On universal differential equations

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Outline

INTRODUCTION

Picard-Vessiot theory of ordinary differential equation

$(\mathbf{k}, \partial)$ a commutative differential ring without zero divisors.
$\mathrm{Const}(\mathbf{k}) = \{c \in \mathbf{k} | \partial c = 0\}$ is supposed to be a field.

$$(ODE) \quad (a_n\partial^n + a_{n-1}\partial^{n-1} + \ldots + a_0)y = 0, \quad a_0, \ldots, a_{n-1}, a_n \in \mathbf{k}.$$

a_n^{-1} is supposed to exist.

Definition 1

1. Let $y_1, \ldots, y_n$ be $\mathrm{Const}(\mathbf{k})$-linearly independent solutions of (ODE). Then $\{y_1, \ldots, y_n\}$ is called a fundamental set of solutions of (ODE) and it generates a $\mathrm{Const}(\mathbf{k})$-vector subspace of dimension at most n.
2. If[1] $M = \mathbf{k}\{y_1, \ldots, y_n\}$ and $\mathrm{Const}(M) = \mathrm{Const}(\mathbf{k})$ then M is called a Picard-Vessiot extension related to (ODE)
3. Let $\mathbf{k} \subset \mathbb{K}_1$ and $\mathbf{k} \subset \mathbb{K}_2$ be differential rings. An isomorphism of rings $\sigma : \mathbb{K}_1 \to \mathbb{K}_2$ is a differential $\mathbf{k}$-isomorphism if
$$\forall a \in \mathbb{K}_1, \quad \partial(\sigma(a)) = \sigma(\partial a) \text{ and, if } a \in \mathbf{k},\ \sigma(a) = a.$$
If $\mathbb{K}_1 = \mathbb{K}_2 = \mathbb{K}$, the differential galois group of $\mathbb{K}$ over $\mathbf{k}$ is by
$$\mathrm{Gal}_{\mathbf{k}}(\mathbb{K}) = \{\sigma | \sigma \text{ is a differential } \mathbf{k}\text{-automorphism of } \mathbb{K}\}.$$

1. Let R_1, R_2 be differential rings s.t. $R_1 \subset R_2$. Let S be a subset of R_2. $R_1\{S\}$ denotes the smallest differential subring of R_2 containing R_1. $R_1\{S\}$ is the ring (over R_1) generated by S and their derivatives of all orders.

Linear differential equations and Dyson series

Let $a_0, \ldots, a_n \in \mathbb{C}(z)$, $\quad (a_n(z)\partial^n + \ldots + a_1(z)\partial + a_0(z))y(z) = 0.$

$$(ED) \quad \begin{cases} \partial q(z) &= A(z)q(z), \qquad A(z) \in \mathcal{M}_{n,n}(\mathbb{C}(z)), \\ q(z_0) &= \eta, \qquad \lambda \in \mathcal{M}_{1,n}(\mathbb{C}), \\ y(z) &= \lambda q(z), \qquad \eta \in \mathcal{M}_{n,1}(\mathbb{C}). \end{cases}$$

By successive Picard iterations, with the initial point $q(z_0) = \eta$, we get [2] $y(z) = \lambda U(z_0; z)\eta$, where $U(z_0; z)$ is the following functional expansion

$$U(z_0; z) = \sum_{k \geq 0} \int_{z_0}^{z} A(z_1)dz_1 \int_{z_0}^{z_1} A(z_2)dz_2 \ldots \int_{z_0}^{z_{k-1}} A(z_k)dz_k, \text{(Dyson series)}$$

and $(z_0, z_1 \ldots, z_k, z)$ is a subdivision of the path of integration $z_0 \rightsquigarrow z$.
In order to find the matrix $\Omega(z_0; z)$ s.t.

$$U(z_0; z) = \exp[\Omega(z_0; z)] = \top \exp \int_{z_0}^{z} A(s)ds, \qquad \text{(Feynman's notation)}$$

Magnus computed $\Omega(z_0; z)$ as limit of the following Lie-integral-functionals

$$\begin{aligned} \Omega_1(z_0; z) &= \int_{z_0}^{z} A(z)ds, \\ \Omega_k(z_0; z) &= \int_{z_0}^{z} [A(z) + [A(z), \Omega_{k-1}(z_0; s)]/2 \\ &\quad + [[A(z), \Omega_{k-1}(z_0; s)], \Omega_{k-1}(z_0; s)]/12 + \ldots)ds. \end{aligned}$$

2. Subject to convergence.

Fuchsian linear differential equations

Let Ω be a simply connected domain and $\mathcal{H}(\Omega)$ be the ring of holomorphic functions over Ω (with $1_{\mathcal{H}(\Omega)}$ as neutral element). Let us consider, here, $\sigma = \{s_i\}_{i=0,..,m}$, $m \geq 1$, as set of simple poles of (ED) and $\Omega = \widetilde{\mathbb{C} \setminus \sigma}$.

$$A(z) = \sum_{i=0}^{m} M_i u_i(z), \quad \text{where} \quad \begin{cases} M_i & \in \mathcal{M}_{n,n}(\mathbb{C}), \\ u_i(z) = (z - s_i)^{-1} & \in \mathbb{C}(z). \end{cases}$$

$$(ED) \quad \begin{cases} \partial q(z) & = & \left(\displaystyle\sum_{i=0}^{m} M_i u_i(z)\right) q(z), \\ q(z_0) & = & \eta, \\ y(z) & = & \lambda q(z). \end{cases}$$

Let X^* be the set of words over $X = \{x_0, \ldots, x_m\}$ and

$$\alpha_{z_0}^{z} \otimes \mathcal{M} : \mathbb{C}\langle X \rangle \otimes \mathbb{C}\langle X \rangle \to \mathcal{M}_{n,n}(\mathcal{H}(\Omega))$$

($z_0 \rightsquigarrow z$ is the path of integration previously introduced) s.t.

$$\mathcal{M}(1_{X^*}) = \mathtt{Id}_n \quad \text{and} \quad \mathcal{M}(x_{i_1} \cdots x_{i_k}) = M_{i_1} \ldots M_{i_k},$$

$$\alpha_{z_0}^{z}(1_{X^*}) = 1_{\mathcal{H}(\Omega)} \quad \text{and} \quad \alpha_{z_0}^{z}(x_{i_1} \cdots x_{i_k}) = \int_{z_0}^{z} \frac{dz_1}{z_1 - s_{i_1}} \ldots \int_{z_0}^{z_{k-1}} \frac{dz_k}{z_k - s_{i_k}}.$$

Then [3] $y(z) = \lambda U(z_0; z) \eta$ with

$$U(z_0; z) = \sum_{w \in X^*} \mathcal{M}(w) \alpha_{z_0}^{z}(w) = (\mathcal{M} \otimes \alpha_{z_0}) \sum_{w \in X^*} w \otimes w.$$

3. Subject to convergence.

Examples of linear dynamical systems

Example 2 (Hypergeometric equation)

Let t_0, t_1, t_2 be parameters and
$$z(1-z)\ddot{y}(z) + [t_2 - (t_0+t_1+1)z]\dot{y}(z) - t_0 t_1 y(z) = 0.$$
Let $q_1(z) = -y(z)$ and $q_2(z) = (1-z)\dot{y}(z)$. Hence, one has
$$y(z) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} q_1(z) \\ q_2(z) \end{pmatrix}$$
and
$$\begin{aligned} \begin{pmatrix} \dot{q}_1(z) \\ \dot{q}_2(z) \end{pmatrix} &= \left(\frac{M_0}{z} + \frac{M_1}{1-z} \right) \begin{pmatrix} q_1(z) \\ q_2(z) \end{pmatrix} \\ &= (u_0(z)M_0 + u_1(z)M_1) \begin{pmatrix} q_1(z) \\ q_2(z) \end{pmatrix}, \end{aligned}$$
where $u_0(z) = z^{-1}, u_1(z) = (1-z)^{-1}$ and
$$M_0 = -\begin{pmatrix} 0 & 0 \\ t_0 t_1 & t_2 \end{pmatrix} \quad \text{and} \quad M_1 = -\begin{pmatrix} 0 & 1 \\ 0 & t_2 - t_0 - t_1 \end{pmatrix}.$$

Nonlinear differential equations

$$(NED) \quad \begin{cases} \partial q(z) &= \Big(\sum_{i=0}^{m} T_i(q)u_i(z)\Big)(q), \\ q(z_0) &= q_0, \\ y(z) &= f(q(z)), \end{cases}$$

where

- $u_i \in (\mathbf{k}, \partial)$,
- the state $q = (q_1, \ldots, q_n)$ belongs the complex analytic manifold Q of dimension n and q_0 is the initial state,
- the observation $f \in \mathcal{O}$, with $\mathcal{O}$ the ring of analytic functions over Q,
- for $i = 0..1$, $T_i = (T_i^1(q)\partial/\partial q_1 + \cdots + T_i^m(q)\partial/\partial q_m)$ is an analytic vector field over Q, with $T_i^j(q) \in \mathcal{O}$, for $j = 1, \ldots, n$.

With X and $\alpha_{z_0}^z$ given as previously, let the morphism τ be defined by $\tau(1_{X^*}) = \mathtt{Id}$ and $\tau(x_{i_1} \cdots x_{i_k}) = T_{i_1} \ldots T_{i_k}$. Then [4] $y(z) = \mathcal{T} \circ f_{|_{q_0}}$ with

$$\mathcal{T} = \sum_{w \in X^*} \tau(w)\alpha_{z_0}^z(w) = (\tau \otimes \alpha_{z_0}) \sum_{w \in X^*} w \otimes w.$$

4. Subject to convergence.

Examples of nonlinear dynamical systems (1/2)

Example 3 (Harmonic oscillator)

Let k_1, k_2 be parameters and $\partial^2 y(z) + k_1 y(z) + k_2 y^2(z) = u_1(z)$ which can be represented by the following state equations (with $n = 1$)

$$\begin{aligned} y(z) &= q(z), \\ \partial q(z) &= A_0(q)u_0(z) + A_1(q)u_1(z), \end{aligned}$$

$$\text{where} \quad A_0 = -(k_1 q + k_2 q^2)\frac{\partial}{\partial q} \quad \text{and} \quad A_1 = \frac{\partial}{\partial q}.$$

Example 4 (Duffing equation)

