and its Applications, Addison-Wesley, 1983.

Hoang Ngoc Minh, G. Jacob.– *Symbolic Integration of meromorphic differential equation via Dirichlet functions*, Disc. Math. 210, pp. 87-116, 2000.

Hoang Ngoc Minh, M. Petitot.– *Lyndon words, polylogarithmic functions and the Riemann ζ function*, Discrete Math., 217, 2000, pp. 273-292.

V. Hoang Ngoc Minh.– On the solutions of the universal differential equation with three regular singularities (On solutions of KZ_3), Confluentes Mathematici (2019).

M. Hoffman.– *Multiple harmonic series*, Pacific J. Math. 152 (1992), pp. 275-290.

D.E. Radford.– *A natural ring basis for shuffle algebra and an application to group schemes Journal of Algebra*, 58, pp. 432-454, 1979.

Ree R.,– *Lie elements and an algebra associated with shuffles Ann. Math* **68** 210–220, 1958.

C. Reutenauer.– *Free Lie Algebras*, London Math. Soc. Monographs (1993).

G. Viennot.– *Algèbres de Lie libres et monoïdes libres*, Lect. N. in Math., Springer-Verlag, 691, 1978.

D. Zagier.– Values of zeta functions and their applications, in "First European Congress of Mathematics", vol. 2, Birkhäuser (1994), pp. 497-512.

Outline

INTRODUCTION

Zeta functions in several variables and monoids

For $r \in \mathbb{N}_+, (s_1, \ldots, s_r) \in \mathbb{C}^r, |z| < 1$, the following functions are well defined

$$\mathrm{Li}_{s_1,\ldots,s_r}(z) := \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}} \quad \text{and} \quad \frac{\mathrm{Li}_{s_1,\ldots,s_r}(z)}{1-z} = \sum_{n \geq 0} \mathrm{H}_{s_1,\ldots,s_r}(n) z^n$$

and, for $\mathcal{H}_r := \{(s_1, \ldots, s_r) \in \mathbb{C}^r | \forall m = 1, \ldots, r, \Re(s_1) + \ldots + \Re(s_m) > m\}$, the following zeta function is convergent

$$\zeta(s_1, \ldots, s_r) := \sum_{n_1 > \ldots > n_r > 0} n_1^{-s_1} \ldots n_r^{-s_r} \quad \text{with} \quad (s_1, \ldots, s_r) \in \mathcal{H}_r.$$

From a theorem by Abel,

$$\lim_{z \to 1} \mathrm{Li}_{s_1,\ldots,s_r}(z) = \zeta(s_1, \ldots, s_r) = \lim_{n \to +\infty} \mathrm{H}_{s_1,\ldots,s_r}(n).$$

$$\mathcal{Z} := \mathrm{span}_{\mathbb{Q}}\{\mathrm{Li}_{s_1,\ldots,s_r}(1)\}_{\substack{s_1,\ldots,s_r \in \mathbb{N}_+ \\ s_1 > 1, r > 0}} = \mathrm{span}_{\mathbb{Q}}\{\mathrm{H}_{s_1,\ldots,s_r}(+\infty)\}_{\substack{s_1,\ldots,s_r \in \mathbb{N}_+ \\ s_1 > 1, r > 0}}.$$

Denoting the (ordered) alphabets $Y := \{y_k\}_{k \geq 1}$ (with $y_1 \succ y_2 \succ \ldots$) or $X := \{x_0, x_1\}$ (with $x_1 \succ x_0$) by $\mathcal{X}$, we use the correspondence among words of the free monoid $(\mathcal{X}^*, \mathrm{conc}, 1_{\mathcal{X}^*})$:

$$(s_1, \ldots, s_r) \in \mathbb{N}_+^r \leftrightarrow y_{s_1} \ldots y_{s_r} \in Y^* \underset{\pi_Y}{\overset{\pi_X}{\rightleftharpoons}} x_0^{s_1 - 1} x_1 \ldots x_0^{s_r - 1} x_1 \in X^* x_1.$$

$\mathcal{L}yn\mathcal{X}$ denotes the set of Lyndon words generated by $\mathcal{X}$.

$$\mathrm{Li}_{s_1,\ldots,s_r}(z) = \alpha_{z_0}^{z}(x_0^{s_1 - 1} x_1 \ldots x_0^{s_k - 1} x_1) \text{ with } \omega_0(z) = \frac{dz}{z}, \omega_1(z) = \frac{dz}{1-z}.$$

Iterated integrals over $\{\omega_i\}_{i\geq 1}$ and along $z_0 \rightsquigarrow z$ on Ω

The iterated integrals, over $\{\omega_i\}_{i\geq 1}$ and along the path $z_0 \rightsquigarrow z$ on a simply connected domain Ω of $\mathbb{C}$, are defined by $\alpha_{z_0}^z(1_{\mathcal{X}^*}) = 1_\Omega$ and

$$\forall x_{i_1}\ldots x_{i_k} \in \mathcal{X}^*, \quad \alpha_{z_0}^z(x_{i_1}\ldots x_{i_k}) = \int_{z_0}^{z} \omega_{i_1}(z_1)\ldots\int_{z_0}^{z_{k-1}} \omega_{i_k}(z_k),$$

with $\omega_i(z) = u_{x_i}(z)dz$ and $u_{x_i} \in \mathcal{C}_0$, being a differential subring of $\mathcal{H}(\Omega)$.
These integrals satisfy, for any $x_i \in \mathcal{X}$ and $w, v \in \mathcal{X}^*$,

$$\partial\alpha_{z_0}^z(x_i w) = u_{x_i}(z)\alpha_{z_0}^z(w) \quad \text{and}^2 \quad \alpha_{z_0}^z(w ⧢ v) = \alpha_{z_0}^z(w)\alpha_{z_0}^z(v).$$

Example 1 (with $\omega_0(z) = z^{-1}dz$ and $\omega_1(z) = (1-z)^{-1}dz$)

$$\alpha_1^z(x_0^k) = \int_1^z \omega_0(z_1)\ldots\int_1^{z_{k-1}} \omega_0(z_{k-1}) = \frac{\log^k(z)}{k!}.$$

$$\alpha_0^z(x_1^k) = \int_0^z \omega_1(z_1)\ldots\int_0^{z_{k-1}} \omega_1(z_{k-1}) = \mathrm{Li}_{\underbrace{1,\ldots,1}_{k\text{ times}}}(z) = \frac{\log^k((1-z)^{-1})}{k!}.$$

$$\alpha_0^z(x_0x_1) = \int_0^z\int_0^s \frac{ds}{s}\frac{dt}{1-t} = \int_0^z \frac{ds}{s}\int_0^s dt\sum_{k\geq 0} t^k = \sum_{k\geq 1}\int_0^z ds\frac{s^{k-1}}{k}$$
$$= \sum_{k\geq 1}\frac{z^k}{k^2} = \mathrm{Li}_2(z).$$

---

[2] For any $x, y \in \mathcal{X}, y_i, y_j \in Y$ and $w, v \in \mathcal{X}^*$ (resp. Y^*),
$w ⧢ 1_{\mathcal{X}^*} = 1_{\mathcal{X}^*} ⧢ w = w$ and $xw ⧢ yv = x(w ⧢ yv) + y(xw ⧢ v)$,
$w \uplus 1_{Y^*} = 1_{Y^*} \uplus w = w$ and $x_iw \uplus y_jv = y_i(w \uplus y_jv) + y_j(y_iw \uplus v) + y_{i+j}(w \uplus v)$.

First structures of polylogarithms and harmonic sums

1. $\{\mathrm{Li}_w\}_{w\in X^*}$ is $\mathbb{C}$-linearly independent. Hence, the following morphism of algebras is injective[3]

$$\begin{array}{rcl}\mathrm{Li}_\bullet : (\mathbb{C}\langle X\rangle, ш, 1_{X^*}) & \to & (\mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}, ., 1),\\ x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1 & \mapsto & \mathrm{Li}_{x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1} \ (\textit{i.e.}\ \mathrm{Li}_{s_1,\ldots,s_r})\\ x_0^k & \mapsto & \log^k(z)/k!.\end{array}$$

Thus, $\{\mathrm{Li}_l\}_{l\in\mathcal{L}ynX}$ is $\mathbb{C}$-algebraically independent.

2. The following morphism of algebras is injective

$$\begin{array}{rcl}\mathrm{P}_\bullet : (\mathbb{C}\langle Y\rangle, ⧢, 1_{Y^*}) & \to & (\mathbb{C}\{\mathrm{P}_w\}_{w\in Y^*}, \odot, 1),\\ w & \mapsto & \mathrm{P}_w(z) := \dfrac{\mathrm{Li}_{\pi_X w}(z)}{1-z} = \displaystyle\sum_{n\geq 0}\mathrm{H}_w(n)z^n.\end{array}$$

Hence, $\{\mathrm{P}_w\}_{w\in Y^*}$ is $\mathbb{C}$-linearly independent. It follows that $\{\mathrm{P}_l\}_{l\in\mathcal{L}ynY}$ is $\mathbb{C}$-algebraically independent, for[4] $\odot$.

3. The following morphism of algebras is injective

$$\begin{array}{rcl}\mathrm{H}_\bullet : (\mathbb{C}\langle Y\rangle, ⧢, 1_{Y^*}) & \to & (\mathbb{C}\{\mathrm{H}_w\}_{w\in Y^*}, ., 1),\\ y_{s_1}\ldots y_{s_r} & \mapsto & \mathrm{H}_{y_{s_1}\ldots y_{s_r}} \ (\textit{i.e.}\ \mathrm{H}_{s_1,\ldots,s_r}).\end{array}$$

Hence, $\{\mathrm{H}_w\}_{w\in Y^*}$ is $\mathbb{C}$-linearly independent. It follows that $\{\mathrm{H}_l\}_{l\in\mathcal{L}ynY}$ is $\mathbb{C}$-algebraically independent.

---

[3] For $(s_1,\ldots,s_r)\in\mathbb{N}_+^r, r\geq 1, k\geq 0$.

[4] For any $u,v\in Y, \mathrm{P}_u\odot\mathrm{P}_v=\mathrm{P}_{u⧢v}$.

Towards more about structure of polyzetas

1. The following polymorphism of algebras is surjective

$$\zeta : \begin{array}{c} (\mathbb{C}[\mathcal{L}ynX - X], ш, 1_{X^*}) \\ (\mathbb{C}[\mathcal{L}ynY - \{y_1\}], ⧢, 1_{Y^*}) \end{array} \twoheadrightarrow (\mathcal{Z}, ., 1),$$

$$\begin{array}{c} x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 \\ y_{s_1} \ldots y_{s_r} \end{array} \mapsto \zeta(s_1, \ldots, s_r),$$

for $(s_1, \ldots, s_r) \in \mathbb{N}_+^r, r \geq 1$.