Let a, b, c be parameters and $\partial^2 y(z) + a\partial y(z) + by(z) + cy^3(z) = u_1(z)$ which can be represented by the following state equations (with $n = 2$)

$$\begin{aligned} y(z) &= q_1(z), \\ \begin{pmatrix} \partial q_1(z) \\ \partial q_2(z) \end{pmatrix} &= \begin{pmatrix} q_2 \\ -(aq_2 + b^2 q_1 + cq_1^3) \end{pmatrix} u_0(z) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_1(z) \\ &= A_0(q)u_0(z) + A_1(q)u_1(z), \end{aligned}$$

$$\text{where} \quad A_0 = -(aq_2 + b^2 q_1 + cq_1^3)\frac{\partial}{\partial q_2} + q_2\frac{\partial}{\partial q_1} \quad \text{and} \quad A_1 = \frac{\partial}{\partial q_2}.$$

Examples of nonlinear dynamical systems (2/2)

Example 5 (Van der Pol oscillator)

Let γ, g be parameters and

$$\partial^2 x(z) - \gamma[1 + x(z)^2]\partial x(z) + x(z) = g\cos(\omega z)$$

which can be tranformed into (with C is some constant of integration)

$$\partial x(z) = \gamma[1 + x(z)^2/3]x(z) - \int_{z_0}^{z} x(s)ds + \frac{g}{\omega}\sin(\omega z) + C.$$

Supposing $x = \partial y$ and $u_1(z) = g\sin(\omega z)/\omega + C$, it leads then to

$$\partial^2 y(z) = \gamma[\partial y(z) + (\partial y(z))^3/3] + y(z) + u_1(z)$$

which can be represented by the following state equations (with $n = 2$)

$$\begin{aligned} y(z) &= q_1(z), \\ \begin{pmatrix} \partial q_1(z) \\ \partial q_2(z) \end{pmatrix} &= \begin{pmatrix} q_2 \\ \gamma(q_2 + q_2^3/3) + q_1 \end{pmatrix} u_0(z) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_1(z) \\ &= A_0(q)u_0(z) + A_1(q)u_1(z), \end{aligned}$$

$$\text{where} \quad A_0 = [\gamma(q_2 + q_2^3/3) + q_1]\frac{\partial}{\partial q_2} + q_2\frac{\partial}{\partial q_1} \quad \text{and} \quad A_1 = \frac{\partial}{\partial q_2}.$$

DUAL LAWS AND REPRESENTATIVE SERIES

Dual law in bialgebra

Startting with a $\mathbf{k}-$ **AAU** ($\mathbf{k}$ is a ring) $\mathcal{A}$. Dualizing $\mu : \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \to \mathcal{A}$, we get the transpose ${}^t\mu : \mathcal{A}^{\vee} \to (\mathcal{A} \otimes_{\mathbf{k}} \mathcal{A})^{\vee}$ so that we do not get a co-multiplication in general.

- Remark that when $\mathbf{k}$ is a field, the following arrow is into (due to the fact that $\mathcal{A}^{\vee} \otimes_{\mathbf{k}} \mathcal{A}^{\vee}$ is torsionfree)
$$\Phi : \mathcal{A}^{\vee} \otimes_{\mathbf{k}} \mathcal{A}^{\vee} \to (\mathcal{A} \otimes_{\mathbf{k}} \mathcal{A})^{\vee}.$$
- One restricts the codomain of ${}^t\mu$ to $\mathcal{A}^{\vee} \otimes_{\mathbf{k}} \mathcal{A}^{\vee}$ and then the domain to $({}^t\mu)^{-1}\Phi(\mathcal{A}^{\vee} \otimes_{\mathbf{k}} \mathcal{A}^{\vee}) =: \mathcal{A}^{\circ}$.

$$\begin{tikzcd}
\mathcal{A}^{\vee} \arrow[r, "{}^t\mu"] & (\mathcal{A} \otimes_{\mathbf{k}} \mathcal{A})^{\vee} \\
\mathcal{A}^{\circ} \arrow[u, hook, "can"] \arrow[r, "\Delta_\mu"] & \mathcal{A}^{\vee} \otimes_{\mathbf{k}} \mathcal{A}^{\vee} \arrow[u, hook, "\Phi"'] \\
\mathcal{A}^{\circ\circ} \arrow[u, hook, "can"] \arrow[r, "\Delta_\mu"] & \mathcal{A}^{\circ} \otimes_{\mathbf{k}} \mathcal{A}^{\circ} \arrow[u, hook, "j\otimes j"']
\end{tikzcd}$$

The descent can stop at first step for a field $\mathbf{k}$ and then $\mathcal{A}^{\circ\circ} = \mathcal{A}^{\circ}$. The coalgebra $(\mathcal{A}^{\circ}, \Delta_\mu)$ is called the Sweedler's dual of $(\mathcal{A}, \mu)$.

Case of algebras noncommutative series

- $\mathcal{X}$ denotes the ordered alphabets $Y := \{y_k\}_{k\geq 1}$ or $X := \{x_0, x_1\}$.
  On the free monoid $(\mathcal{X}^*, \mathrm{conc}, 1_{\mathcal{X}^*})$, we use the correspondences
  $$x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 \in X^*x_1 \underset{\pi_X}{\overset{\pi_Y}{\rightleftharpoons}} y_{s_1} \ldots y_{s_r} \in Y^* \leftrightarrow (s_1, \ldots, s_r) \in \mathbb{N}_+^r.$$
  Let $\mathcal{L}yn\mathcal{X}$ denote the set of Lyndon words generated by $\mathcal{X}$.

- Let $(\mathcal{L}ie_A\langle\langle\mathcal{X}\rangle\rangle, [.])$ and $(A\langle\langle\mathcal{X}\rangle\rangle, \mathrm{conc})$ (resp. $(\mathcal{L}ie_A\langle\mathcal{X}\rangle, [.])$ and $(A\langle\mathcal{X}\rangle, \mathrm{conc})$) denote the algebras of (Lie) series (resp. polynomials) with coefficients in the ring A, over $\mathcal{X}$.

  $\{P_l\}_{l\in\mathcal{L}yn\mathcal{X}}$ (resp. $\{\Pi_l\}_{l\in\mathcal{L}ynY}$) is a basis of Lie algebra of primitive elements and $\{S_l\}_{l\in\mathcal{L}yn\mathcal{X}}$ (resp. $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$) is a transcendence basis of $(A\langle\mathcal{X}\rangle, ш, 1_{\mathcal{X}^*})$ (resp. $(A\langle Y\rangle, \stackrel{-}{ш}, 1_{Y^*})$).

- $\mathcal{H}_ш(\mathcal{X}) := (A\langle\mathcal{X}\rangle, \mathrm{conc}, 1_{\mathcal{X}^*}, \Delta_ш, \mathrm{e})$ and $\mathcal{H}_{\stackrel{-}{ш}}(Y) := (A\langle Y\rangle, \mathrm{conc}, 1_{Y^*}, \Delta_{\stackrel{-}{ш}}, \mathrm{e})$ with [5] (for $x \in \mathcal{X}, y_i \in Y$)
  $$\begin{aligned} \Delta_ш x &= x \otimes 1_{\mathcal{X}^*} + 1_{\mathcal{X}^*} \otimes x, \\ \Delta_{\stackrel{-}{ш}} y_i &= y_i \otimes 1_{Y^*} + 1_{Y^*} \otimes y_i + \textstyle\sum_{k+l=i} y_k \otimes y_l. \end{aligned}$$

- The dual law associated to conc is defined, for $w \in \mathcal{X}^*$, by
  $$\Delta_{\mathrm{conc}}(w) = \textstyle\sum_{u,v\in\mathcal{X}^*, uv=w} u \otimes v.$$

5. Or equivalently, for $x, y \in \mathcal{X}, y_i, y_j \in Y$ and $u, v \in \mathcal{X}^*$ (resp. Y^*), $u \,ш\, 1_{\mathcal{X}^*} = 1_{\mathcal{X}^*} \,ш\, u = u$ and $xu \,ш\, yv = x(u \,ш\, yv) + y(xu \,ш\, v)$, $u \stackrel{-}{ш} 1_{Y^*} = 1_{Y^*} \stackrel{-}{ш} u = u$ and $x_iu \stackrel{-}{ш} y_jv = y_i(u \stackrel{-}{ш} y_jv) + y_j(y_iu \stackrel{-}{ш} v) + y_{i+j}(u \stackrel{-}{ш} v)$.

Dualizable laws in conc-shuffle bialgebras (1/2)

We can exploit the basis of words as follows

1. Any bilinear law (shuffle, stuffle or any) $\mu : A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle \to A\langle\mathcal{X}\rangle$ can be decribed through its structure constants wrt to the basis of words, *i.e.* for $u, v, w \in \mathcal{X}^*$, $\Gamma_{u,v}^{w} := \langle\mu(u \otimes v)|w\rangle$ so that
$$\mu(u \otimes v) = \sum_{w\in\mathcal{X}^*} \Gamma_{u,v}^{w} w.$$
2. In the case when $\Gamma_{u,v}^{w}$ is locally finite in w, we say that the given law is dualizable, the arrow ${}^t\mu$ restricts nicely to $A\langle\mathcal{X}\rangle \hookrightarrow A\langle\langle\mathcal{X}\rangle\rangle$ and one can define on the polynomials a comultiplication by
$$\Delta_\mu(w) := \sum_{u,v\in\mathcal{X}^*} \Gamma_{u,v}^{w} u \otimes v.$$
3. When the law μ is dualizable, we have

$$\begin{tikzcd}
A\langle\langle\mathcal{X}\rangle\rangle \arrow[r, "{}^t\mu"] & A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle \\
A\langle\mathcal{X}\rangle \arrow[u, hook, "can"] \arrow[r, "\Delta_\mu"] & A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle \arrow[u, hook, "\Phi|_{A\langle\mathcal{X}\rangle\otimes_A A\langle\mathcal{X}\rangle}"']
\end{tikzcd}$$

The arrow Δ_μ is unique to be able to close the rectangle and $\Delta_\mu(P)$ is defined as above.

Dualizable laws in conc-shuffle bialgebras (2/2)

4. Proof that the arrow $A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle \longrightarrow A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle$ is into :
   Let $T = \sum_{i=1}^{n} P_i \otimes_A Q_i$ such that $\Phi(T) = 0$. Rewriting T as a finitely supported sum $T = \sum_{u,v\in\mathcal{X}^*} c_{u,v}\, u \otimes v$ (this is indeed the iso between $A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle$ and $A[\mathcal{X}^* \times \mathcal{X}^*]$), $\Phi(T)$ is by definition of Φ the double series (here a polynomial) s.t. $\langle\Phi(T)|u \otimes v\rangle = c_{u,v}$. If $\Phi(T) = 0$, then for all $(u, v) \in \mathcal{X}^* \times \mathcal{X}^*$, $c_{u,v} = 0$ entailing $T = 0$.