2. For any l_1 and $l_2 \in \mathcal{L}ynX - X$ (hence, $\pi_Y l_1$ and $\pi_Y l_2 \in \mathcal{L}ynY - \{y_1\}$),

$$\zeta(l_1)\zeta(l_2) = \zeta(l_1 ш l_2) = \zeta((\pi_Y l_1) ⧢ (\pi_Y l_2)) = \zeta(\pi_Y l_1)\zeta(\pi_Y l_2).$$

3. ζ can be extended as characters:

$$\zeta_ш : (\mathbb{C}\langle X\rangle, ш, 1_{X^*}) \to (\mathcal{Z}, ., 1),$$
$$\zeta_⧢ : (\mathbb{C}\langle Y\rangle, ⧢, 1_{Y^*}) \to (\mathcal{Z}, ., 1),$$
$$\zeta_ш(x_0) = 0 = \log(1),$$
$$\zeta_ш(x_1) = 0 = \mathrm{f.p.}_{z\to 1}\log(1-z), \quad \{(1-z)^a \log^b(1-z)\}_{a\in\mathbb{Z}_{\leq -1}, b\in\mathbb{N}},$$
$$\zeta_⧢(y_1) = 0 = \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_1(n), \quad \{n^a \mathrm{H}_1^b(n)\}_{a\in\mathbb{Z}_{\leq -1}, b\in\mathbb{N}}.$$

Conjecture 1 (Zagier's dimension conjecture)

$\forall k \geq 1, \mathcal{A}_k := \mathrm{span}_{\mathbb{Z}}\{\zeta(s_1, \ldots, s_r), s_1 + \ldots + s_r = k\}_{s_1+\ldots+s_r \in \mathcal{H}_r \cap \mathbb{N}^r, r\geq 0}$, *and* $d_k := \dim_{\mathbb{Z}} \mathcal{A}_k$. *Then* $d_k = d_{k-2} + d_{k-3}$ *with* $d_0 = 1, d_1 = 0, d_2 = 1$.

$\bigoplus_{k\geq 0} \mathcal{A}_k \to \mathcal{Z}$ is injective ? $\mathcal{Z}$ is graded ?

ALGEBRAIC COMBINATORIAL ASPECTS

conc-shuffle and conc-stuffle bialgebras

Let $(A\langle\mathcal{X}\rangle, \mathtt{conc})$ (resp. $(A\langle\langle\mathcal{X}\rangle\rangle, \mathtt{conc})$) be the algebra of polynomials (resp. series) and $(\mathcal{L}ie_A\langle\mathcal{X}\rangle, [.])$ (resp. $\mathcal{L}ie_A\langle\langle\mathcal{X}\rangle\rangle, [.])$ be the algebra of Lie polynomials (resp. series) over $\mathcal{X}$ with coefficients in the commutative ring $A \supset \mathbb{Q}$.

The dual law associated to conc is defined by

$$\forall w \in \mathcal{X}^*, \quad \Delta_{\mathtt{conc}}(w) = \sum_{u,v\in\mathcal{X}^*, uv=w} u \otimes v.$$

On $(A\langle\mathcal{X}\rangle, \mathtt{conc}, 1_{\mathcal{X}^*}, \Delta_{\sqcup\!\sqcup}, \varepsilon)$ and $(A\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\uplus}, \varepsilon)$, one defines also, as morphisms for conc, on letters by

$$\begin{aligned}
\forall x \in \mathcal{X} \quad \Delta_{\sqcup\!\sqcup}\, x &= x \otimes 1_{\mathcal{X}^*} + 1_{\mathcal{X}^*} \otimes x,\\
\forall y_i \in Y \quad \Delta_{\uplus}\, y_i &= y_i \otimes 1_{Y^*} + 1_{Y^*} \otimes y_i + \sum_{k+l=i} y_k \otimes y_l,
\end{aligned}$$

and extends by linearity and infinite sums, for $S \in A\langle\langle Y\rangle\rangle$ (resp. $A\langle\langle\mathcal{X}\rangle\rangle$), by

$$\begin{aligned}
\Delta_{\uplus} S &= \sum_{w\in Y^*} \langle S|w\rangle \Delta_{\uplus}\, w \quad \in A\langle\langle Y^* \otimes Y^*\rangle\rangle,\\
\Delta_{\mathtt{conc}} S &= \sum_{w\in\mathcal{X}^*} \langle S|w\rangle \Delta_{\mathtt{conc}}\, w \quad \in A\langle\langle \mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle,\\
\Delta_{\sqcup\!\sqcup} S &= \sum_{w\in\mathcal{X}^*} \langle S|w\rangle \Delta_{\sqcup\!\sqcup}\, w \quad \in A\langle\langle \mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle.
\end{aligned}$$

Dual laws in conc-shuffle bialgebras

Startting with a $\mathbf{k}-\mathbf{AAU}$ ($\mathbf{k}$ is a ring) $\mathcal{A}$. Dualizing $\mu : \mathcal{A}\otimes_{\mathbf{k}}\mathcal{A}\to\mathcal{A}$, we get the transpose ${}^t\mu : \mathcal{A}^\vee \to (\mathcal{A}\otimes_{\mathbf{k}}\mathcal{A})^\vee$ so that we do not get a co-multiplication in general.

- Remark that when $\mathbf{k}$ is a field, the following arrow is into (due to the fact that $\mathcal{A}^\vee\otimes_{\mathbf{k}}\mathcal{A}^\vee$ is torsionfree)
$$\Phi : \mathcal{A}^\vee\otimes_{\mathbf{k}}\mathcal{A}^\vee \to (\mathcal{A}\otimes_{\mathbf{k}}\mathcal{A})^\vee.$$
- One restricts the codomain of ${}^t\mu$ to $\mathcal{A}^\vee\otimes_{\mathbf{k}}\mathcal{A}^\vee$ and then the domain to $({}^t\mu)^{-1}\Phi(\mathcal{A}^\vee\otimes_{\mathbf{k}}\mathcal{A}^\vee) =: \mathcal{A}^\circ$.

$$\begin{array}{ccc}
\mathcal{A}^\vee & \xrightarrow{\quad {}^t\mu \quad} & (\mathcal{A}\otimes_{\mathbf{k}}\mathcal{A})^\vee \\
{\scriptstyle can}\uparrow & & \uparrow{\scriptstyle \Phi} \\
\mathcal{A}^\circ & \xrightarrow{\quad \Delta_\mu \quad} & \mathcal{A}^\vee\otimes_{\mathbf{k}}\mathcal{A}^\vee \\
{\scriptstyle can}\uparrow & & \uparrow{\scriptstyle j\otimes j} \\
\mathcal{A}^{\circ\circ} & \xrightarrow{\quad \Delta_\mu \quad} & \mathcal{A}^\circ\otimes_{\mathbf{k}}\mathcal{A}^\circ
\end{array}$$

The descent can stop at first step for a field $\mathbf{k}$ and then $\mathcal{A}^{\circ\circ}=\mathcal{A}^\circ$. The coalgebra $(\mathcal{A}^\circ,\Delta_\mu)$ is called the Sweedler's dual of $(\mathcal{A},\mu)$.

Dualizable laws in conc-shuffle bialgebras (1/2)

We can exploit the basis of words as follows

1. Any bilinear law (shuffle, stuffle or any) $\mu : A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle \to A\langle\mathcal{X}\rangle$ can be decribed through its structure constants wrt to the basis of words, *i.e.* for $u, v, w \in \mathcal{X}^*$, $\Gamma_{u,v}^w := \langle \mu(u \otimes v) | w \rangle$ so that
$$\mu(u \otimes v) = \sum_{w \in \mathcal{X}^*} \Gamma_{u,v}^w w.$$
2. In the case when $\Gamma_{u,v}^w$ is locally finite in w, we say that the given law is dualizable, the arrow ${}^t\mu$ restricts nicely to $A\langle\mathcal{X}\rangle \hookrightarrow A\langle\langle\mathcal{X}\rangle\rangle$ and one can define on the polynomials a comultiplication by
$$\Delta_\mu(w) := \sum_{u,v \in \mathcal{X}^*} \Gamma_{u,v}^w u \otimes v.$$
3. When the law μ is dualizable, we have

$$\begin{array}{ccc}
A\langle\langle\mathcal{X}\rangle\rangle & \xrightarrow{\quad {}^t\mu \quad} & A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle \\
{\scriptstyle can}\uparrow & & \uparrow{\scriptstyle \Phi|_{A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle}} \\
A\langle\mathcal{X}\rangle & \xrightarrow{\quad \Delta_\mu \quad} & A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle
\end{array}$$

The arrow Δ_μ is unique to be able to close the rectangle and $\Delta_\mu(P)$ is defined as above.

Dualizable laws in $\mathtt{conc}$-shuffle bialgebras (2/2)

4. Proof that the arrow $A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle \longrightarrow A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle$ is into:

   Let $T = \sum_{i=1}^{n} P_i \otimes_A Q_i$ such that $\Phi(T) = 0$. We can rewrite T as a finitely supported sum $T = \sum_{u,v\in\mathcal{X}^*} c_{u,v}\, u \otimes v$ (this is indeed the iso between $A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle$ and $A[\mathcal{X}^* \times \mathcal{X}^*]$), then $\Phi(T)$ is by definition of Φ the double series (here a polynomial) such that $\langle\Phi(T)|u \otimes v\rangle = c_{u,v}$. If $\Phi(T) = 0$, then for all $(u, v) \in \mathcal{X}^* \times \mathcal{X}^*$, $c_{u,v} = 0$ entailing $T = 0$.

In the sequel,

1. $A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ denotes the algebraic closure by $\{\mathtt{conc}, +, *\}$ of $\widehat{A.\mathcal{X}}$ in $A\langle\langle\mathcal{X}\rangle\rangle$. Let $S \in A\langle\langle\mathcal{X}\rangle\rangle$ s.t. $\langle S|1_{\mathcal{X}^*}\rangle = 0$. Then $S^* = \sum_{n\geq 0} S^n$, so called Kleene star of S.

   $A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ is closed under $\sqcup\!\sqcup$. $A^{\mathrm{rat}}\langle\langle Y\rangle\rangle$ is also closed under $\uplus$.

2. $A_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$ denotes the set of (syntactically) exchangeable[5] series and $A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$ the set of series admitting a linear representation with commuting matrices (hence, exchangeable).

[5] *i.e.* if $S \in A_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$ then $(\forall u, v \in \mathcal{X}^*)((\forall x \in \mathcal{X})(|u|_x = |v|_x) \Rightarrow \langle S|u\rangle = \langle S|v\rangle)$.

Case of rational series and of $\Delta_{\texttt{conc}}$

$$\begin{array}{ccc}
A\langle\langle\mathcal{X}\rangle\rangle & \xrightarrow{\ {}^t\texttt{conc}\ } & A\langle\langle\mathcal{X}^*\otimes\mathcal{X}^*\rangle\rangle \\
{\scriptstyle can}\uparrow & & \uparrow{\scriptstyle \Phi|_{A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle\otimes_A A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle}} \\
A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle & \dashrightarrow & A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle\otimes_A A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle
\end{array}$$

The dashed arrow may not exist in general, but for any $R\in A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ admitting (λ,μ,η) as linear representation of dimension n, we can however obtain expressions of the type

$${}^t\texttt{conc}(R)=\Phi(\sum_{i=1}^{n}G_i\otimes D_i).$$

Indeed, since $\langle R|xy\rangle=\lambda\mu(xy)\eta=\lambda\mu(x)\mu(y)\eta\ (x,y\in\mathcal{X})$ then, letting e_i is the vector such that ${}^te_i=\begin{pmatrix}0&\ldots&0&1&0&\ldots&0\end{pmatrix}$, one has

$$\langle R|xy\rangle=\sum_{i=1}^{n}\lambda\mu(x)e_i{}^te_i\mu(y)\eta=\sum_{i=1}^{n}\langle G_i|x\rangle\langle D_i|y\rangle=\sum_{i=1}^{n}\langle G_i\otimes D_i|x\otimes y\rangle.$$

G_i (resp. D_i) admits then (λ,μ,e_i) (resp. $({}^te_i,\mu,\eta)$) as linear representation.
If $A=\mathbf{k}$ being a field then, due to the injectivity of Φ, all expressions of the type $\sum_{i=1}^{n}G_i\otimes D_i$, of course, coincide. Hence, the dashed arrow (a restriction of $\Delta_{\texttt{conc}}$) in the above diagram is well-defined.

Extension by continuity (infinite sums)

Now, suppose that the ring A (containing $\mathbb{Q}$) is a field $\mathbf{k}$. Then

$$\Delta_{\sqcup\!\sqcup} : \mathbf{k}\langle\mathcal{X}\rangle \to \mathbf{k}\langle\mathcal{X}\rangle \otimes \mathbf{k}\langle\mathcal{X}\rangle \text{ and } \Delta_{\uplus} : \mathbf{k}\langle Y\rangle \to \mathbf{k}\langle Y\rangle \otimes \mathbf{k}\langle Y\rangle$$

are graded for the multidegree. Then $\Delta_{\uplus}$ is graded for the length. Their extension to the completions (*i.e.* $\mathbf{k}\langle\langle\mathcal{X}\rangle\rangle$ and[6] $\mathbf{k}\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle$) are continuous and then, when exist, commute with infinite sums. Hence[7],

$$\forall c \in \mathbf{k}, \quad \Delta_{\sqcup\!\sqcup}(cx)^* = \sum_{n\geq 0} c^n \Delta_{\sqcup\!\sqcup} x^n = \sum_{n\geq 0} c^n \sum_{j=0}^{n} \binom{n}{j} x^j \otimes x^{n-j}.$$

For $c \in \mathbb{N}_{\geq 2}$ which is neither a field nor a ring (containing $\mathbb{Q}$), we also get

$$(cx)^* = (c-1)^{-1} \sum_{a,b\in\mathbb{N}_{\geq 1}, a+b=c} (ax)^* \sqcup\!\sqcup (bx)^* \quad \in \mathbb{N}_{\geq 2}\langle\langle\mathcal{X}\rangle\rangle,$$

$$\Delta_{\sqcup\!\sqcup}(cx)^* \neq (c-1)^{-1} \sum_{a,b\in\mathbb{N}_{\geq 1}, a+b=c} (ax)^* \otimes (bx)^* \quad \in \mathbb{Q}\langle\langle\mathcal{X}\rangle\rangle \otimes \mathbb{Q}\langle\langle\mathcal{X}\rangle\rangle,$$

because $\langle \mathrm{LHS} | x \otimes 1_{\mathcal{X}^*}\rangle = c$, $\langle \mathrm{RHS} | x \otimes 1_{\mathcal{X}^*}\rangle = (c-1)^{-1} \sum_{a=1}^{c-1} a = c/2$.
For $c \in \mathbb{Z}$ (or even $\mathbb{Q}, \mathbb{R}, \mathbb{C}$), the such decomposition is not finite.

[6] $\mathbf{k}\langle\langle\mathcal{X}\rangle\rangle \otimes_{\mathbf{k}} \mathbf{k}\langle\langle\mathcal{X}\rangle\rangle$ embeds injectively in $\mathbf{k}\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle \cong [\mathbf{k}\langle\langle\mathcal{X}\rangle\rangle]\langle\langle\mathcal{X}\rangle\rangle$.
Indeed, $\mathbf{k}\langle\langle\mathcal{X}\rangle\rangle \otimes_{\mathbf{k}} \mathbf{k}\langle\langle\mathcal{X}\rangle\rangle$ contains the elements of the form $\sum_{i\in I \text{ finite}} G_i \otimes D_i$, for $(G_i, D_i) \in \mathbf{k}\langle\langle\mathcal{X}\rangle\rangle \times \mathbf{k}\langle\langle\mathcal{X}\rangle\rangle$. But since elements of $M \otimes N$ are finite combination of $m_i \otimes n_i, m_i \in M, n_i \in N$ then, for any $u, v \in \mathcal{X}^{\geq 1}$, $\sum_{i\geq 0} u^i \otimes v^i$ belongs to $\mathbf{k}\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle$ and does not belong to $\mathbf{k}\langle\langle\mathcal{X}\rangle\rangle \otimes_{\mathbf{k}} \mathbf{k}\langle\langle\mathcal{X}\rangle\rangle$.

[7] $\Delta_{\sqcup\!\sqcup} x^n = (\Delta_{\sqcup\!\sqcup} x)^n = (1_{\mathcal{X}^*} \otimes x + x \otimes 1_{\mathcal{X}^*})^n = \sum_{j=0}^{n} \binom{n}{j} x^j \otimes x^{n-j}$.

Representative series and Sweedler's dual

Theorem 2 (rational series)

Let $S \in A\langle\langle\mathcal{X}\rangle\rangle$. The following assertions are equivalent

1. *The series S belongs to $A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$.*
2. *There exists a linear representation (ν, μ, η) (of rank n) for S with $\nu \in M_{1,n}(A), \eta \in M_{n,1}(A)$ and a morphism of monoids $\mu : \mathcal{X}^* \to M_{n,n}(A)$ s.t. $S = \sum_{w \in \mathcal{X}^*} (\nu\mu(w)\eta) w$.*
3. *The shifts[8] $\{S \triangleleft w\}_{w \in \mathcal{X}^*}$ (resp. $\{w \triangleright S\}_{w \in \mathcal{X}^*}$) lie within a finitely generated shift-invariant A-module.*

Moreover, if A is a field $\mathbf{k}$, the previous assertions are equivalent to

4. *There exists $(G_i, D_i)_{i \in F\,finite}$ s.t. $\Delta_{\mathrm{conc}}(S) = \sum_{i \in F\,finite} G_i \otimes D_i$.*

Hence,

$$\begin{aligned} \mathcal{H}^\circ_{ш}(\mathcal{X}) &= (\mathbf{k}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle, ш, 1_{\mathcal{X}^*}, \Delta_{\mathrm{conc}}, \mathrm{e}), \\ (\textit{resp. } \mathcal{H}^\circ_{\text{⧢}}(Y) &= (\mathbf{k}^{\mathrm{rat}}\langle\langle Y\rangle\rangle, \text{⧢}, 1_{\mathcal{X}^*}, \Delta_{\mathrm{conc}}, \mathrm{e})). \end{aligned}$$

[8] The *left* (resp. *right*) shift of S by P is $P \triangleright S$ (resp. $S \triangleleft P$) defined by, for $w \in \mathcal{X}^*$, $\langle P \triangleright S | w\rangle = \langle S | wP\rangle$ (resp. $\langle S \triangleleft P | w\rangle = \langle S | Pw\rangle$).

Kleene stars of the plane and conc-characters

Theorem 3 (rational exchangeable series)

One has

1. *If the* $\mathbb{Q}$*-algebra* A *is a field* $\mathbf{k}$ *then, for any* $S \in \mathbf{k}\langle\langle\mathcal{X}\rangle\rangle$,
   $\Delta_{\mathtt{conc}}(S) = S \otimes S, \langle S|1_{\mathcal{X}^*}\rangle = 1 \iff S = (\sum_{x\in\mathcal{X}} c_x x)^*$ *with* $c_x \in \mathbf{k}$.
2. $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \subset A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \cap A_{\mathrm{exc}}^{\mathrm{synt}}\langle\langle\mathcal{X}\rangle\rangle$. *If* A *is a field then the equality holds and* $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle X\rangle\rangle = A^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \sqcup\!\sqcup A^{\mathrm{rat}}\langle\langle x_1\rangle\rangle$ *and, for the algebra of series over finite subalphabets* $A_{\mathrm{fin}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle = \cup_{F\subset_{finite} Y} A^{\mathrm{rat}}\langle\langle F\rangle\rangle$, *we get*[9]
   $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle \cap A_{\mathrm{fin}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle = \cup_{k\geq 0} A^{\mathrm{rat}}\langle\langle y_1\rangle\rangle \sqcup\!\sqcup \ldots \sqcup\!\sqcup A^{\mathrm{rat}}\langle\langle y_k\rangle\rangle \subsetneq A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle$.
3. $\forall x \in \mathcal{X}, A^{\mathrm{rat}}\langle\langle x\rangle\rangle = \{P(1-xQ)^{-1}\}_{P,Q\in A[x]}$. *If* $\mathbf{k}$ *is an algebraically closed field then* $\mathbf{k}^{\mathrm{rat}}\langle\langle x\rangle\rangle = \mathrm{span}_{\mathbf{k}}\{(ax)^* \sqcup\!\sqcup \mathbf{k}\langle x\rangle | a \in K\}$.
4. *If* A *is a* $\mathbb{Q}$*-algebra without zero divisors,* $\{x^*\}_{x\in\mathcal{X}}$ *(resp.* $\{y^*\}_{y\in Y}$*) are* conc*-character and are algebraically independent over* $(A\langle\mathcal{X}\rangle, \sqcup\!\sqcup, 1_{\mathcal{X}^*})$ *(resp.* $(A\langle Y\rangle, \uplus, 1_{Y^*})$*) within* $(A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle, \sqcup\!\sqcup, 1_{\mathcal{X}^*})$ *(resp.* $(A^{\mathrm{rat}}\langle\langle Y\rangle\rangle, \uplus, 1_{Y^*})$*).*

---

[9] The following identity lives in $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle$ but not in $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle \cap A_{\mathrm{fin}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle$,
$(y_1 + \ldots)^* = \lim_{k\to+\infty}(y_1 + \ldots + y_k)^* = \lim_{k\to+\infty} y_1^* \sqcup\!\sqcup \ldots \sqcup\!\sqcup y_k^* = \sqcup\!\sqcup_{k\geq 1} y_k$.

Triangular sub bialgebras of $(A^{\rm rat}\langle\langle X\rangle\rangle, ⧢, 1_{X^*}, \Delta_{\tt conc}, {\rm e})$

Let (ν,μ,η) be a linear representation of $R\in A^{\rm rat}\langle\langle X\rangle\rangle$ and $\mathcal{L}$ be the Lie algebra generated by $\{\mu(x)\}_{x\in X}$.
Let $M(x):=\mu(x)x$, for $x\in X$. Then $R=\nu M(X^*)\eta$. If $\{\mu(x)\}_{x\in X}$ are triangular then let $D(X)$ (resp. $N(X)$) be the diagonal (resp. nilpotent) letter matrix s.t. $M(X)=D(X)+N(X)$ then $M(X^*)=((D(X^*)T(X))^*D(X^*))$. Moreover, if $X=\{x_0,x_1\}$ then $M(X^*)=(M(x_1^*)M(x_0))^*M(x_1^*)=(M(x_0^*)M(x_1))^*M(x_0^*)$.

If A is an algabraically closed field, the modules generated by the following families are closed by ${\tt conc}$, ⧢ and coproducts :

$(F_0)\quad E_1x_1\ldots E_jx_1E_{j+1},\quad$ where $\quad E_k\in A^{\rm rat}\langle\langle x_0\rangle\rangle,$

$(F_1)\quad E_1x_0\ldots E_jx_0E_{j+1},\quad$ where $\quad E_k\in A^{\rm rat}\langle\langle x_1\rangle\rangle,$

$(F_2)\quad E_1x_{i_1}\ldots E_jx_{i_j}E_{j+1},\quad$ where $\quad E_k\in A^{\rm rat}_{\rm exc}\langle\langle X\rangle\rangle, x_{i_k}\in X.$

It follows then that

1. R is a linear combination of expressions in the form (F_0) (resp. (F_1)) iff $M(x_1^*)M(x_0)$ (resp. $M(x_0^*)M(x_1)$) is nilpotent,
2. R is a linear combination of expressions in the form (F_2) iff $\mathcal{L}$ is solvable. Thus, if $R\in A^{\rm rat}_{\rm exc}\langle\langle X\rangle\rangle ⧢ A\langle X\rangle$ then $\mathcal{L}$ is nilpotent.