We extend by linearity and infinite sums, for $S \in A\langle\langle Y\rangle\rangle$ (resp. $A\langle\langle\mathcal{X}\rangle\rangle$), by

$$\begin{aligned}
\Delta_{\uplus} S &= \sum_{w\in Y^*} \langle S|w\rangle \Delta_{\uplus} w &&\in A\langle\langle Y^* \otimes Y^*\rangle\rangle,\\
\Delta_{\mathrm{conc}} S &= \sum_{w\in\mathcal{X}^*} \langle S|w\rangle \Delta_{\mathrm{conc}} w &&\in A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle,\\
\Delta_{\sqcup\mathchoice{\mkern-7mu}{\mkern-7mu}{\mkern-3.2mu}{\mkern-3.8mu}\sqcup} S &= \sum_{w\in\mathcal{X}^*} \langle S|w\rangle \Delta_{\sqcup\mathchoice{\mkern-7mu}{\mkern-7mu}{\mkern-3.2mu}{\mkern-3.8mu}\sqcup} w &&\in A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle.
\end{aligned}$$

$A\langle\langle\mathcal{X}\rangle\rangle \otimes A\langle\langle\mathcal{X}\rangle\rangle$ embeds injectively in[6] $A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle \cong [A\langle\langle\mathcal{X}\rangle\rangle]\langle\langle\mathcal{X}\rangle\rangle$.

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6. $A\langle\langle\mathcal{X}\rangle\rangle \otimes A\langle\langle\mathcal{X}\rangle\rangle$ contains the elements of the form $\sum_{i\in I \text{ finite}} G_i \otimes D_i$, for $(G_i, D_i) \in A\langle\langle\mathcal{X}\rangle\rangle \times A\langle\langle\mathcal{X}\rangle\rangle$. But since elements of $M \otimes N$ are finite combination of $m_i \otimes n_i, m_i \in M, n_i \in N$ then $\sum_{i\geq 0} u^i \otimes v^i$ belongs to $A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle$ and does not belong to $A\langle\langle\mathcal{X}\rangle\rangle \otimes A\langle\langle\mathcal{X}\rangle\rangle$, for $u, v \in \mathcal{X}^{\geq 1}$.

Extended Ree's theorem

Let $S \in A\langle\langle Y \rangle\rangle$ (resp. $A\langle\langle \mathcal{X} \rangle\rangle$), A is a commutative ring containing $\mathbb{Q}$. The series S is said to be

1. a $⧢$ (resp. conc, $⧢$)-character iff, for any $w, v \in Y^*$ (resp. $\mathcal{X}^*$), $\langle S|w\rangle\langle S|v\rangle = \langle S|w ⧢ v\rangle$ (resp. $\langle S|wv\rangle, \langle S|w ⧢ v\rangle$) and $\langle S|1\rangle = 1$.
2. an infinitesimal $⧢$ (resp. conc, $⧢$)-character iff, for any $w, v \in Y^*$ (resp. $\mathcal{X}^*$), $\langle S|w ⧢ v\rangle = \langle S|w\rangle\langle v|1_{Y^*}\rangle + \langle w|1_{Y^*}\rangle\langle S|v\rangle$ (resp. $\langle S|wv\rangle = \langle S|w\rangle\langle v|1_{\mathcal{X}^*}\rangle + \langle w|1_{\mathcal{X}^*}\rangle\langle S|v\rangle$, $\langle S|w ⧢ v\rangle = \langle S|w\rangle\langle v|1_{\mathcal{X}^*}\rangle + \langle w|1_{\mathcal{X}^*}\rangle\langle S|v\rangle$).
3. a group-like series iff $\langle S|1_{\mathcal{X}^*}\rangle = 1$ and $\Delta_{⧢} S = \Phi(S \otimes S)$ (resp. $\Delta_{\mathrm{conc}} S = \Phi(S \otimes S), \Delta_{⧢} S = \Phi(S \otimes S)$).
4. a primitive series iff $\Delta_{⧢} S = 1_{Y^*} \otimes S + S \otimes 1_{Y^*}$ (resp. $\Delta_{\mathrm{conc}} S = 1_{\mathcal{X}^*} \otimes S + S \otimes 1_{\mathcal{X}^*}, \Delta_{⧢} S = 1_{\mathcal{X}^*} \otimes S + S \otimes 1_{\mathcal{X}^*}$).

Then the following assertions are equivalent

1. S is a $⧢$ (resp. conc and $⧢$)-character.
2. $\log S$ an infinitesimal $⧢$ (resp. conc and $⧢$)-character.
3. S is group-like, for $\Delta_{⧢}$ (resp. Δ_{conc} and $\Delta_{⧢}$).
4. $\log S$ is primitive, for $\Delta_{⧢}$ (resp. Δ_{conc} and $\Delta_{⧢}$).

Extension by continuity (infinite sums)

Now, suppose that the ring A (containing $\mathbb{Q}$) is a field $\mathbf{k}$. Then

$$\Delta_{\sqcup\!\sqcup} : \mathbf{k}\langle\mathcal{X}\rangle \to \mathbf{k}\langle\mathcal{X}\rangle \otimes \mathbf{k}\langle\mathcal{X}\rangle \text{ and } \Delta_{\uplus\!\!\!\sqcup} : \mathbf{k}\langle Y\rangle \to \mathbf{k}\langle Y\rangle \otimes \mathbf{k}\langle Y\rangle$$

are graded for the multidegree. Then $\Delta_{\uplus\!\!\!\sqcup}$ is graded for the length. Their extension to the completions (*i.e.* $\mathbf{k}\langle\langle\mathcal{X}\rangle\rangle$ and $\mathbf{k}\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle$) are continuous and then, when exist, commute with infinite sums. Hence [7, 8],

$$\forall c \in \mathbf{k}, \quad \Delta_{\sqcup\!\sqcup}(cx)^* = \sum_{n\geq 0} c^n \Delta_{\sqcup\!\sqcup} x^n = \sum_{n\geq 0} c^n \sum_{j=0}^{n} \binom{n}{j} x^j \otimes x^{n-j}.$$

For $c \in \mathbb{N}_{\geq 2}$ which is neither a field nor a ring (containing $\mathbb{Q}$), we also get

$$(cx)^* = (c-1)^{-1} \sum_{a,b\in\mathbb{N}_{\geq 1}, a+b=c} (ax)^* \sqcup\!\sqcup (bx)^* \quad \in \mathbb{N}_{\geq 2}\langle\langle\mathcal{X}\rangle\rangle,$$

$$\Delta_{\sqcup\!\sqcup}(cx)^* {\color{red}\neq} (c-1)^{-1} \sum_{a,b\in\mathbb{N}_{\geq 1}, a+b=c} (ax)^* \otimes (bx)^* \quad \in \mathbb{Q}\langle\langle\mathcal{X}\rangle\rangle \otimes \mathbb{Q}\langle\langle\mathcal{X}\rangle\rangle,$$

because

$$\langle \mathrm{LHS} | x \otimes 1_{\mathcal{X}^*}\rangle = c \quad \text{and} \quad \langle \mathrm{RHS} | x \otimes 1_{\mathcal{X}^*}\rangle = (c-1)^{-1} \sum_{a=1}^{c-1} a = \frac{c}{2}.$$

For $c \in \mathbb{Z}$ (or even $\mathbb{Q}, \mathbb{R}, \mathbb{C}$), the such decomposition is not finite.

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7. For $S \in A\langle\langle\mathcal{X}\rangle\rangle$ s.t. $\langle S | 1_{\mathcal{X}^*}\rangle = 0$, $S^* = \sum_{n\geq 0} S^n$ is called Kleene star of S.
8. $\Delta_{\sqcup\!\sqcup} x^n = (\Delta_{\sqcup\!\sqcup} x)^n = (1_{\mathcal{X}^*} \otimes x + x \otimes 1_{\mathcal{X}^*})^n = \sum_{j=0}^{n} \binom{n}{j} x^j \otimes x^{n-j}$.

Case of rational series and of Δ_{conc}

$A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ denotes the algebraic closure by[9] $\{\mathrm{conc}, +, *\}$ of $\widehat{A.\mathcal{X}}$ in $A\langle\langle\mathcal{X}\rangle\rangle$.

$$\begin{tikzcd}
A\langle\langle\mathcal{X}\rangle\rangle \arrow[rr, "{}^t\mathrm{conc}"] & & A\langle\langle\mathcal{X}^*\otimes\mathcal{X}^*\rangle\rangle \\
A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \arrow[u, hook, "can"] \arrow[rr, dashed] & & A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \otimes_A A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \arrow[u, "\Phi|_{A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle\otimes_A A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle}"']
\end{tikzcd}$$

The dashed arrow may not exist in general, but for any $R \in A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ admitting (λ,μ,η) as linear representation of dimension n, we can get

$${}^t\mathrm{conc}(R) = \Phi\left(\sum_{i=1}^n G_i \otimes D_i\right).$$

Indeed, since $\langle R|xy\rangle = \lambda\mu(xy)\eta = \lambda\mu(x)\mu(y)\eta$ $(x,y\in\mathcal{X})$ then, letting e_i is the vector such that ${}^te_i = \begin{pmatrix}0 & \ldots & 0 & 1 & 0 & \ldots & 0\end{pmatrix}$, one has

$$\langle R|xy\rangle = \sum_{i=1}^n \lambda\mu(x)e_i{}^te_i\mu(y)\eta = \sum_{i=1}^n \langle G_i|x\rangle\langle D_i|y\rangle = \sum_{i=1}^n \langle G_i\otimes D_i|x\otimes y\rangle.$$

G_i (resp. D_i) admits then (λ,μ,e_i) (resp. $({}^te_i,\mu,\eta)$) as linear representation.

If $A = \mathbf{k}$ being a field then, due to the injectivity of Φ, all expressions of the type $\sum_{i=1}^n G_i\otimes D_i$, of course, coincide. Hence, the dashed arrow (a restriction of Δ_{conc}) in the above diagram is well-defined.

9. $A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ is closed under $\shuffle$. $A^{\mathrm{rat}}\langle\langle Y\rangle\rangle$ is also closed under $\stuffle$.

Representative series and Sweedler's dual

Theorem 6 (representative series)

Let $S \in A\langle\langle\mathcal{X}\rangle\rangle$. The following assertions are equivalent

1. *The series S belongs to $A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$.*
2. *There exists a linear representation (ν, μ, η), of rank n, for S with $\nu \in M_{1,n}(A), \eta \in M_{n,1}(A)$ and a morphism of monoids $\mu : \mathcal{X}^* \to M_{n,n}(A)$ s.t., for any $w \in \mathcal{X}^*$, $\langle S|w\rangle = \nu\mu(w)\eta$.*
3. *The shifts[10] $\{S \triangleleft w\}_{w\in\mathcal{X}^*}$ (resp. $\{w \triangleright S\}_{w\in\mathcal{X}^*}$) lie within a finitely generated shift-invariant A-module.*

Moreover, if A is a field $\mathbf{k}$, the previous assertions are equivalent to

4. *There exist $(G_i, D_i)_{i\in F\,finite}$ s.t. $\Delta_{\mathtt{conc}}(S) = \sum_{i\in F\,finite} G_i \otimes D_i$.*

Hence, $\mathcal{H}^{\circ}_{\sqcup\!\sqcup}(\mathcal{X}) = (\mathbf{k}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle, \sqcup\!\sqcup, 1_{\mathcal{X}^*}, \Delta_{\mathtt{conc}}, \mathrm{e})$ and $\mathcal{H}^{\circ}_{\uplus}(Y) = (\mathbf{k}^{\mathrm{rat}}\langle\langle Y\rangle\rangle, \uplus, 1_{\mathcal{X}^*}, \Delta_{\mathtt{conc}}, \mathrm{e})$.

Now, let $A_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$ (resp. $A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$) be the set of exchangeable[11] series (resp. series admitting a linear representation with commuting matrices).

10. The *left* (resp. *right*) shift of S by P is $P \triangleright S$ (resp. $S \triangleleft P$) defined by, for $w \in \mathcal{X}^*$, $\langle P \triangleright S|w\rangle = \langle S|wP\rangle$ (resp. $\langle S \triangleleft P|w\rangle = \langle S|Pw\rangle$).

11. *i.e.* if $S \in A_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$ then $(\forall u, v \in \mathcal{X}^*)((\forall x \in \mathcal{X})(|u|_x = |v|_x) \Rightarrow \langle S|u\rangle = \langle S|v\rangle)$.

Kleene stars of the plane and conc-characters

For any $S \in A\langle\langle\mathcal{X}\rangle\rangle$, let ∇S denotes $S - 1_{\mathcal{X}^*}$.

Theorem 7 (rational exchangeable series)

1. $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \subset A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \cap A_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$. *If A is a field then the equality holds and* $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle X\rangle\rangle = A^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \sqcup\!\sqcup A^{\mathrm{rat}}\langle\langle x_1\rangle\rangle$ *and, for the algebra of series over subalphabets* $A_{\mathrm{fin}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle := \cup_{F \subset_{finite} Y} A^{\mathrm{rat}}\langle\langle F\rangle\rangle$, *we get*[12] $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle \cap A_{\mathrm{fin}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle = \cup_{k\geq 0} A^{\mathrm{rat}}\langle\langle y_1\rangle\rangle \sqcup\!\sqcup \ldots \sqcup\!\sqcup A^{\mathrm{rat}}\langle\langle y_k\rangle\rangle \subsetneq A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle$.
2. $\forall x \in \mathcal{X}, A^{\mathrm{rat}}\langle\langle x\rangle\rangle = \{P(1 - xQ)^{-1}\}_{P,Q \in A[x]}$. *If* $\mathbf{k}$ *is an algebraically closed field then* $\mathbf{k}^{\mathrm{rat}}\langle\langle x\rangle\rangle = \mathrm{span}_{\mathbf{k}}\{(ax)^* \sqcup\!\sqcup \mathbf{k}\langle x\rangle | a \in K\}$.
3. *If A is a $\mathbb{Q}$-algebra without zero divisors,* $\{x^*\}_{x\in\mathcal{X}}$ *(resp.* $\{y^*\}_{y\in Y}$*) are* conc*-character and algebraically independent over* $(A\langle\mathcal{X}\rangle, \sqcup\!\sqcup)$ *(resp.* $(A\langle Y\rangle, \uplus)$*) within* $(A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle, \sqcup\!\sqcup)$ *(resp.* $(A^{\mathrm{rat}}\langle\langle Y\rangle\rangle, \sqcup\!\sqcup)$*).*
4. *Let* $S \in A\langle\langle\mathcal{X}\rangle\rangle$. *If* $A = \mathbf{k}$, *a field, then t.f.a.e.*
   a) S *is groupe-like, for* Δ_{conc}.
   b) *There exists* $M := \sum_{x\in\mathcal{X}} c_x x \in \widehat{\mathbf{k}.\mathcal{X}}$ *s.t.* $S = M^*$.
   c) *There exists* $M := \sum_{x\in\mathcal{X}} c_x x \in \widehat{\mathbf{k}.\mathcal{X}}$ *s.t.* $\nabla S = MS = SM$.

12. The following identity lives in $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle$ but not in $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle \cap A_{\mathrm{fin}}^{\mathrm{rat}}\langle\langle Y\rangle\rangle$,
$(y_1 + \ldots)^* = \lim_{k\to+\infty}(y_1 + \ldots + y_k)^* = \lim_{k\to+\infty} y_1^* \sqcup\!\sqcup \ldots \sqcup\!\sqcup y_k^* = \sqcup\!\sqcup_{k\geq 1} y_k^*$.

Triangular sub bialgebras of $(A^{\mathrm{rat}}\langle\langle X\rangle\rangle, ⧢, 1_{X^*}, \Delta_{\mathtt{conc}}, \mathrm{e})$

Let (ν, μ, η) be a linear representation of $R \in A^{\mathrm{rat}}\langle\langle X\rangle\rangle$ and $\mathcal{L}$ be the Lie algebra generated by $\{\mu(x)\}_{x\in X}$.
Let $M(x) := \mu(x)x$, for $x \in X$. Then $R = \nu M(X^*)\eta$. If $\{\mu(x)\}_{x\in X}$ are triangular then let $D(X)$ (resp. $N(X)$) be the diagonal (resp. nilpotent) letter matrix s.t. $M(X) = D(X) + N(X)$ then
$M(X^*) = ((D(X^*)T(X))^* D(X^*))$. Moreover, if $X = \{x_0, x_1\}$ then
$M(X^*) = (M(x_1^*)M(x_0))^* M(x_1^*) = (M(x_0^*)M(x_1))^* M(x_0^*)$.

If A is an algabraically closed field, the modules generated by the following families are closed by conc, ⧢ and coproducts :

(F_0) $\quad E_1x_1 \ldots E_jx_1E_{j+1}$, $\quad$ where $\quad E_k \in A^{\mathrm{rat}}\langle\langle x_0\rangle\rangle$,
(F_1) $\quad E_1x_0 \ldots E_jx_0E_{j+1}$, $\quad$ where $\quad E_k \in A^{\mathrm{rat}}\langle\langle x_1\rangle\rangle$,
(F_2) $\quad E_1x_{i_1} \ldots E_jx_{i_j}E_{j+1}$, $\quad$ where $\quad E_k \in A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle, x_{i_k} \in X$.

It follows then that

1. R is a linear combination of expressions in the form (F_0) (resp. (F_1)) iff $M(x_1^*)M(x_0)$ (resp. $M(x_0^*)M(x_1)$) is nilpotent,
2. R is a linear combination of expressions in the form (F_2) iff $\mathcal{L}$ is solvable. Thus, if $R \in A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle ⧢ A\langle X\rangle$ then $\mathcal{L}$ is nilpotent.

CONTINUITY OVER CHEN SERIES

Iterated integrals over $\omega_i(z) = u_{x_i}(z)dz$ and along $z_0 \rightsquigarrow z$

Let Ω be a simply connected domain admitting $1_{\mathcal{H}(\Omega)}$ as neutral element.

Let $\mathcal{A} := (\mathcal{H}(\Omega), \partial)$ and let $\mathcal{C}_0$ be a differential subring of $\mathcal{A}$ ($\partial\mathcal{C}_0 \subset \mathcal{C}_0$) which is an integral domain containing $\mathbb{C}$.

$\mathbb{C}\{\{(g_i)_{i\in I}\}\}$ denotes the differential subalgebra of $\mathcal{A}$ generated by $(g_i)_{i\in I}$, *i.e.* the $\mathbb{C}$-algebra generated by g_i's and their derivatives

$\{u_x\}_{x\in\mathcal{X}}$: elements in $\mathcal{C}_0 \cap \mathcal{A}^{-1}$ in correspondence with $\{\theta_x\}_{x\in\mathcal{X}}$ ($\theta_x = u_x^{-1}\partial$).

The iterated integral associated to $x_{i_1}\ldots x_{i_k} \in \mathcal{X}^*$, over the differential forms $\omega_i(z) = u_{x_i}(z)dz$, and along a path $z_0 \rightsquigarrow z$ on Ω, is defined by

$$\begin{aligned}
\alpha_{z_0}^{z}(1_{\mathcal{X}^*}) &= 1_\Omega,\\
\alpha_{z_0}^{z}(x_{i_1}\ldots x_{i_k}) &= \int_{z_0}^{z}\omega_{i_1}(z_1)\ldots\int_{z_0}^{z_{k-1}}\omega_{i_k}(z_k).\\
\partial\alpha_{z_0}^{z}(x_{i_1}\ldots x_{i_k}) &= u_{x_{i_1}}(z)\int_{z_0}^{z}\omega_{i_2}(z_2)\ldots\int_{z_0}^{z_{k-1}}\omega_{i_k}(z_k).
\end{aligned}$$

$$\begin{aligned}
\mathrm{span}_{\mathbb{C}}\{\partial^l\alpha_{z_0}^{z}(w)\}_{w\in\mathcal{X}^*,l\geq 0} &\subset \mathrm{span}_{\mathbb{C}\{\{(u_x)_{x\in\mathcal{X}}\}\}}\{\alpha_{z_0}^{z}(w)\}_{w\in\mathcal{X}^*}\\
&\subset \mathrm{span}_{\mathbb{C}\{\{(u_x^{\pm 1})_{x\in\mathcal{X}}\}\}}\{\alpha_{z_0}^{z}(w)\}_{w\in\mathcal{X}^*}\\
&\cong \mathbb{C}\{\{(u_x^{\pm 1})_{x\in\mathcal{X}}\}\}\otimes_{\mathbb{C}}\mathrm{span}_{\mathbb{C}}\{\alpha_{z_0}^{z}(w)\}_{w\in\mathcal{X}^*}\,?
\end{aligned}$$

Iterated integrals and integro differential operators

Let $\mathcal{C} = \mathbb{C}\{\{(u_x^{\pm 1})_{x\in\mathcal{X}}\}\}$. One has $\theta_x \in \mathcal{C}\langle\partial\rangle$, for $x \in \mathcal{X}$, and

$$\forall x, y \in \mathcal{X}, \quad \forall w \in \mathcal{X}^*, \quad \theta_x \alpha_{z_0}^z(yw) = u_x^{-1}(z)u_y(z)\alpha_{z_0}^z(w).$$

Now, let Θ be the morphism $\mathbb{C}\langle\mathcal{X}\rangle \longrightarrow \mathcal{C}\langle\partial\rangle$ defined as follows

$$\Theta(w) = \begin{cases} \mathrm{Id} & \text{if} \quad w = 1_{\mathcal{X}^*}, \\ \Theta(u)\theta_x & \text{if} \quad w = ux \in \mathcal{X}^*\mathcal{X}. \end{cases}$$

One has, for any $w \in \mathcal{X}^*$,

1. $\Theta(\tilde{w})\alpha_{z_0}^z(w) = 1_\Omega$, and then $\partial(\Theta(\tilde{w})\alpha_{z_0}^z(w)) = 0$.
2. $L_w\alpha_{z_0}^z(\tilde{w}) = 0$, where $L_w := \partial\Theta(w) \in \mathcal{C}\langle\partial\rangle$.