Extended Ree's theorem

Let $S \in A\langle\langle Y \rangle\rangle$ (resp. $A\langle\langle \mathcal{X} \rangle\rangle$), A is a commutative ring containing $\mathbb{Q}$. The series S is said to be

1. a $\uplus$ (resp. $\mathtt{conc}$, $\sqcup\!\sqcup$)-character iff, for any $w, v \in Y^*$ (resp. $\mathcal{X}^*$), $\langle S|w\rangle\langle S|v\rangle = \langle S|w \uplus v\rangle$ (resp. $\langle S|wv\rangle, \langle S|w \sqcup\!\sqcup v\rangle$) and $\langle S|1\rangle = 1$.
2. an infinitesimal $\uplus$ (resp. $\mathtt{conc}$, $\sqcup\!\sqcup$)-character iff, for any $w, v \in Y^*$ (resp. $\mathcal{X}^*$), $\langle S|w \uplus v\rangle = \langle S|w\rangle\langle v|1_{Y^*}\rangle + \langle w|1_{Y^*}\rangle\langle S|v\rangle$ (resp. $\langle S|wv\rangle = \langle S|w\rangle\langle v|1_{\mathcal{X}^*}\rangle + \langle w|1_{\mathcal{X}^*}\rangle\langle S|v\rangle$, $\langle S|w \sqcup\!\sqcup v\rangle = \langle S|w\rangle\langle v|1_{\mathcal{X}^*}\rangle + \langle w|1_{\mathcal{X}^*}\rangle\langle S|v\rangle$).
3. a group-like element iff $\langle S|1_{\mathcal{X}^*}\rangle = 1$ and $\Delta_{\uplus} S = \Phi(S \otimes S)$ (resp. $\Delta_{\mathtt{conc}} S = \Phi(S \otimes S), \Delta_{\uplus} S = \Phi(S \otimes S)$).
4. a primitive element iff $\Delta_{\uplus} S = 1_{Y^*} \otimes S + S \otimes 1_{Y^*}$ (resp. $\Delta_{\mathtt{conc}} S = 1_{\mathcal{X}^*} \otimes S + S \otimes 1_{\mathcal{X}^*}, \Delta_{\sqcup\!\sqcup} S = 1_{\mathcal{X}^*} \otimes S + S \otimes 1_{\mathcal{X}^*}$).

Then the following assertions are equivalent

1. S is a $\uplus$ (resp. $\mathtt{conc}$ and $\sqcup\!\sqcup$)-character.
2. $\log S$ an infinitesimal $\uplus$ (resp. $\mathtt{conc}$ and $\sqcup\!\sqcup$)-character.
3. S is group-like, for $\Delta_{\uplus}$ (resp. $\Delta_{\mathtt{conc}}$ and $\Delta_{\sqcup\!\sqcup}$).
4. $\log S$ is primitive, for $\Delta_{\uplus}$ (resp. $\Delta_{\mathtt{conc}}$ and $\Delta_{\sqcup\!\sqcup}$).

ABEL LIKE THEOREMS VIA BIALGEBRAS

Chen series and $(NCDE)$

On $(A\langle\mathcal{X}\rangle, \mathtt{conc}, 1_{\mathcal{X}^*}, \Delta_{\sqcup\!\sqcup}, \varepsilon)$ and $(A\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\uplus}, \varepsilon)$, we also get

$$\mathcal{D}_{\mathcal{X}} := \sum_{w\in\mathcal{X}^*} w\otimes w = \sum_{w\in\mathcal{X}^*} S_w\otimes P_w = \prod_{l\in\mathcal{L}yn\mathcal{X}}^{\searrow} e^{S_l\otimes P_l},$$

$$\mathcal{D}_{Y} := \sum_{w\in Y^*} w\otimes w = \sum_{w\in Y^*} \Sigma_w\otimes \Pi_w = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\Sigma_l\otimes \Pi_l},$$

where $\{P_l\}_{l\in\mathcal{L}yn\mathcal{X}}$ (resp. $\{\Pi_l\}_{l\in\mathcal{L}ynY}$) is a basis of Lie algebra of primitive elements and $\{S_l\}_{l\in\mathcal{L}yn\mathcal{X}}$ (resp. $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$) is a transcendence basis of $(A\langle\mathcal{X}\rangle, \sqcup\!\sqcup, 1_{\mathcal{X}^*})$ (resp. $(A\langle Y\rangle, \uplus, 1_{Y^*})$).

The Chen series of $\{\omega_i\}_{i\geq 1}$ and along $z_0 \rightsquigarrow z$ is defined as follows

$$C_{z_0\rightsquigarrow z} := \sum_{w\in\mathcal{X}^*} \alpha_{z_0}^{z}(w)w = (\alpha_{z_0}^{z}\otimes \mathtt{Id})\mathcal{D}_{\mathcal{X}} = \prod_{l\in\mathcal{L}yn\mathcal{X}}^{\searrow} e^{\alpha_{z_0}^{z}(S_l)P_l}.$$

It belongs to $\mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle$ and satisfies the following equation[10]

$$(NCDE)\quad \mathbf{d}S = MS,\quad \text{with}^{11}\quad M = \sum_{x\in\mathcal{X}} u_x x.$$

[10] Considering $A = (\mathcal{H}(\Omega), \partial)$ as the differential ring of holomorphic functions on Ω, equipped 1_Ω as the neutral element, the differential ring $(\mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle, \mathbf{d})$ is defined, for any $S\in\mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle$, by $\mathbf{d}S = \sum_{w\in\mathcal{X}^*}(\partial\langle S|w\rangle)w \in \mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle$.

[11] For $\Delta_{\sqcup\!\sqcup}$, the multiplier M is primitive and the series $C_{z_0\rightsquigarrow z}$ is group-like.

Noncommutative generating series

$$\mathrm{L}(z) := \sum_{w\in X^*} \mathrm{Li}_w(z)w = (\mathrm{Li}_\bullet \otimes \mathrm{Id})\mathcal{D}_X = e^{-\log(1-z)x_1}\mathrm{L}_{\mathrm{reg}}(z)e^{\log(z)x_0},$$

$$\mathrm{H}(n) := \sum_{w\in Y^*} \mathrm{H}_w(n)w = (\mathrm{H}_\bullet \otimes \mathrm{Id})\mathcal{D}_Y = e^{\mathrm{H}_{y_1}(n)y_1}\mathrm{H}_{\mathrm{reg}}(n),$$

where $\mathrm{L}_{\mathrm{reg}} := \prod^{\searrow}_{l\in\mathcal{L}ynX\setminus X} e^{\mathrm{Li}_{S_l} P_l}$ and $\mathrm{H}_{\mathrm{reg}} := \prod^{\searrow}_{l\in\mathcal{L}ynY\setminus\{y_1\}} e^{\mathrm{H}_{\Sigma_l}\Pi_l}$.

We put also[12]

$$Z_{ш} := \mathrm{L}_{\mathrm{reg}}(1) = \prod^{\searrow}_{\substack{l\in\mathcal{L}ynX\\ l\neq x_0,x_1}} e^{\zeta(S_l)P_l} \quad\text{and}\quad Z_{ш\!\!\!+} := \mathrm{H}_{\mathrm{reg}}(+\infty) = \prod^{\searrow}_{\substack{l\in\mathcal{L}ynY\\ l\neq y_1}} e^{\zeta(\Sigma_l)\Pi_l}.$$

L satisfies[13]

$$(DE)\quad \mathbf{d}S = \left(\frac{x_0}{z} + \frac{x_1}{1-z}\right)S \quad\text{and}^{14}\ \mathrm{L}(z) \sim_0 e^{x_0\log(z)}.$$

L and $Z_{ш}$ (resp. H and $Z_{ш\!\!\!+}$) are group-like, for $\Delta_{ш}$ (resp. $\Delta_{ш\!\!\!+}$).

[12] The polynomials S_l and P_l (resp. Σ_l and Π_l) are homogenous in weight and $\zeta(S_l)$ (resp. $\zeta(\Sigma_l)$) is convergent, for $l \in \mathcal{L}ynX \setminus X$ (resp. $\mathcal{L}ynY \setminus \{y_1\}$).

[13] For $x_0 = A/2\mathrm{i}\pi$ and $x_1 = -B/2\mathrm{i}\pi$, (DE) is nothing else (KZ_3) and $Z_{ш}$ corresponds to the Drindfel'd associator, Φ_{KZ}.

[14] A Drindfel'd asymptotic condition for (KZ_3).

Gradation of L and $Z_{⧢}$

Let $\mathcal{J}$ be the Lie ideal freely generated by $\{\mathrm{ad}_{x_0}^l x_1\}_{l\geq 0}$. Let the operation $\circ$ be defined by $x_1x_0^l \circ P = x_1(x_0^l ⧢ P)$, for $l \in \mathbb{N}, P \in \mathbb{C}\langle X\rangle$. Then[15]

$$\begin{aligned}
\mathrm{L}(z) &= \sum_{k\geq 0}\sum_{w\in x_0^* ⧢ x_1^k} \mathrm{Li}_w(z)w \\
&= e^{x_0\log(z)}\Big(1_{X^*} + \sum_{k\geq 1}\sum_{l_1,\cdots,l_k\geq 0} \mathrm{Li}_{x_1x_0^{l_1}\circ\cdots\circ x_1x_0^{l_k}}(z)\prod_{i=1}^{k}\mathrm{ad}_{-x_0}^{l_i}x_1\Big) \\
&= \sum_{k\geq 0}\int_0^z \omega_1(t_k)\cdots\int_0^{t_{k-1}}\omega_1(t_1)\kappa_k(z,t_1,\cdots,t_k),
\end{aligned}$$

where, for any $k\geq 0, \kappa_k(z,t_1,\cdots,t_k)$ is the formal power series given by

$$\begin{aligned}
\kappa_k(z,t_1,\cdots,t_k) &= e^{x_0[\log(z)-\log(t_1)]}x_1\cdots e^{x_0[\log(t_{k-1})-\log(t_k)]}x_1 e^{x_0\log(t_k)} \\
&= e^{x_0\log(z)}e^{\mathrm{ad}_{-x_0}\log(t_1)}x_1\cdots e^{\mathrm{ad}_{-x_0}\log(t_k)}x_1 \\
&= e^{x_0\log(z)}\sum_{l_1,\cdots,l_k\geq 0}\prod_{i=1}^{k}\frac{\log^{l_i}(t_i)}{l_i!}\mathrm{ad}_{-x_0}^{l_i}x_1.
\end{aligned}$$

$$Z_{⧢} = \sum_{k\geq 0}\sum_{l_1,\cdots,l_k\geq 0}\zeta_{⧢}(x_1x_0^{l_1}\circ\cdots\circ x_1x_0^{l_k})\prod_{i=0}^{k}\mathrm{ad}_{-x_0}^{l_i}x_1.$$

---

[15] Since $\mathrm{Li}_\bullet$ is injective then $\mathcal{U}(\mathcal{J})$ (resp. $\mathcal{U}(\mathcal{J})^\vee$) is freely generated by $\{\mathrm{ad}_{-x_0}^{l_1}x_1\cdots\mathrm{ad}_{-x_0}^{l_k}x_1\}_{k\geq 0}^{l_1,\cdots,l_k\geq 0}$ (resp. $\{x_1x_0^{l_1}\circ\cdots\circ x_1x_0^{l_k}\}_{k\geq 0}^{l_1,\cdots,l_k\geq 0}$) and one has $\mathrm{supp}(x_1x_0^{l_1}\circ\cdots\circ x_1x_0^{l_k}) = \{w\in x_1X^* \mid |w|_{x_1}=k, |w|_{x_0}=l_1+\cdots+l_k\}$.