For any $x_i \in \mathcal{X}$, let us consider a section of θ_{x_i} : $\theta_{x_i}\iota_{x_i}^{z_0} = \mathrm{Id}$, *i.e.*

$$\forall f \in \mathcal{H}(\Omega), \quad \iota_{x_i}^{z_0} f(z) = \int_{z_0}^{z} \omega_i(s)f(s).$$

The operator $\theta_y\iota_x^{z_0}$, for $x \neq y$, admits $u_y u_x^{-1}$ as eigenvalue, *i.e.*

$$\forall f \in \mathcal{H}(\Omega), \quad (\theta_y\iota_x^{z_0})f = u_y u_x^{-1} f, \quad \text{in particular}, \quad (\theta_y\iota_x^{z_0})1_\Omega = u_y u_x^{-1}.$$

Now, let $\Im^{z_0}$ be the morphism defined as follows

$$\Im^{z_0}(w) = \begin{cases} \mathrm{Id} & \text{if} \quad w = 1_{\mathcal{X}^*}, \\ \Im^{z_0}(u)\iota_x^{z_0} & \text{if} \quad w = ux \in \mathcal{X}^*\mathcal{X}. \end{cases}$$

Hence, for any $w \in X^*, \Im^{z_0}(w)1_\Omega = \alpha_{z_0}^z(w)$.

Practical example (polylogarithms)

For $X = \{x_0, x_1\}$ and $\Omega = \widetilde{\mathbb{C} \setminus \{0,1\}}$, let us consider

$$u_{x_0}(z) = z^{-1} \quad \text{and} \quad u_{x_1}(z) = (1-z)^{-1}.$$

Then, on the other hand,

$$\begin{aligned} \omega_0(z) = u_{x_0}(z)dz = z^{-1}dz \quad &\text{and} \quad \omega_1(z) = u_{x_1}(z)dz = (1-z)^{-1}dz, \\ \theta_{x_0} = u_{x_0}^{-1}(z)\partial = z\partial \quad &\text{and} \quad \theta_{x_1} = u_{x_1}^{-1}(z)\partial = (1-z)\partial. \end{aligned}$$

On the other hand [13], $\mathcal{C} = \mathbb{C}\{\{(u_x^{\pm 1})_{x \in X}\}\} = \mathbb{C}[z, z^{-1}, (1-z)^{-1}]$ being closed by $\theta_{x_0}, \theta_{x_1}$ and then by $\partial = \theta_{x_0} + \theta_{x_1} = \Theta(x_0 + x_1)$. One also has

1. $\Theta([x_1, x_0]) = [\theta_{x_1}, \theta_{x_0}] = \partial$.
2. $\forall w \in X^* x_1, \Im^0(w)1_\Omega = \alpha_0^z(w) = \mathrm{Li}_w(z)$.
3. $(\theta_{x_0}\iota_{x_1}^{z_0})1_\Omega = z(1-z)^{-1}$ and $(\theta_{x_1}\iota_{x_0}^{z_0})1_\Omega = z^{-1} - 1$.
4. $[\theta_{x_0}\iota_{x_1}^{z_0}, \theta_{x_1}\iota_{x_0}^{z_0}] = 0$.
5. $(\theta_{x_0}\iota_{x_1}^{z_0})(\theta_{x_1}\iota_{x_0}^{z_0}) = (\theta_{x_1}\iota_{x_0}^{z_0})(\theta_{x_0}\iota_{x_1}^{z_0}) = \mathrm{Id}$.

For any $L \in \mathcal{C}\langle\partial\rangle$, there is $P \in \mathcal{C}\langle X\rangle$ s.t $L = \Theta(P)$, meaning that Θ is surjective and non injective. Moreover, $\ker\Theta$ is the ideal generated by $[x_1, x_0] - x_0 - x_1$.

13. Any $p \in \mathcal{C}$ is polynomial on z, z^{-1} and $(1-z)^{-1}$ and admits 0 and 1 as poles.

Structure of iterated integrals

Proposition 1

The following assertions are equivalent

1. *The morphism* $(\mathcal{C}\langle\mathcal{X}\rangle, ⧢, 1_{\mathcal{X}^*}) \to (\mathrm{span}_{\mathcal{C}}\{\alpha_{z_0}^z(w)\}_{w\in\mathcal{X}^*}, \times, 1_\Omega)$ *is injective.*
2. $\{\alpha_{z_0}^z(w)\}_{w\in\mathcal{X}^*}$ *is* $\mathcal{C}$*-linearly independent.*
3. $\{\alpha_{z_0}^z(l)\}_{l\in\mathcal{L}yn\mathcal{X}}$ *is* $\mathcal{C}$*-algebraically independent.*
4. $\{\alpha_{z_0}^z(x)\}_{x\in\mathcal{X}}$ *is* $\mathcal{C}$*-algebraically independent.*
5. $\{\alpha_{z_0}^z(x)\}_{x\in\mathcal{X}\cup\{1_{\mathcal{X}^*}\}}$ *is* $\mathcal{C}$*-linearly independent.*

If one of the above assertions holds then

1. $\mathcal{C}[\{\alpha_{z_0}^z(w)\}_{w\in\mathcal{X}^*}]$ forms the universal $\mathcal{C}$-module of solutions of all differential equations $Ly = 0$,
2. $\mathcal{C}\{\alpha_{z_0}^z(w)\}_{w\in\mathcal{X}^*}$ forms the universal Picard-Vessiot extension related to all differential equations $Ly = 0$,

where[14] L's are linear differential operators belonging to $\mathcal{C}\langle\partial\rangle$.

14. For any $w \in X^*$, let $\mathcal{I}_w := \{L \in \mathcal{C}\langle\partial\rangle \text{ s.t. } L\alpha_{z_0}^z(w) = 0\}$. Then $\mathcal{I}_w$ is a left ideal.

Examples of linear differential equation

Example 8 (with $\mathcal{C} = \mathbb{C}(z)$)

$$(\partial - z)y = 0. \tag{1}$$

1. $e^{z^2/2}$ is solution of (1).
2. $ce^{z^2/2} = e^{z^2/2}e^{\log c}$ is an other solution ($c \in \mathbb{R} \setminus \{0\}$).
3. $\{e^{z^2/2}\}$ is a fundamental set of solutions of (1).
4. $\mathcal{C}\{e^{z^2/2}\}$ is a Picard-Vessiot extension related to (1).

For $\theta_{x_0} = z\partial$ and $\theta_{x_1} = (1-z)\partial$, since $L_{x_1x_0} = \partial\theta_{x_1}\theta_{x_0} \in \mathcal{C}\langle\partial\rangle$ then let

$$L_{x_1x_0}y = (z(1-z)\partial^3 + (2-3z)\partial^2 - 1)y = 0. \tag{2}$$

1. $L_{x_1x_0}\,\mathrm{Li}_2 = 0$ meaning that Li_2 is solution of (2).
2. $c\,\mathrm{Li}_2 = \mathrm{Li}_2\, e^{\log c}$ is an other solution ($c \in \mathbb{R} \setminus \{0\}$) but it is not independent to Li_2.
3. $\{\mathrm{Li}_2, \log, 1_\Omega\}$ is a fundamental set of solutions of (2).
4. $\mathcal{C}\{\mathrm{Li}_2, \log, 1_\Omega\}$ is a Picard-Vessiot extension [15] related to (2).

15. $\mathcal{C}\{\mathrm{Li}_2(z)\} = \mathcal{C} \otimes \mathbb{C}[\mathrm{Li}_2(z), \log(1-z), \log(z)]$.

Chen series of $\{\omega_i\}_{i\ge 1}$ and along $z_0 \rightsquigarrow z$

We get on the bialgebras $\mathcal{H}_{\sqcup\!\sqcup}(\mathcal{X})$ and $\mathcal{H}_{\uplus}(Y)$ (over a commutative ring A containing $\mathbb{Q}$)

$$\mathcal{D}_{\mathcal{X}} := \sum_{w\in\mathcal{X}^*} w\otimes w = \prod_{l\in\mathcal{L}yn\mathcal{X}}^{\searrow} e^{S_l\otimes P_l} \text{ and } \mathcal{D}_Y := \sum_{w\in Y^*} w\otimes w = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\Sigma_l\otimes\Pi_l}.$$

Hence, since $\alpha_{z_0}^z(u \sqcup\!\sqcup v) = \alpha_{z_0}^z(u)\alpha_{z_0}^z(v)$, for $u,v\in\mathcal{X}^*$, then the Chen series, $C_{z_0\rightsquigarrow z}\in\mathcal{H}(\Omega)\langle\!\langle\mathcal{X}\rangle\!\rangle$, is given by

$$C_{z_0\rightsquigarrow z} := \sum_{w\in\mathcal{X}^*}\alpha_{z_0}^z(w)w = (\alpha_{z_0}^z\otimes\mathtt{Id})\mathcal{D}_{\mathcal{X}} = \prod_{l\in\mathcal{L}yn\mathcal{X}}^{\searrow} e^{\alpha_{z_0}^z(S_l)P_l}$$

and then [16] $\Delta_{\sqcup\!\sqcup} C_{z_0\rightsquigarrow z} = C_{z_0\rightsquigarrow z}\otimes C_{z_0\rightsquigarrow z}$ and $\langle C_{z_0\rightsquigarrow z}|1_{\mathcal{X}^*}\rangle = 1$.

Note that $C_{z_0\rightsquigarrow z}$ only depends on the homotopy class of $z_0\rightsquigarrow z$ and the endpoints z_0, z. One has $C_{z_0\rightsquigarrow z}C_{z_1\rightsquigarrow z_0} = C_{z_1\rightsquigarrow z}$. Or equivalently,

$$\forall w\in\mathcal{X}^*,\quad \langle C_{z_1\rightsquigarrow z}|w\rangle = \sum_{u,v\in\mathcal{X}^*, uv=w}\langle C_{z_0\rightsquigarrow z}|u\rangle\langle C_{z_1\rightsquigarrow z_0}|v\rangle.$$

Although $\Delta_{\mathrm{conc}} w = \sum_{u,v\in\mathcal{X}^*, uv=w} u\otimes v$ but $\Delta_{\mathrm{conc}} C_{z_1\rightsquigarrow z} \neq C_{z_0\rightsquigarrow z}\otimes C_{z_1\rightsquigarrow z_0}$.

16. $\langle C_{z_0\rightsquigarrow z}|u \sqcup\!\sqcup v\rangle = \langle C_{z_0\rightsquigarrow z}|u\rangle\langle C_{z_0\rightsquigarrow z}|v\rangle$ and on the other hand, $\langle C_{z_0\rightsquigarrow z}|u \sqcup\!\sqcup v\rangle = \langle \Delta_{\sqcup\!\sqcup} C_{z_0\rightsquigarrow z}|u\otimes v\rangle, \langle C_{z_0\rightsquigarrow z}|u\rangle\langle C_{z_0\rightsquigarrow z}|v\rangle = \langle C_{z_0\rightsquigarrow z}\otimes C_{z_0\rightsquigarrow z}|u\otimes v\rangle$.

More about Chen series

Note also that, for $g \in \mathcal{H}(\Omega)$, one has $C_{g(z_0)\rightsquigarrow g(z)} = g_* C_{z_0 \rightsquigarrow z}$, *i.e.* the Chen series of $\{g^*\omega_i\}_{i\geq 1}$ along the path $g^*(z_0 \rightsquigarrow z)$.

Example 9 (with $\omega_0(z) = z^{-1}dz$ and $\omega_1(z) = (1-z)^{-1}dz$)

$g(z)$	z	z^{-1}	$(z-1)z^{-1}$	$z(z-1)^{-1}$	$(1-z)^{-1}$	$1-z$
$g^*\omega_0$	ω_0	$-\omega_0$	$-\omega_1 - \omega_0$	$\omega_1 + \omega_0$	ω_1	$-\omega_1$
$g^*\omega_1$	ω_1	$\omega_1 + \omega_0$	$-\omega_0$	$-\omega_1$	$-\omega_1 - \omega_0$	$-\omega_0$

For any $n \geq 0$, one has

$$\mathbf{d}^n C_{z_0 \rightsquigarrow z} = p_n C_{z_0 \rightsquigarrow z},$$

where, for any $S \in \mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle$, $\mathbf{d}S \in \mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle$ is defined as follows

$$\mathbf{d}S = \sum_{w \in \mathcal{X}^*} (\partial\langle S|w\rangle)w,$$

$p_n \in \mathcal{C}\langle\mathcal{X}\rangle$ is defined as follows

$$p_n = \sum_{\mathrm{wgt}\,\mathbf{r}=n} \sum_{w \in \mathcal{X}^n} \prod_{i=1}^{\deg \mathbf{r}} \binom{\sum_{j=1}^{i} r_j + j - 1}{r_i} \tau_{\mathbf{r}}(w)$$

and, for $w = x_{i_1} \ldots x_{i_k} \in \mathcal{X}^*$ associated to the derivation multiindex $\mathbf{r} = (r_1, \ldots, r_k) \in \mathbb{N}^k$ of weight $\mathrm{wgt}\,\mathbf{r} = |w| + \sum_{i=1}^{k} r_i$ and of degree $\deg \mathbf{r} = |w|$, $\tau_{\mathbf{r}}(w) := \tau_{r_1}(x_{i_1}) \ldots \tau_{r_k}(x_{i_k}) = (\partial^{r_1} u_{x_{i_1}})x_{i_1} \ldots (\partial^{r_k} u_{x_{i_k}})x_{i_k}$.

Continuity, indiscernability and growth condition

For $i = 0, 2$, let $(\mathbf{k}_i, \|.\|_i)$ be a semi-normed space and $g_i \in \mathbb{Z}$.

Definition 10

1. Let $\mathcal{C}l$ be a class of $\mathbf{k}_1\langle\langle\mathcal{X}\rangle\rangle$. Let $S \in \mathbf{k}_2\langle\langle\mathcal{X}\rangle\rangle$ and it is said to be
   a) *continuous* over $\mathcal{C}l$ if, for $\Phi \in \mathcal{C}l$, the following sum is convergent
   $$\sum_{w\in\mathcal{X}^*} \|\langle S|w\rangle\|_2 \|\langle\Phi|w\rangle\|_1 .$$
   We will denote $\langle S\|\Phi\rangle$ the sum $\sum_{w\in\mathcal{X}^*}\langle S|w\rangle\langle\Phi|w\rangle$ and $\mathbf{k}_2\langle\langle\mathcal{X}\rangle\rangle^{\mathrm{cont}}$ the set of continuous power series over $\mathcal{C}l$.
   b) *indiscernable* over $\mathcal{C}l$ iff, for any $\Phi \in \mathcal{C}l$, $\langle S\|\Phi\rangle = 0$.
2. Let χ_1 and χ_2 be real positive functions over $\mathcal{X}^*$. Let $S \in \mathbf{k}_1\langle\langle\mathcal{X}\rangle\rangle$.
   a) S satisfies the χ_1-*growth condition* of order g_1 if it satisfies
   $$\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in \mathcal{X}^{\geq n}, \quad \|\langle S|w\rangle\|_1 \leq K\chi_1(w)\ |w|!^{g_1}.$$
   We denote by $\mathbf{k}_1^{(\chi_1,g_1)}\langle\langle\mathcal{X}\rangle\rangle$ the set of formal power series in $\mathbf{k}_1\langle\langle\mathcal{X}\rangle\rangle$ satisfying the χ_1-growth condition of order g_1.
   b) If S is continuous over $\mathbf{k}_2^{(\chi_2,g_2)}\langle\langle\mathcal{X}\rangle\rangle$ then it will be said to be (χ_2, g_2)-*continuous*. The set of formal power series which are (χ_2, g_2)-continuous is denoted by $\mathbf{k}_2^{(\chi_2,g_2)}\langle\langle\mathcal{X}\rangle\rangle^{\mathrm{cont}}$.

Convergence condition

Proposition 2

Let χ_1 and χ_2 be real positive functions over $\mathcal{X}^*$.
Let g_1 and $g_2 \in \mathbb{Z}$ such that $g_1 + g_2 \leq 0$.

1. *Let $\mathbf{k}_1^{(\chi_1,g_1)}\langle\langle\mathcal{X}\rangle\rangle$ and let $P \in \mathbf{k}_1\langle\mathcal{X}\rangle$.*
   The right residual of S by P belongs to $\mathbf{k}_1^{(\chi_1,g_1)}\langle\langle\mathcal{X}\rangle\rangle$.
2. *Let $R \in \mathbf{k}_2^{(\chi_2,g_2)}\langle\langle\mathcal{X}\rangle\rangle$ and let $Q \in \mathbf{k}_2\langle\mathcal{X}\rangle$.*
   The concatenation QR belongs to $\mathbf{k}_2^{(\chi_2,g_2)}\langle\langle\mathcal{X}\rangle\rangle$.
3. *χ_1, χ_2 are morphisms over $\mathcal{X}^*$ satisfying $\sum_{x\in\mathcal{X}} \chi_1(x)\chi_2(x) < 1$.*
   If $F_1 \in \mathbf{k}_1^{(\chi_1,g_1)}\langle\langle\mathcal{X}\rangle\rangle$ (resp. $F_2 \in \mathbf{k}_2^{(\chi_2,g_2)}\langle\langle\mathcal{X}\rangle\rangle$) then F_1 (resp. F_2) is continuous over $\mathbf{k}_2^{(\chi_2,g_2)}\langle\langle\mathcal{X}\rangle\rangle$ (resp. $\mathbf{k}_1^{(\chi_1,g_1)}\langle\langle\mathcal{X}\rangle\rangle$).

Proposition 3

Let $\mathcal{C}l \subset \mathbf{k}_1\langle\langle\mathcal{X}\rangle\rangle$ be a monoid containing $\{e^{tx}\}_{x\in\mathcal{X}}^{t\in\mathbf{k}_1}$. Let $S \in \mathbf{k}_2\langle\langle\mathcal{X}\rangle\rangle^{cont}$.

1. *If S is indiscernable over $\mathcal{C}l$ then for any $x \in \mathcal{X}$, $x \triangleleft S$ and $S \triangleright x$ belong to $\mathbf{k}_2\langle\langle\mathcal{X}\rangle\rangle^{cont}$ and they are indiscernable over $\mathcal{C}l$.*
2. *S is indiscernable over $\mathcal{C}l$ iff $S = 0$.*

Chen series and differential equations

Let K be a compact on Ω. There is $c_K \in \mathbb{R}_{\geq 0}$ and a morphism M_K s.t.

$$\forall w \in \mathcal{X}^*, \quad \|\langle C_{z_0 \rightsquigarrow z}|w\rangle\|_K \leq c_K M_K(w)\,|w|!^{-1}.$$

Let $R \in \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ of minimal representation (λ, μ, η) of dimension n. Then

$$\forall w \in \mathcal{X}^*, \quad |\langle R|w\rangle| \leq \|\lambda\|_\infty^{1,n}\|\mu(w)\|_\infty^{n,n}\|\eta\|_\infty^{n,1}.$$

With these data, we have

Theorem 11

If $c_K\|\lambda\|_\infty^{1,n}\|\eta\|_\infty^{n,1}\sum_{x\in\mathcal{X}} M_K(x)\|\mu(x)\|_\infty^{n,n} < 1$ then $\alpha_{z_0}^z(R) = \langle R\| C_{z_0 \rightsquigarrow z}\rangle$ and

$$\forall x \in \mathcal{X}, \quad \theta_x\alpha_{z_0}^z(R) = \sum_{x'\in\mathcal{X}} u_x^{-1}(z)u_{x'}(z)\alpha_{z_0}^z(R \triangleleft x').$$

Letting $y(z_0, z) := \langle R\| C_{z_0 \rightsquigarrow z}\rangle$, the following assertions are equivalent :

1. *There is $p \in \mathcal{C}_0\langle\mathcal{X}\rangle$ s.t. $\langle R\|pC_{z_0 \rightsquigarrow z}\rangle = \langle R \triangleleft p\| C_{z_0 \rightsquigarrow z}\rangle = 0$.*
2. *There is $l = 0, .., n-1$ s.t. $\{\partial^k y\}_{0\leq k\leq l}$ is $\mathcal{C}_0$-linearly independent and $a_l, \ldots, a_1, a_0 \in \mathcal{C}_0$ s.t. $(a_l\partial^l + \ldots + a_1\partial + a_0)y = 0$.*