More about generating series

Let $\gamma_\bullet$ be the character on $(\mathbb{C}\langle Y\rangle, ⧢, 1_{Y^*})$ defined by $\gamma_{1_{Y^*}} = 1$ and[16]
$\forall l \in \mathcal{L}ynY, \quad \gamma_{\Sigma_l} := \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_{\Sigma_l}(n) = \zeta(\Sigma_l), \quad \{n^a \log^b(n)\}_{a\in\mathbb{Z}_{\le -1}, b\in\mathbb{N}}.$

$$Z_\gamma := \sum_{w\in Y^*} \gamma_w w = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\gamma_{\Sigma_l}\Pi_l} = e^{\gamma y_1} Z_{⧢}.$$

Let us consider

$$\mathtt{Mono}(z) := \sum_{n\ge 0} \mathrm{P}_{y_1^n} y_1^n \in \mathcal{H}(\Omega)\langle\langle y_1\rangle\rangle \quad \text{and} \quad \mathtt{Const} := \sum_{k\ge 0} \mathrm{H}_{y_1^k} y_1^k.$$

Then[17]

$$\mathtt{Mono}(z) = (1-z)^{-1} e^{-\log(1-z)y_1} \quad \text{and}^{18} \quad \mathtt{Const} = \exp\left(-\sum_{k\ge 0} \mathrm{H}_{y_k} \frac{(-y_1)^k}{k}\right).$$

Let us also consider[19]

$$B'(y_1) := \exp\left(\sum_{k\ge 2} \zeta(k) \frac{(-y_1)^k}{k}\right) \text{ and } B(y_1) := \exp\left(\gamma y_1 - \sum_{k\ge 2} \zeta(k) \frac{(-y_1)^k}{k}\right).$$

---

[16] In particular, $\gamma_{\Sigma_{y_1}} = \gamma_{y_1} = \gamma$.
[17] Because $\mathrm{P}_{y_1^k}(z) = (1-z)^{-1}\,\mathrm{Li}_{x_1^k}(z)$ with $\mathrm{Li}_{x_1^k}(z) = (-\log(1-z))^k/k!$, $k \ge 1$.
[18] By Newton-Girard identity, or by $(ty_k)^* = \exp_{⧢}(-\sum_{n\ge 0} y_{nk}(-t)^n/n)$, $k \ge 1$.
Note also that $\mathtt{Const}^{-1} = \sum_{n\ge 0} \mathrm{H}_{y_1^n}(-y_1)^n = \exp(\sum_{k\ge 0} \mathrm{H}_{y_k}(-y_1)^k/k)$.
[19] $B'(y_1)$ corresponds to the Ecalle's mould Mono. $\mathbb{C}\langle\langle y_1\rangle\rangle \ni B(y_1) = \Gamma^{-1}(1+y_1)$.

Chen series of ω_0 and ω_1 along a path $z_0 \rightsquigarrow z$

$$C_{z_0 \rightsquigarrow z} := \sum_{w \in X^*} \alpha_{z_0}^{z}(w) w \quad \text{with} \quad \begin{cases} \omega_0(z) = z^{-1} dz, \\ \omega_1(z) = (1-z)^{-1} dz. \end{cases}$$

Here, $C_{z_0 \rightsquigarrow z}$ is also solution[20] of (DE).

Let g be the transformation $z \mapsto 1 - z$. Then $g^* \omega_0 = -\omega_1$ and $g^* \omega_1 = -\omega_0$. Hence,

$$C_{g(z_0) \rightsquigarrow g(z)} = \sum_{w \in X^*} \alpha_{g(z_0)}^{g(z)}(w) w = \sum_{w \in X^*} \alpha_{z_0}^{z}(w) \sigma(w) = \sigma(C_{z_0 \rightsquigarrow z}),$$

where σ is the morphism defined by $\sigma(x_0) = -x_1$ and $\sigma(x_1) = -x_0$.

On the other hand, one has

$$\mathrm{L}(z) = C_{z_0 \rightsquigarrow z} \mathrm{L}(z_0) \quad \text{and} \quad \mathrm{L}(g(z)) = C_{g(z_0) \rightsquigarrow g(z)} \mathrm{L}(g(z_0)).$$

Since $\mathrm{L}(z) \sim_0 e^{x_0 \log(z)}$ then

$$C_{g(z_0) \rightsquigarrow g(z)} = \sigma(\mathrm{L}(z) \mathrm{L}^{-1}(z_0)) \sim_{z_0 \to 0} \sigma(\mathrm{L}(z)) e^{x_1 \log(z_0)}.$$

Proposition 1

Let σ be the letter morphism s.t. $\sigma(x_0) = -x_1$ and $\sigma(x_1) = -x_0$. Then

$$\mathrm{L}(1-z) = \sigma(\mathrm{L}(z)) Z_{\sqcup\!\sqcup}.$$

[20] It can be obtained by a convergent Picard iteration, for a discrete topology, initialized at $\langle C_{z_0 \rightsquigarrow z} | 1_{\mathcal{X}^*} \rangle = 1_\Omega 1_{\mathcal{X}^*}$.

Abel like results and bridge equations

Since[21] $\mathrm{L}(z) = \sigma(\mathrm{L}(1-z))Z_{ш} = e^{x_0\log(z)}\sigma(\mathrm{L}_{\mathtt{reg}}(1-z))e^{-x_1\log(1-z)}Z_{ш}$
then[22] $\mathrm{L}(z) \sim_1 e^{-x_1\log(1-z)}Z_{ш}$ and then $\mathrm{H}(n) \sim_{+\infty} \mathtt{Const(n)}\pi_Y Z_{ш}$.

Theorem 4 (first Abel like theorem)

$$\lim_{z\to 1} e^{y_1\log(1-z)}\pi_Y\mathrm{L}(z) = \pi_Y Z_{ш} = \lim_{n\to\infty} \mathtt{Const(n)}^{-1}\mathrm{H}(n).$$

Corollary 5 (bridge equations)

$$Z_\gamma = B(y_1)\pi_Y Z_{ш} \iff Z_{⧢} = B'(y_1)\pi_Y Z_{ш}.$$

Remark 1

On the one hand, by identification coefficients, for $w \in X^*x_1$,
$\zeta_{ш}(w) = \langle Z_{ш}|w\rangle = \mathrm{f.p.}_{z\to 1}\mathrm{Li}_w(z)$, $\{(1-z)^a\log^b(1-z)\}_{a\in\mathbb{Z}_{\le -1},b\in\mathbb{N}}$.
On the other hand, by an ⧢-modified Radford theorem, for $w \in Y^*$,
$\zeta_{⧢}(w) = \langle Z_{⧢}|w\rangle = \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_w(n)$, $\{n^a\mathrm{H}_1^b(n)\}_{a\in\mathbb{Z}_{\le -1},b\in\mathbb{N}}$.
In particular[23], $\zeta_{ш}(x_1) = \zeta_{⧢}(y_1) = 0$.

[21] By Hoffman's duality, *i.e.* $\zeta(\rho(\tilde{w})) = \zeta(w)$ (where ρ is the morphism defined by $\rho(x_0) = x_1, \rho(x_1) = x_0$ and $\tilde{w}$ is mirror of w), we get $\sigma(Z_{ш}^{-1}) = Z_{ш}$.

[22] An another Drindfel'd asymptotic condition for (KZ_3).

[23] These coefficients of singular and asymptotic expansions can be changed if we use other comparison scales.

Cloned Abel like results and cloned bridge equations

Let $e^C \in \mathrm{Gal}_{\mathbb{C}}(DE) = \{e^C\}_{C \in \mathcal{L}ie_{\mathbb{C}}\langle\langle X\rangle\rangle}$ and $\overline{\mathrm{L}} := \mathrm{L}e^C, \overline{Z}_{\sqcup\!\sqcup} := Z_{\sqcup\!\sqcup}\, e^C$.
Hence, $\overline{\mathrm{L}}(z) \sim_1 e^{-x_1 \log(1-z)}\overline{Z}_{\sqcup\!\sqcup}$ and then $\overline{\mathrm{H}}(n) \sim_{+\infty} \mathtt{Const(n)}\pi_Y \overline{Z}_{\sqcup\!\sqcup}$.

Theorem 6 (cloned first Abel like theorem)

$$\lim_{z\to 1} e^{y_1 \log(1-z)}\pi_Y \overline{\mathrm{L}}(z) = \pi_Y \overline{Z}_{\sqcup\!\sqcup} = \lim_{n\to\infty} \mathtt{Const(n)}^{-1}\overline{\mathrm{H}}(n).$$

If[24] $\overline{Z}_{\sqcup\!\sqcup} \in dm(A) := \{Z_{\sqcup\!\sqcup}\, e^C \mid C \in \mathcal{L}ie_A\langle\langle X\rangle\rangle, \langle e^C|x_0\rangle = \langle e^C|x_1\rangle = 0\}$
then[25] $\overline{Z}_\gamma = e^{\gamma y_1}\overline{Z}_{\,\uplus}$ and it follows that

Corollary 7 (cloned bridge equations)

If $\overline{Z}_{\sqcup\!\sqcup} \in dm(A)$ *then* $(\overline{Z}_\gamma = B(y_1)\pi_Y \overline{Z}_{\sqcup\!\sqcup} \iff \overline{Z}_{\,\uplus} = B'(y_1)\pi_Y\overline{Z}_{\sqcup\!\sqcup})$.

Remark 2

The local coordinates of $\overline{Z}_{\,\uplus}$ *and* $\overline{Z}_{\sqcup\!\sqcup}$ *are homogenous polynomial on convergent polyzetas, with coefficients in* A. *Hence, if* $\gamma \notin A$ *then* γ *is transcendent over the* A*-algebra generated by convergent polyzetas.*

---

[24] $dm(A)$ contains $DM(A)$ introduced by P. Cartier and G. Racinet and it is a strict normal subgroup of $\mathrm{Gal}_A(DE)$ (recall that $\mathbb{Q} \subset A \subset \mathbb{C}$).

[25] For $w \in Y^*$, one has $\langle \overline{Z}_{\,\uplus}|w\rangle = \mathrm{f.p.}_{n\to+\infty}\overline{\mathrm{H}}_w(n)$, $\{n^a \mathrm{H}_1^b(n)\}_{a\in\mathbb{Z}_{\le -1}, b\in\mathbb{N}}$ and $\langle \overline{Z}_\gamma|w\rangle = \mathrm{f.p.}_{n\to+\infty}\overline{\mathrm{H}}_w(n)$, $\{n^a \log^b(n)\}_{a\in\mathbb{Z}_{\le -1}, b\in\mathbb{N}}$.

COMPUTATIONAL EXAMPLES[26]

[26] Examples, in the sequel, use maple packages developed in the PhD theses of C. Bùi (2016), C. Costermans (2008) and H. Ngô (2016).

Generalized Euler's gamma constant

Identifying the coefficients of $y_1^k w, w \in X^*, k \in \mathbb{N}$ in $Z_\gamma = B(y_1)\pi_Y Z_{ш}$, one has

1. $$\gamma_{y_1^k} = \sum_{\substack{s_1,\ldots,s_k>0 \\ s_1+\ldots+ks_k=k}} \frac{(-1)^k}{s_1!\ldots s_k!}(-\gamma)^{s_1}\left(-\frac{\zeta(2)}{2}\right)^{s_2}\ldots\left(-\frac{\zeta(k)}{k}\right)^{s_k}.$$

Example 8

$$\gamma_{1,1} = \frac{1}{2}(\gamma^2 - \zeta(2)), \qquad \gamma_{1,1,1} = \frac{1}{6}(\gamma^3 - 3\zeta(2)\gamma + 2\zeta(3)).$$

2. $$\gamma_{y_1^k w} = \sum_{i=0}^{k} \frac{\zeta(x_0(-x_1)^{k-i} ш \pi_X w])}{i!}\left(\sum_{j=1}^{i} b_{i,j}(\gamma, -\zeta(2), 2\zeta(3), \ldots)\right),$$

where $k \in \mathbb{N}_+, w \in Y^+$ and $b_{n,k}(t_1, \ldots, t_k)$ are Bell polynomials.