Proposition 4

Let $G \in \mathbb{C}\langle\langle X\rangle\rangle$ and $H \in \mathbb{C}_{\mathrm{exc}}\langle\langle X\rangle\rangle$ s.t. $\alpha_{z_0}^z(G) = \langle G\| C_{z_0 \rightsquigarrow z}\rangle$ and $h(\alpha_{z_0}^z(x_0), \alpha_{z_0}^z(x_1)) := \alpha_{z_0}^z(H) = \langle H\| C_{z_0 \rightsquigarrow z}\rangle$ exist ($X = \{x_0, x_1\}$). Then

$$\alpha_{z_0}^z(HG) = \langle G|1_{X^*}\rangle\alpha_{z_0}^z(H) + \int_{z_0}^z h(\alpha_s^z(x_0), \alpha_s^z(x_1))d\alpha_{z_0}^s(G).$$

Practical examples (eulerian functions)

For any $y \in \mathcal{X}^*, n \in \mathbb{N}$ and $t \in \mathbb{C}, |t| < 1$, since $y^n = y^{ш n}/n!$ then

$$\alpha_{z_0}^z(y^n) = \frac{[\alpha_{z_0}^z(y)]^n}{n!} \quad \text{and} \quad \alpha_{z_0}^z((ty)^*) = e^{t\alpha_{z_0}^z(y)}.$$

Example 12 (extension of eulerian functions)

For any $z \in \Omega = \mathbb{C}, |z| < 1$, let us consider

$$\ell_1(z) := \gamma z - \sum_{k\geq 2} \zeta(k)\frac{(-z)^k}{k} \quad \text{and } \forall r \geq 2, \quad \ell_r(z) := -\sum_{k\geq 1} \zeta(kr)\frac{(-z^r)^k}{k}.$$

Hence, for any $k \geq 1$, letting $\omega_k = \partial \ell_k$, one has

$$\alpha_0^z(y_1^*) = e^{\ell_1(z)} =: \frac{1}{\Gamma_{y_1}(1+z)} \quad \text{and } \forall r \geq 2, \quad \alpha_0^z(y_k^*) = e^{\ell_k(z)} =: \frac{1}{\Gamma_{y_k}(1+z)}.$$

Example 13 (more about extented eulerian functions)

Let us consider $\omega_k = e^{\ell_k}\partial \ell_k (k \geq 1)$, where ℓ_k is defined as in Ex. 12.

Then $\quad \alpha_0^z(y_k) = e^{\ell_k(z)} = \Gamma_{y_k}^{-1}(1+z) \quad$ and $\quad \alpha_0^z(y_k^*) = e^{e^{\ell_k(z)}-1}, \quad k \geq 1.$

Remark 1

In Examples 12, 13, Γ_{y_1} is nothing else the eulerian Gamma function, Γ. What are $\{\alpha_0^z(w)\}_{w \in Y^*Y}$? Similarly, in the case of $\{\alpha_0^z(w)\}_{w \in (Y \cup \{y_0\})^*}$ and with the new differential form $\omega_0(z) = z^{-1}dz$?

First properties of extented eulerian functions

Let G_r (resp. $\mathcal{G}_r$) denote the set (resp. group) of solutions, $\{\xi_0, \dots, \xi_{r-1}\}$, of $z^r = (-1)^{r-1}$ (resp. $z^r = 1$), for $r \geq 1$. If r is odd, it is a group as $G_r = \mathcal{G}_r$ otherwise it is an orbit as $G_r = \xi\mathcal{G}_r$, where ξ is any solution of $\xi^r = -1$ (or equivalently, $\xi \in \mathcal{G}_{2r}$ and $\xi \notin \mathcal{G}_r$).

Proposition 5 (Weierstrass factorization)

1. *For $r \geq 1, \chi \in \mathcal{G}_r$ and $z \in \mathbb{C}, |z| < 1$, the functions ℓ_r and e^{ℓ_r} have the symmetry, $\ell_r(z) = \ell_r(\chi z)$ and $e^{\ell_r(z)} = e^{\ell_r(\chi z)}$. In particular, for r even, as $-1 \in \mathcal{G}_r$, these functions are even.*

2. *For $|z| < 1$, we have*
$$\ell_r(z) = \sum_{\chi \in G_r} \log \frac{1}{\Gamma(1+\chi z)} \text{ and } e^{\ell_r(z)} = \prod_{\chi \in G_r} e^{\gamma \chi z} \prod_{n \geq 1} (1 + \frac{\chi z}{n}) e^{-\frac{\chi z}{n}}.$$

3. *For any odd $r \geq 2$, $\Gamma_{y_r}^{-1}(1+z) = e^{\ell_r(z)} = \Gamma^{-1}(1+z) \prod_{\chi \in G_r \setminus \{1\}} e^{\ell_1(\chi z)}$.*

4. *In general, for any odd or even $r \geq 2$,*
$$\ell_r(z) = \prod_{\chi \in G_r} e^{\ell_1(\chi z)} = \prod_{n \geq 1} (1 + \frac{z^r}{n^r}).$$

Other practical examples (1/2)

Example 14 ($\omega_1(z) = (1-z)^{-1}dz$ and $\omega_0(z) = z^{-1}dz$)

1. For any $a, z \in \mathbb{C}$ s.t. $|a| < 1, |z| < 1$, one has
$$\begin{aligned}\mathrm{Li}_{(ax_0)^* x_1}(z) &= \alpha_0^z((ax_0)^* x_1)\\ &= \int_0^z e^{a\log(\frac{z}{s})}\omega_1(s) = z^a \int_0^z \sum_{n\geq 0} s^{n-a} ds = \sum_{n\geq 1} \frac{z^n}{n-a}.\end{aligned}$$

2. For any $n \in \mathbb{N}$ and $a, b \in \mathbb{C}$ s.t. $|a| < 1, |b| < 1$, one has
$$\mathrm{Li}_{x_0^n}(z) = \alpha_1^z(x_0^n) = \log^n(z)/n!, \quad \mathrm{Li}_{x_1^n}(z) = \alpha_0^z(x_1^n) = \log^n((1-z)^{-1})/n!,$$
$$\mathrm{Li}_{(ax_0)^*}(z) = \alpha_1^z((ax_0)^*) = z^a, \quad \mathrm{Li}_{(bx_1)^*}(z) = \alpha_0^z((bx_1)^*) = (1-z)^{-b}.$$
Let $\mathcal{C} = \mathbb{C}[z^a, (1-z)^b]_{a,b\in\mathbb{C}}$ and $S \in \mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle \sqcup\!\sqcup\, \mathbb{C}\langle X\rangle$ (resp. $\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle = \mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle x_0\rangle\rangle \sqcup\!\sqcup\, \mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle x_1\rangle\rangle$), we get
$$\mathrm{Li}_S(z) \in \mathcal{C}[\{\mathrm{Li}_l\}_{l\in\mathcal{L}ynX}] \text{ (resp. } \mathcal{C}[\log(z), \log(1-z)]).$$

3. For any $z, a, b \in \mathbb{C}$ s.t. $|z| < 1$ and $\Re a > 0, \Re b > 0$, we get the partial Beta function and the eulerian
Beta function, $\mathrm{B}(a,b) = \mathrm{B}(1;a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, as follows[17]
$$\mathrm{B}(z;a,b) := \int_0^z dt\ t^{a-1}(1-t)^{b-1} = \begin{Bmatrix} \mathrm{Li}_{x_0[(ax_0)^* \sqcup\!\sqcup\, ((1-b)x_1)^*]}(z) \\ \mathrm{Li}_{x_1[((a-1)x_0)^* \sqcup\!\sqcup\, (-bx_1)^*]}(z)\end{Bmatrix}.$$

17. $x_0[(ax_0)^* \sqcup\!\sqcup\, ((1-b)x_1)^*$ and $x_1[((a-1)x_0)^* \sqcup\!\sqcup\, (-bx_1)^*]$ are of the form (F_2). What is $\alpha_0^z(S)$, for S of the form (F_2) ?

Other on practical examples (2/2)

Example 15 (Polylogarithms indexed by non positive integers)

Now, let us use the noncommutative multivariate exponential transforms, *i.e.*, for any rational exchangeable series, we get the following transform

$$\sum_{i_0,i_1\geq 0} s_{i_0,i_1} x_0^{i_0} \sqcup\!\sqcup\, x_1^{i_1} \quad \longmapsto \quad \sum_{i_0,i_1\geq 0} \frac{s_{i_0,i_1}}{i_0!\,i_1!} \log^{i_0}(z) \log^{i_1}((1-z)^{-1}).$$

In particular, for any $n \in \mathbb{N}$, we have $x_0^n \mapsto \log^n(z)/n!$ and $x_1^n \mapsto \log^n((1-z)^{-1})/n!$. Then $(tx_0)^* \mapsto z^t$ and $(tx_1)^* \mapsto (1-z)^{-t}$. We then obtain the following polylogarithms indexed by rational series

$$\mathrm{Li}_{x_0^*}(z) = z, \quad \mathrm{Li}_{x_1^*}(z) = (1-z)^{-1}, \quad \mathrm{Li}_{(ax_0+bx_1)^*}(z) = z^a(1-z)^{-b}$$

Thus, for any $(s_1,\ldots,s_r) \in \mathbb{N}_+^r$, there exists an unique series $R_{y_{s_1}\ldots y_{s_r}}$ belonging to $(\mathbb{Z}[x_1^*], \sqcup\!\sqcup\,, 1_{X^*})$ s.t. $\mathrm{Li}_{-s_1,\ldots,-s_r} = \mathrm{Li}_{R_{y_{s_1}\ldots y_{s_r}}}$. More precisely,

$$R_{y_{s_1}\ldots y_{s_r}} = \sum_{k_1=0}^{s_1} \ldots \sum_{k_r=0}^{\substack{(s_1+\ldots+s_r)-\\(k_1+\ldots+k_{r-1})}} \binom{s_1}{k_1} \ldots \binom{\sum_{i=1}^{r} s_i - \sum_{i=1}^{r-1} k_i}{k_r} \rho_{k_1} \sqcup\!\sqcup\, \ldots \sqcup\!\sqcup\, \rho_{k_r},$$

where, for any $i = 1,\ldots,r$, if $k_i = 0$ then $\rho_{k_i} = x_1^* - 1_{X^*}$ else

$$\rho_{k_i} = x_1^* \sqcup\!\sqcup \sum_{j=1}^{k_i} S_2(k_i,j) j! (x_1^* - 1_{X^*})^{\sqcup\!\sqcup\, j}$$

the $S_2(k_i,j)$ being the Stirling numbers of second kind.