Example 9

$$\begin{aligned}
\gamma_{1,7} &= \zeta(7)\gamma + \zeta(3)\zeta(5) - \tfrac{54}{175}\zeta(2)^4, \\
\gamma_{1,1,6} &= \frac{4}{35}\zeta(2)^3\gamma^2 + (\zeta(2)\zeta(5) + \frac{2}{5}\zeta(3)\zeta(2)^2 - 4\zeta(7))\gamma \\
&\quad + \zeta(6,2) + \frac{19}{35}\zeta(2)^4 + \frac{1}{2}\zeta(2)\zeta(3)^2 - 4\zeta(3)\zeta(5).
\end{aligned}$$

Rewriting rules and irreducible local coordinates

$\mathcal{Z}_{irr}^{\infty}(Y) := \{\}$ and $\mathcal{Z}_{irr}^{\infty}(X) := \{\}$;
$\mathcal{L}_{irr}^{\infty}(Y) := \{\}$ and $\mathcal{L}_{irr}^{\infty}(X) := \{\}$;
$\mathcal{R}_{irr}(Y) := \{\}$ and $\mathcal{R}_{irr}(X) := \{\}$;
for p range in $2, \ldots, \infty$ do
  for l range in the totally ordered[27] $\mathcal{L}yn^p(\mathcal{X})$ do
    identify the coefficients of Π_l in $Z_\gamma = B(y_1)\pi_Y Z_{\sqcup\!\sqcup}$;
    identify the coefficients of P_l in $\pi_X Z_\gamma = B(x_1) Z_{\sqcup\!\sqcup}$
  end_for;
  for l range in the totally ordered $\mathcal{L}yn^p(\mathcal{X})$ do
    express the local coordinate $\zeta(\Sigma_l)$ as rewriting rule;
    if $\zeta(\Sigma_l) \to \zeta(\Sigma_l)$
      then $\mathcal{Z}_{irr}^{\infty}(Y) := \mathcal{Z}_{irr}^{\infty}(Y) \cup \{\zeta(\Sigma_l)\}$ and $\mathcal{L}_{irr}^{\infty}(Y) := \mathcal{L}_{irr}^{\infty}(Y) \cup \{\Sigma_l\}$
      else $\mathcal{R}_{irr}(Y) := \mathcal{R}_{irr}(Y) \cup \{\Sigma_l \to \Upsilon_l\}$;
    express the local coordinate $\zeta(S_l)$ as rewriting rule;
    if $\zeta(S_l) \to \zeta(S_l)$
      then $\mathcal{Z}_{irr}^{\infty}(X) := \mathcal{Z}_{irr}^{\infty}(X) \cup \{\zeta(S_l)\}$ and $\mathcal{L}_{irr}^{\infty}(X) := \mathcal{L}_{irr}^{\infty}(X) \cup \{S_l\}$
      else $\mathcal{R}_{irr}(X) := \mathcal{R}_{irr}(X) \cup \{S_l \to U_l\}$
  end_for
end_for

[27] $\mathcal{L}yn^p(\mathcal{X})$ denotes the set of Lyndon words over $\mathcal{X}$ of weight p.

Homogenous polynomials relations[28] on local coordinates

Identifying the local coordinates in $Z_\gamma = B(y_1)\pi_Y Z_{\sqcup\!\sqcup}$, one has

	Polynomial relations on $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$	Polynomial relations on $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1}) = \frac{3}{2}\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0x_1^2}) = \zeta(S_{x_0^2x_1})$
4	$\zeta(\Sigma_{y_4}) = \frac{2}{5}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^3x_1}) = \frac{2}{5}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_3y_1}) = \frac{3}{10}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^2x_1^2}) = \frac{1}{10}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_2y_1^2}) = \frac{2}{3}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0x_1^3}) = \frac{2}{5}\zeta(S_{x_0x_1})^2$
5	$\zeta(\Sigma_{y_3y_2}) = 3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - 5\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^3x_1^2}) = -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_4y_1}) = -\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{2}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1x_0x_1}) = -\frac{3}{2}\zeta(S_{x_0^4x_1}) + \zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})$
	$\zeta(\Sigma_{y_2^2y_1}) = \frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - \frac{25}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1^3}) = -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_3y_1^2}) = \frac{5}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1x_0x_1^2}) = \frac{1}{2}\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_2y_1^3}) = \frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{4}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1^4}) = \zeta(S_{x_0^4x_1})$
6	$\zeta(\Sigma_{y_6}) = \frac{8}{35}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^5x_1}) = \frac{8}{35}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_2}) = \zeta(\Sigma_{y_3})^2 - \frac{4}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^4x_1^2}) = \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_5y_1}) = \frac{2}{7}\zeta(\Sigma_{y_2})^3 - \frac{1}{2}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1x_0x_1}) = \frac{4}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_3y_1y_2}) = -\frac{17}{30}\zeta(\Sigma_{y_2})^3 + \frac{9}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1^3}) = \frac{23}{70}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_2y_1}) = 3\zeta(\Sigma_{y_3})^2 - \frac{9}{10}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^2x_1x_0x_1^2}) = \frac{2}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_1^2}) = \frac{3}{10}\zeta(\Sigma_{y_2})^3 - \frac{3}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^2x_0x_1}) = -\frac{89}{210}\zeta(S_{x_0x_1})^3 + \frac{3}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2^2y_1^2}) = \frac{11}{63}\zeta(\Sigma_{y_2})^3 - \frac{1}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^4}) = \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_1^3}) = \frac{1}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0x_1x_0x_1^3}) = \frac{8}{21}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2y_1^4}) = \frac{17}{50}\zeta(\Sigma_{y_2})^3 + \frac{3}{16}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0x_1^5}) = \frac{8}{35}\zeta(S_{x_0x_1})^3$

[28] These polynomials relations are independent from γ and similarly for the case where the ring of their coefficients is the ring A.

Homogenous polynomials generating inside $\ker\zeta$

	$\{Q_l\}_{l\in\mathcal{L}ynY-\{y_1\}}$	$\{Q_l\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1}-\frac{3}{2}\Sigma_{y_3})=0$	$\zeta(S_{x_0x_1^2}-S_{x_0^2x_1})=0$
4	$\zeta(\Sigma_{y_4}-\frac{2}{5}\Sigma_{y_2}^{\uplus 2})=0$ $\zeta(\Sigma_{y_3y_1}-\frac{3}{10}\Sigma_{y_2}^{\uplus 2})=0$ $\zeta(\Sigma_{y_2y_1^2}-\frac{2}{3}\Sigma_{y_2}^{\uplus 2})=0$	$\zeta(S_{x_0^3x_1}-\frac{2}{5}S_{x_0x_1}^{\sqcup\!\sqcup 2})=0$ $\zeta(S_{x_0^2x_1^2}-\frac{1}{10}S_{x_0x_1}^{\sqcup\!\sqcup 2})=0$ $\zeta(S_{x_0x_1^3}-\frac{2}{5}S_{x_0x_1}^{\sqcup\!\sqcup 2})=0$
5	$\zeta(\Sigma_{y_3y_2}-3\Sigma_{y_3}\uplus\Sigma_{y_2}-5\Sigma_{y_5})=0$ $\zeta(\Sigma_{y_4y_1}-\Sigma_{y_3}\uplus\Sigma_{y_2})+\frac{5}{2}\Sigma_{y_5})=0$ $\zeta(\Sigma_{y_2^2y_1}-\frac{3}{2}\Sigma_{y_3}\uplus\Sigma_{y_2}-\frac{25}{12}\Sigma_{y_5})=0$ $\zeta(\Sigma_{y_3y_1^2}-\frac{5}{12}\Sigma_{y_5})=0$ $\zeta(\Sigma_{y_2y_1^3}-\frac{1}{4}\Sigma_{y_3}\uplus\Sigma_{y_2})+\frac{5}{4}\Sigma_{y_5})=0$	$\zeta(S_{x_0^3x_1^2}-S_{x_0^2x_1}\sqcup\!\sqcup S_{x_0x_1}+2S_{x_0^4x_1})=0$ $\zeta(S_{x_0^2x_1x_0x_1}-\frac{3}{2}S_{x_0^4x_1}+S_{x_0^2x_1}\sqcup\!\sqcup S_{x_0x_1})=0$ $\zeta(S_{x_0^2x_1^3}-S_{x_0^2x_1}\sqcup\!\sqcup S_{x_0x_1}+2S_{x_0^4x_1})=0$ $\zeta(S_{x_0x_1x_0x_1^2}-\frac{1}{2}S_{x_0^4x_1})=0$ $\zeta(S_{x_0x_1^4}-S_{x_0^4x_1})=0$
6	$\zeta(\Sigma_{y_6}-\frac{8}{35}\Sigma_{y_2}^{\uplus 3})=0$ $\zeta(\Sigma_{y_4y_2}-\Sigma_{y_3}^{\uplus 2}-\frac{4}{21}\Sigma_{y_2}^{\uplus 3})=0$ $\zeta(\Sigma_{y_5y_1}-\frac{2}{7}\Sigma_{y_2}^{\uplus 3}-\frac{1}{2}\Sigma_{y_3}^{\uplus 2})=0$ $\zeta(\Sigma_{y_3y_1y_2}-\frac{17}{30}\Sigma_{y_2}^{\uplus 3}+\frac{9}{4}\Sigma_{y_3}^{\uplus 2})=0$ $\zeta(\Sigma_{y_3y_2y_1}-3\Sigma_{y_3}^{\uplus 2}-\frac{9}{10}\Sigma_{y_2}^{\uplus 3})=0$ $\zeta(\Sigma_{y_4y_1^2}-\frac{3}{10}\Sigma_{y_2}^{\uplus 2}-\frac{3}{4}\Sigma_{y_3}^{\uplus 2})=0$ $\zeta(\Sigma_{y_2^2y_1^2}-\frac{11}{63}\Sigma_{y_2}^{\uplus 2}-\frac{1}{4}\Sigma_{y_3}^{\uplus 2})=0$ $\zeta(\Sigma_{y_3y_1^3}-\frac{1}{21}\Sigma_{y_2}^{\uplus 3})=0$ $\zeta(\Sigma_{y_2y_1^4}-\frac{17}{50}\Sigma_{y_2}^{\uplus 3}+\frac{3}{16}\Sigma_{y_3}^{\uplus 2})=0$	$\zeta(S_{x_0^5x_1}-\frac{8}{35}S_{x_0x_1}^{\sqcup\!\sqcup 3})=0$ $\zeta(S_{x_0^4x_1^2}-\frac{6}{35}S_{x_0x_1}^{\sqcup\!\sqcup 3}-\frac{1}{2}S_{x_0^2x_1}^{\sqcup\!\sqcup 2})=0$ $\zeta(S_{x_0^3x_1x_0x_1}-\frac{4}{105}S_{x_0x_1}^{\sqcup\!\sqcup 3})=0$ $\zeta(S_{x_0^3x_1^3}-\frac{23}{70}S_{x_0x_1}^{\sqcup\!\sqcup 3}-S_{x_0^2x_1}^{\sqcup\!\sqcup 2})=0$ $\zeta(S_{x_0^2x_1x_0x_1^2}-\frac{2}{105}S_{x_0x_1}^{\sqcup\!\sqcup 3})=0$ $\zeta(S_{x_0^2x_1^2x_0x_1}-\frac{89}{210}S_{x_0x_1}^{\sqcup\!\sqcup 3}+\frac{3}{2}S_{x_0^2x_1}^{\sqcup\!\sqcup 2})=0$ $\zeta(S_{x_0^2x_1^4}-\frac{6}{35}S_{x_0x_1}^{\sqcup\!\sqcup 3}-\frac{1}{2}S_{x_0^2x_1}^{\sqcup\!\sqcup 2})=0$ $\zeta(S_{x_0x_1x_0x_1^3}-\frac{8}{21}S_{x_0x_1}^{\sqcup\!\sqcup 3}-S_{x_0^2x_1}^{\sqcup\!\sqcup 2})=0$ $\zeta(S_{x_0x_1^5}-\frac{8}{35}S_{x_0x_1}^{\sqcup\!\sqcup 3})=0$

One has $\mathcal{R}_{\mathcal{X}}\subseteq\ker\zeta$, where $\begin{cases}\mathcal{R}_Y := (\mathrm{span}_{\mathbb{Q}}\{Q_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}},\uplus,1_{Y^*}),\\ \mathcal{R}_X := (\mathrm{span}_{\mathbb{Q}}\{Q_l\}_{l\in\mathcal{L}ynX\setminus X},\sqcup\!\sqcup,1_{X^*}).\end{cases}$

Noetherian rewriting system & irreducible coordinates[29]

	Rewriting among $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$	Rewriting among $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1}) \to \frac{3}{2}\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0x_1^2}) \to \zeta(S_{x_0^2x_1})$
4	$\zeta(\Sigma_{y_4}) \to \frac{2}{5}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^3x_1}) \to \frac{2}{5}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_3y_1}) \to \frac{3}{10}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^2x_1^2}) \to \frac{1}{10}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_2y_1^2}) \to \frac{2}{3}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0x_1^3}) \to \frac{2}{5}\zeta(S_{x_0x_1})^2$
5	$\zeta(\Sigma_{y_3y_2}) \to 3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - 5\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^3x_1^2}) \to -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_4y_1}) \to -\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{2}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1x_0x_1}) \to -\frac{3}{2}\zeta(S_{x_0^4x_1}) + \zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})$
	$\zeta(\Sigma_{y_2^2y_1}) \to \frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - \frac{25}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1^3}) \to -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_3y_1^2}) \to \frac{5}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1x_0x_1^2}) \to \frac{1}{2}\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_2y_1^3}) \to \frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{4}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1^4}) \to \zeta(S_{x_0^4x_1})$
6	$\zeta(\Sigma_{y_6}) \to \frac{8}{35}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^5x_1}) \to \frac{8}{35}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_2}) \to \zeta(\Sigma_{y_3})^2 - \frac{4}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^4x_1^2}) \to \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_5y_1}) \to \frac{2}{7}\zeta(\Sigma_{y_2})^3 - \frac{1}{2}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1x_0x_1}) \to \frac{4}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_3y_1y_2}) \to -\frac{17}{30}\zeta(\Sigma_{y_2})^3 + \frac{9}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1^3}) \to \frac{23}{70}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_2y_1}) \to 3\zeta(\Sigma_{y_3})^2 - \frac{9}{10}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^2x_1x_0x_1^2}) \to \frac{2}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_1^2}) \to \frac{3}{10}\zeta(\Sigma_{y_2})^3 - \frac{3}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^2x_0x_1}) \to -\frac{89}{210}\zeta(S_{x_0x_1})^3 + \frac{3}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2^2y_1^2}) \to \frac{11}{63}\zeta(\Sigma_{y_2})^3 - \frac{1}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^4}) \to \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_1^3}) \to \frac{1}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0x_1x_0x_1^3}) \to \frac{8}{21}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2y_1^4}) \to \frac{17}{50}\zeta(\Sigma_{y_2})^3 + \frac{3}{16}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0x_1^5}) \to \frac{8}{35}\zeta(S_{x_0x_1})^3$

$$\mathcal{Z}_{irr}^{\leq 2}(\mathcal{X}) \subset \cdots \mathcal{Z}_{irr}^{\leq p}(\mathcal{X}) \subset \cdots \subset \mathcal{Z}_{irr}^{\infty}(\mathcal{X}) = \cup_{p\geq 2}\mathcal{Z}_{irr}^{\leq p}(\mathcal{X}).$$

[29] The set of irreducible local coordinates forms algebraic generator system for $\mathcal{Z}$.

Noetherian rewriting system & totally ordered $\mathcal{L}_{irr}^{\infty}(\mathcal{X})$

	Rewriting among $\{\Sigma_l\}_{l\in\mathcal{L}ynY-\{y_1\}}$	Rewriting among $\{S_l\}_{\mathcal{L}ynX-X}$
3	$\Sigma_{y_2y_1} \to \frac{3}{2}\Sigma_{y_3}$	$S_{x_0x_1^2} \to S_{x_0^2x_1}$
4	$\Sigma_{y_4} \to \frac{2}{5}\Sigma_{y_2}^2$	$S_{x_0^3x_1} \to \frac{2}{5}S_{x_0x_1}^2$
	$\Sigma_{y_3y_1} \to \frac{3}{10}\Sigma_{y_2}^2$	$S_{x_0^2x_1^2} \to \frac{1}{10}S_{x_0x_1}^2$
	$\Sigma_{y_2y_1^2} \to \frac{2}{3}\Sigma_{y_2}^2$	$S_{x_0x_1^3} \to \frac{2}{5}S_{x_0x_1}^2$
5	$\Sigma_{y_3y_2} \to 3\Sigma_{y_3}\Sigma_{y_2} - 5\Sigma_{y_5}$	$S_{x_0^3x_1^2} \to -S_{x_0^2x_1}S_{x_0x_1} + 2S_{x_0^4x_1}$
	$\Sigma_{y_4y_1} \to -\Sigma_{y_3}\Sigma_{y_2} + \frac{5}{2}\Sigma_{y_5}$	$S_{x_0^2x_1x_0x_1} \to -\frac{3}{2}S_{x_0^4x_1} + S_{x_0^2x_1}S_{x_0x_1}$
	$\Sigma_{y_2^2y_1} \to \frac{3}{2}\Sigma_{y_3}\Sigma_{y_2} - \frac{25}{12}\Sigma_{y_5}$	$S_{x_0^2x_1^3} \to -S_{x_0^2x_1}S_{x_0x_1} + 2S_{x_0^4x_1}$
	$\Sigma_{y_3y_1^2} \to \frac{5}{12}\Sigma_{y_5}$	$S_{x_0x_1x_0x_1^2} \to \frac{1}{2}S_{x_0^4x_1}$
	$\Sigma_{y_2y_1^3} \to \frac{1}{4}\Sigma_{y_3}\Sigma_{y_2} + \frac{5}{4}\Sigma_{y_5}$	$S_{x_0x_1^4} \to S_{x_0^4x_1}$
6	$\Sigma_{y_6} \to \frac{8}{35}\Sigma_{y_2}^3$	$S_{x_0^5x_1} \to \frac{8}{35}S_{x_0x_1}^3$
	$\Sigma_{y_4y_2} \to \Sigma_{y_3}^2 - \frac{4}{21}\Sigma_{y_2}^3$	$S_{x_0^4x_1^2} \to \frac{6}{35}S_{x_0x_1}^3 - \frac{1}{2}S_{x_0^2x_1}^2$
	$\Sigma_{y_5y_1} \to \frac{2}{7}\Sigma_{y_2}^3 - \frac{1}{2}\Sigma_{y_3}^2$	$S_{x_0^3x_1x_0x_1} \to \frac{4}{105}S_{x_0x_1}^3$
	$\Sigma_{y_3y_1y_2} \to -\frac{17}{30}\Sigma_{y_2}^3 + \frac{9}{4}\Sigma_{y_3}^2$	$S_{x_0^3x_1^3} \to \frac{23}{70}S_{x_0x_1}^3 - S_{x_0^2x_1}^2$
	$\Sigma_{y_3y_2y_1} \to 3\Sigma_{y_3}^2 - \frac{9}{10}\Sigma_{y_2}^3$	$S_{x_0^2x_1x_0x_1^2} \to \frac{2}{105}S_{x_0x_1}^3$
	$\Sigma_{y_4y_1^2} \to \frac{3}{10}\Sigma_{y_2}^3 - \frac{3}{4}\Sigma_{y_3}^2$	$S_{x_0^2x_1^2x_0x_1} \to -\frac{89}{210}S_{x_0x_1}^3 + \frac{3}{2}S_{x_0^2x_1}^2$
	$\Sigma_{y_2^2y_1^2} \to \frac{11}{63}\Sigma_{y_2}^3 - \frac{1}{4}\Sigma_{y_3}^2$	$S_{x_0^2x_1^4} \to \frac{6}{35}S_{x_0x_1}^3 - \frac{1}{2}S_{x_0^2x_1}^2$
	$\Sigma_{y_3y_1^3} \to \frac{1}{21}\Sigma_{y_2}^3$	$S_{x_0x_1x_0x_1^3} \to \frac{8}{21}S_{x_0x_1}^3 - S_{x_0^2x_1}^2$
	$\Sigma_{y_2y_1^4} \to \frac{17}{50}\Sigma_{y_2}^3 + \frac{3}{16}\Sigma_{y_3}^2$	$S_{x_0x_1^5} \to \frac{8}{35}S_{x_0x_1}^3$

$$\mathcal{L}_{irr}^{\leq 2}(\mathcal{X}) \subset \cdots \mathcal{L}_{irr}^{\leq p}(\mathcal{X}) \subset \cdots \subset \mathcal{L}_{irr}^{\infty}(\mathcal{X}) = \cup_{p\geq 2}\mathcal{L}_{irr}^{\leq p}(\mathcal{X}).$$

$$\forall l \in \left\{\begin{array}{c}\mathcal{L}ynY \setminus \{y_1\}\\ \mathcal{L}ynX \setminus X\end{array}\right\}, \quad \left\{\begin{array}{c}\Sigma_l \in \mathcal{L}_{irr}^{\infty}(Y)\\ S_l \in \mathcal{L}_{irr}^{\infty}(X)\end{array}\right\} \Leftrightarrow \left\{\begin{array}{c}\Sigma_l \to \Sigma_l\\ S_l \to S_l\end{array}\right\} \Leftrightarrow Q_l = 0.$$

STRUCTURE OF POLYZETAS

Identification of local coordinates $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX\setminus X}$ $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}$

The identification of local coordinates in $Z_\gamma = B(y_1)\pi_Y Z_{\sqcup\!\sqcup}$, leads to

1. A family of algebraic generators $\mathcal{Z}_{irr}^{\infty}(\mathcal{X})$ of $\mathcal{Z}$ constructed as follows
$$\mathcal{Z}_{irr}^{\leq 2}(\mathcal{X}) \subset \cdots \subset \mathcal{Z}_{irr}^{\leq p}(\mathcal{X}) \subset \cdots \subset \mathcal{Z}_{irr}^{\infty}(\mathcal{X}) = \bigcup_{p\geq 2} \mathcal{Z}_{irr}^{\leq p}(\mathcal{X})$$
and their inverse image, by a section of ζ,
$$\mathcal{L}_{irr}^{\leq 2}(\mathcal{X}) \subset \cdots \subset \mathcal{L}_{irr}^{\leq p}(\mathcal{X}) \subset \cdots \subset \mathcal{L}_{irr}^{\infty}(\mathcal{X}) = \bigcup_{p\geq 2} \mathcal{L}_{irr}^{\leq p}(\mathcal{X})$$
such that the following restriction is bijective
$$\zeta : \mathbb{Q}[\mathcal{L}_{irr}^{\infty}(\mathcal{X})] \to \mathcal{Z} = \mathbb{Q}[\mathcal{Z}_{irr}^{\infty}(\mathcal{X})] = \mathbb{Q}[\{\zeta(p)\}_{p\in\mathcal{L}_{irr}^{\infty}(\mathcal{X})}].$$
2. A ideal $\mathcal{R}_{\mathcal{X}}$ generated by the polynomials $\{Q_l\}_{\substack{l\in\mathcal{L}yn\mathcal{X}\\ l\neq y_1,x_0,x_1}}$ homogenous in weight $(=(l))$ such that the following assertions are equivalent
   i. $Q_l = 0$,
   ii. $\Sigma_l \to \Sigma_l$ (resp. $S_l \to S_l$),
   iii. $\Sigma_l \in \mathcal{L}_{irr}^{\infty}(Y)$ (resp. $S_l \in \mathcal{L}_{irr}^{\infty}(X)$).

   $0 \neq Q_l$ is led by Σ_l (resp. S_l), being transcendent over $\mathbb{Q}[\mathcal{L}_{irr}^{\infty}(\mathcal{X})]$, and $\Sigma_l \to \Upsilon_l$ (resp. $S_l \to U_l$) being homogenous of weight $p = (l)$ and belonging to $\mathbb{Q}[\mathcal{L}_{irr}^{\leq p}(\mathcal{X})]$. In other terms, $\Sigma_l = Q_l + \Upsilon_l$ (resp. $S_l = Q_l + U_l$), *i.e.* $\mathrm{span}_{\mathbb{Q}} \{\Sigma_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}$ ($\{S_l\}_{l\in\mathcal{L}ynX\setminus X}$) $= \mathcal{R}_{\mathcal{X}} \oplus \mathrm{span}_{\mathbb{Q}} \mathcal{L}_{irr}^{\infty}(\mathcal{X})$.

Im and ker of $\zeta : \begin{array}{c}(\mathbb{Q}[\{S_l\}_{l\in\mathcal{L}ynX\setminus x}], ш, 1_{X^*})\\ (\mathbb{Q}[\{\Sigma_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}], \text{⊎}, 1_{Y^*})\end{array} \twoheadrightarrow (\mathcal{Z}, ., 1)$

For $w \in \begin{array}{c}x_0X^*x_1\\(Y\setminus\{y_1\})Y^*\end{array}$, by the Radford's theorem, $\zeta(w) \in \mathbb{Q}[\mathcal{Z}_{irr}^{\infty}(\mathcal{X})]$. Thus, for $P \in \begin{array}{c}\mathbb{Q}[\{S_l\}_{l\in\mathcal{L}ynX\setminus x}]\\\mathbb{Q}[\{\Sigma_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}]\end{array}$, $P \notin \ker\zeta \supseteq \mathcal{R}_{\mathcal{X}}$, by linearity, $\zeta(P) \in \mathbb{Q}[\mathcal{Z}_{irr}^{\infty}(\mathcal{X})]$. Let $Q \in \mathcal{R}_{\mathcal{X}} \cap \mathbb{Q}[\mathcal{L}_{irr}^{\infty}(\mathcal{X})]$. Since $\mathcal{R}_{\mathcal{X}} \subseteq \ker\zeta$ then $\zeta(Q) = 0$. Restricted on $\mathbb{Q}\mathcal{L}_{irr}^{\infty}(\mathcal{X})]$, the polymorphism ζ is bijective and then $Q = 0$. It follows that

Proposition 2

$$\begin{aligned}\mathbb{Q}[\{S_l\}_{l\in\mathcal{L}ynX\setminus x}] &= \mathcal{R}_X \oplus \mathbb{Q}[\mathcal{L}_{irr}^{\infty}(X)],\\ \mathbb{Q}[\{\Sigma_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}] &= \mathcal{R}_Y \oplus \mathbb{Q}[\mathcal{L}_{irr}^{\infty}(Y)],\end{aligned}$$

(as v.s. associated to ш or ⊎ -subalgebras). By duality & CQMM,

$$\begin{aligned}\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle X\rangle \setminus X) &= \mathcal{J}_X \oplus \mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle\{P_l\}_{l\in\mathcal{L}ynX:S_l\in\mathcal{L}_{irr}^{\infty}(X)}\rangle),\\ \mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle Y\rangle \setminus \{y_1\}) &= \mathcal{J}_Y \oplus \mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle\{\Pi_l\}_{l\in\mathcal{L}ynX:\Sigma_l\in\mathcal{L}_{irr}^{\infty}(Y)}\rangle),\end{aligned}$$

where $\mathcal{J}_X$ (resp. $\mathcal{J}_Y$) is a Lie ideal generated by $\{P_l\}_{l\in\mathcal{L}ynX:S_l\notin\mathcal{L}_{irr}^{\infty}(X)}$ (resp. $\{\Pi_l\}_{l\in\mathcal{L}ynY:\Sigma_l\notin\mathcal{L}_{irr}^{\infty}(Y)}$).

Now, let $Q \in \ker\zeta$, $\langle Q|1_{\mathcal{X}^*}\rangle = 0$. Then $Q = Q_1 + Q_2$ with $Q_1 \in \mathcal{R}_{\mathcal{X}}$ and $Q_2 \in \mathbb{Q}[\mathcal{L}_{irr}^{\infty}(\mathcal{X})]$. Thus, $Q \equiv_{\mathcal{R}_{\mathcal{X}}} Q_1 \in \mathcal{R}_{\mathcal{X}}$.

Corollary 10

$\mathbb{Q}[\{\zeta(p)\}_{p\in\mathcal{L}_{irr}^{\infty}(\mathcal{X})}] = \mathcal{Z} = \mathrm{Im}\,\zeta$ *and* $\mathcal{R}_{\mathcal{X}} = \ker\zeta$.

Structure of polyzetas

$$\begin{aligned} \mathcal{Z} &\cong \mathbb{Q}1_{Y^*} \oplus (Y - \{y_1\})\mathbb{Q}\langle Y \rangle / \ker \zeta \\ &\cong \mathbb{Q}1_{X^*} \oplus x_0 \mathbb{Q}\langle X \rangle x_1 / \ker \zeta. \\ \forall k \geq 0, \quad \mathcal{Z}_k &:= \mathrm{span}_{\mathbb{Q}}\{\zeta(w), \mid w \mid = k\}_{w \in x_0 X^* x_1} \\ &= \mathrm{span}_{\mathbb{Q}}\{\zeta(w), (w) = k\}_{w \in (Y - \{y_1\})Y^*}, \end{aligned}$$

where, for any $w = x_{s_1} \ldots x_{s_r} \in \mathcal{X}^*, \mid w \mid = r$. If $\mathcal{X} = X$ then $(w) = \mid w \mid$ and if $\mathcal{X} = Y$ then $(w) = \mid \pi_X w \mid = s_1 + \ldots + s_r$ being weight of $(s_r, \ldots, s_r)$. Hence,

Corollary 11

As an ideal generated by homogenous polynomials, $\ker \zeta$ *is graded and then* $\mathcal{Z}$ *is also graded:*

$$\mathcal{Z} = \mathbb{Q}1 \oplus \bigoplus_{k \geq 2} \mathcal{Z}_k.$$

Now, let $\xi := \zeta(P)$, where $\mathbb{Q}\langle \mathcal{X} \rangle \ni P \notin \ker \zeta$, homogenous in weight. Each monomial $\xi^n, n \geq 1$, is of different weight (because $\mathcal{Z}_p \mathcal{Z}_n \subset \mathcal{Z}_{p+n}$). Thus ξ could not satisfy $\xi^n + a_{n-1}\xi^{n-1} + \ldots = 0$, with $a_{n-1}, \ldots \in \mathbb{Q}$.

Any $s \in \mathcal{L}_{irr}^{\infty}(\mathcal{X})$ is homogenous in weight then $\zeta(s)$ is transcendent over $\mathbb{Q}$.

Concluding remarks

For any $l \in \mathcal{L}yn\mathcal{X}, l \neq y_1, x_0, x_1$, one has $l \succeq y_n$ (resp. $l \succeq x_0^{n-1}x_1$). In particular, $\Sigma_{y_n} = y_n \in \mathcal{L}ynY$ and $S_{x_0^{n-1}x_1} = x_0^{n-1}x_1 \in \mathcal{L}ynX$. Next,

1. $\zeta(2) = \zeta(\Sigma_{y_2}) = \zeta(S_{x_0x_1})$ is then irreducible and, by the Euler's identity about the ratio $\zeta(2k)/\pi^{2k}$, one deduces then, for $k > 1$, $\Sigma_{y_{2k}} = y_{2k} \notin \mathcal{L}_{irr}^{\infty}(Y)$ and $S_{x_0^{2k-1}x_1} = x_0^{2k-1}x_1 \notin \mathcal{L}_{irr}^{\infty}(X)$,
2. $\Sigma_{y_{2n+1}} = y_{2n+1} \in \mathcal{L}_{irr}^{\infty}(Y)$ and $S_{x_0^{2n}x_1} = x_0^{2n}x_1 \in \mathcal{L}_{irr}^{\infty}(X)$.

Up to weight 12, the Zagier's dimension conjecture holds meaning that $\mathcal{Z}_{irr}^{\leq 12}(\mathcal{X})$ is algebraically independent over $\mathbb{Q}$:

$$\mathcal{Z}_{irr}^{\leq 12}(X) = \{\zeta(S_{x_0x_1}), \zeta(S_{x_0^2x_1}), \zeta(S_{x_0^4x_1}), \zeta(S_{x_0^6x_1}), \zeta(S_{x_0x_1^2x_0x_1^4}), \zeta(S_{x_0^8x_1}), \zeta(S_{x_0x_1^2x_0x_1^6}), \zeta(S_{x_0^{10}x_1}), \zeta(S_{x_0x_1^3x_0x_1^7}), \zeta(S_{x_0x_1^2x_0x_1^8}), \zeta(S_{x_0x_1^4x_0x_1^6})\}.$$

$$\mathcal{L}_{irr}^{\leq 12}(X) = \{S_{x_0x_1}, S_{x_0^2x_1}, S_{x_0^4x_1}, S_{x_0^6x_1}, S_{x_0x_1^2x_0x_1^4}, S_{x_0^8x_1}, S_{x_0x_1^2x_0x_1^6}, S_{x_0^{10}x_1}, S_{x_0x_1^3x_0x_1^7}, S_{x_0x_1^2x_0x_1^8}, S_{x_0x_1^4x_0x_1^6}\}.$$

$$\mathcal{Z}_{irr}^{\leq 12}(Y) = \{\zeta(\Sigma_{y_2}), \zeta(\Sigma_{y_3}), \zeta(\Sigma_{y_5}), \zeta(\Sigma_{y_7}), \zeta(\Sigma_{y_3y_1^5}), \zeta(\Sigma_{y_9}), \zeta(\Sigma_{y_3y_1^7}), \zeta(\Sigma_{y_{11}}), \zeta(\Sigma_{y_2y_1^9}), \zeta(\Sigma_{y_3y_1^9}), \zeta(\Sigma_{y_2^2y_1^8})\}.$$

$$\mathcal{L}_{irr}^{\leq 12}(Y) = \{\Sigma_{y_2}, \Sigma_{y_3}, \Sigma_{y_5}, \Sigma_{y_7}, \Sigma_{y_3y_1^5}, \Sigma_{y_9}, \Sigma_{y_3y_1^7}, \Sigma_{y_{11}}, \Sigma_{y_2y_1^9}, \Sigma_{y_3y_1^9}, \Sigma_{y_2^2y_1^8}\}.$$

THANK YOU FOR YOUR ATTENTION