NONCOMMUTATIVE PV THEORY AND INDEPENDENCE VIA WORDS

First step of noncommutative PV theory

The Chen series $C_{z_0 \rightsquigarrow z}$ of $\{\omega_k\}_{k\geq 1}$ and along the path $z_0 \rightsquigarrow z$ over Ω satisfies the following differential equation

$$(NCDE) \quad \mathbf{d}S = MS, \quad \text{with} \quad M = \sum_{x\in\mathcal{X}} u_x x \quad \text{and} \quad u_x \in \mathcal{C}_0 \cap \mathcal{A}^{-1}.$$

$$\Delta_{\sqcup\!\sqcup} M = \sum_{x\in\mathcal{X}} u_x(1_{\mathcal{X}^*} \otimes x + x \otimes 1_{\mathcal{X}^*}) = 1_{\mathcal{X}^*} \otimes M + M \otimes 1_{\mathcal{X}^*}.$$

The space of solutions of $(NCDE)$ is a right free $\mathbb{C}\langle\langle X\rangle\rangle$-module of rank 1. By a theorem of Ree, $C_{z_0 \rightsquigarrow z}$ is a $\sqcup\!\sqcup$ $-$group-like solution [18] of $(NCDE)$. Moreover, if G, H are $\sqcup\!\sqcup$ $-$group-like solutions there is a constant Lie series C s.t. $G = He^C$ (and conversely). From this, it follows that

- the Hausdorff group $\{e^C\}_{C\in\mathcal{L}ie_{\mathbb{C}}\langle\langle\mathcal{X}\rangle\rangle}$, group of characters of $\mathcal{H}_{\sqcup\!\sqcup}(\mathcal{X})$, plays the role of the differential Galois group of $(NCDE)+$ $\sqcup\!\sqcup$ $-$group-like.

Which leads us to the following definition

- the PV extension related to $(NCDE)$ is $\widehat{\mathcal{C}_0.\mathcal{X}}\{C_{z_0 \rightsquigarrow z}\}$.

It, of course, is such that $\mathrm{Const}(\mathcal{C}_0\langle\langle\mathcal{X}\rangle\rangle) = \ker \mathbf{d} = \mathbb{C}.1_\Omega\langle\langle\mathcal{X}\rangle\rangle$.

18. It can be obtained as the limit of a convergent Picard iteration, initialized at $\langle C_{z_0 \rightsquigarrow z} | 1_{\mathcal{X}^*}\rangle = 1_{\mathcal{H}(\Omega)}$, for ultrametric distance.

Basic triangular theorem over a differential ring (BTT)

Let $S \in \mathcal{A}\langle\langle\mathcal{X}\rangle\rangle$ be a group-like solution of $(NCDE)$ in the following form

$$S = \sum_{w \in \mathcal{X}^*} \langle S|w\rangle w = \sum_{w \in \mathcal{X}^*} \langle S|S_w\rangle P_w = \prod_{l \in \mathcal{L}yn\mathcal{X}}^{\searrow} e^{\langle S|S_l\rangle P_l}.$$

Then

1. If $H \in \mathcal{A}\langle\langle\mathcal{X}\rangle\rangle$ is another grouplike solution then there exists $C \in \mathcal{L}ie_{\mathcal{A}}\langle\langle\mathcal{X}\rangle\rangle$ such that $S = He^C$ (and conversely).
2. The following assertions are equivalent [19]
   a) $\{\langle S|w\rangle\}_{w \in \mathcal{X}^*}$ is $\mathcal{C}_0$-linearly independent,
   b) $\{\langle S|S_l\rangle\}_{l \in \mathcal{L}yn\mathcal{X}}$ is $\mathcal{C}_0$-algebraically independent,
   c) $\{\langle S|x\rangle\}_{x \in \mathcal{X}}$ is $\mathcal{C}_0$-algebraically independent,
   d) $\{\langle S|x\rangle\}_{x \in \mathcal{X} \cup \{1_{\mathcal{X}^*}\}}$ is $\mathcal{C}_0$-linearly independent,
   e) $\{u_x\}_{x \in \mathcal{X}}$ is such that, for $f \in \mathrm{Frac}(\mathcal{C}_0)$ and $(c_x)_{x \in \mathcal{X}} \in \mathbb{C}^{(\mathcal{X})}$,
   $$\sum_{x \in \mathcal{X}} c_x u_x = \partial f \quad \Longrightarrow \quad (\forall x \in \mathcal{X})(c_x = 0).$$
   f) $(u_x)_{x \in \mathcal{X}}$ is free over $\mathbb{C}$ and $\partial \mathrm{Frac}(\mathcal{C}_0) \cap \mathrm{span}_{\mathbb{C}}\{u_x\}_{x \in \mathcal{X}} = \{0\}$.

19. In particular, for $S = C_{z_0 \rightsquigarrow z} = \sum_{w \in \mathcal{X}^*} \alpha_{z_0}^z(w)w$.

Examples of positive cases over $\mathcal{X} = \{x\}, \mathcal{A} = (\mathcal{H}(\Omega), \partial)$

1. $\Omega = \mathbb{C}, u_x(z) = 1_\Omega, \mathcal{C}_0 = \mathbb{C}\{\{u_x^{\pm 1}\}\} = \mathbb{C}$.
   $\alpha_0^z(x^n) = z^n/n!$, for $n \geq 1$. Thus, $\mathbf{d}S = xS$ and
   $$S = \sum_{n\geq 0} \alpha_0^z(x^n)x^n = \sum_{n\geq 0} \frac{z^n}{n!}x^n = e^{zx}.$$
   Moreover, $\alpha_0^z(x) = z$ which is transcendent over $\mathcal{C}_0$
   and the family $\{\alpha_0^z(x^n)\}_{n\geq 0}$ is $\mathcal{C}_0$-free. Let $f \in \mathcal{C}_0$ then $\partial f = 0$. Thus,
   if $\partial f = cu_x$ then $c = 0$.
2. $\Omega = \mathbb{C}\backslash]-\infty, 0], u_x(z) = z^{-1}, \mathcal{C}_0 = \mathbb{C}\{\{z^{\pm 1}\}\} = \mathbb{C}[z^{\pm 1}] \subset \mathbb{C}(z)$.
   $\alpha_1^z(x^n) = \log^n(z)/n!$, for $n \geq 1$. Thus $\mathbf{d}S = z^{-1}xS$ and
   $$S = \sum_{n\geq 0} \alpha_1^z(x^n)x^n = \sum_{n\geq 0} \frac{\log^n(z)}{n!}x^n = z^x.$$
   Moreover, $\alpha_1^z(x) = \log(z)$ which is transcendent over $\mathbb{C}(z)$ then over $\mathbb{C}[z^{\pm 1}]$. The family the family $\{\alpha_1^z(x^n)\}_{n\geq 0}$ is $\mathbb{C}(z)$-free and then $\mathcal{C}_0$-free. Let $f \in \mathcal{C}_0$ then $\partial f \in \mathrm{span}_{\mathbb{C}}\{z^{\pm n}\}_{n\neq 1}$. Thus,
   if $\partial f = cu_x$ then $c = 0$.

Examples of negative cases over $\mathcal{X} = \{x\}, \mathcal{A} = (\mathcal{H}(\Omega), \partial)$

1. $\Omega = \mathbb{C}, u_x(z) = e^z, \mathcal{C}_0 = \mathbb{C}\{\{e^{\pm z}\}\} = \mathbb{C}[e^{\pm z}]$.
   $\alpha_0^z(x^n) = (e^z - 1)^n / n!$, for $n \geq 1$. Thus, $\mathbf{d}S = e^z x S$ and
   $$S = \sum_{n\geq 0} \alpha_0^z(x^n) x^n = \sum_{n\geq 0} \frac{(e^z-1)^n}{n!} x^n = e^{(e^z-1)x}.$$
   Moreover, $\alpha_0^z(x) = e^z - 1$ which is not transcendent over $\mathcal{C}_0$ and and $\{\alpha_0^z(x^n)\}_{n\geq 0}$ is not $\mathcal{C}_0$-free. If $f(z) = ce^z \in \mathcal{C}_0$ $(c \neq 0)$ then $\partial f(z) = ce^z = cu_x(z)$.
2. $\Omega = \mathbb{C}\backslash]-\infty, 0], u_x(z) = z^a (a \notin \mathbb{Q})$,
   $\mathcal{C}_0 = \mathbb{C}\{\{z, z^{\pm a}\}\} = \mathrm{span}_{\mathbb{C}}\{z^{ka+l}\}_{k,l\in\mathbb{Z}}$.
   $\alpha_0^z(x^n) = (a+1)^{-n} z^{n(a+1)} / n!$, for $n \geq 1$. Thus, $\mathbf{d}S = z^a x S$ and
   $$S = \sum_{n\geq 0} \alpha_0^z(x^n) x^n = \sum_{n\geq 0} \frac{z^{n(a+1)}}{(a+1)^n n!} x^n = e^{(a+1)^{-1} z^{(a+1)} x}.$$
   Moreover, $\alpha_0^z(x) = z^{a+1}/(a+1)$ which is not transcendent over $\mathcal{C}_0$ and $\{\alpha_0^z(x^n)\}_{n\geq 0}$ is not $\mathcal{C}_0$-free. If $f(z) = cz^{a+1}/(a+1) \in \mathcal{C}_0$ $(c \neq 0)$ then $\partial f(z) = cz^a = cu_x(z)$.

Independence over $\mathbb{C}$ of extended eulerian functions

Proposition 6

Let $L := \mathrm{span}_{\mathbb{C}}\{\ell_r\}_{r\geq 1}$ *and* $E := \mathrm{span}_{\mathbb{C}}\{e^{\ell_r}\}_{r\geq 1}$. *One has*

1. *The families* $(\ell_r)_{r\geq 1}$ *and* $(e^{\ell_r})_{r\geq 1}$ *are* $\mathbb{C}$*-linearly free and free from* $1_{\mathcal{H}(\Omega)}$. *Hence, with the differential forms* $\{u_{y_r}dz\}_{r\geq 1}$ *and*[20]
   a) $u_{y_r} = e^{\ell_r}\partial\ell_r$, *the restriction* $\alpha_0^z : \mathbb{C}Y \to E$ *is injective.*
   b) $u_{y_r} = \partial\ell_r$, *the restrictions of* α_0^z, $\mathrm{span}_{\mathbb{C}}\{y_r\}_{r\geq 1} \to L$ *and* $\mathrm{span}_{\mathbb{C}}\{y_r^*\}_{r\geq 1} \to E$ *are injective.*
2. *The families* $(\ell_r)_{r\geq 1}$ *and* $(e^{\ell_r})_{r\geq 1}$ *are* $\mathbb{C}$*-algebraically independent.*
3. *For any* $r \geq 1$, *one has*
   a) *The functions* ℓ_r *and* e^{ℓ_r} $\mathbb{C}$*-algebraically independent.*
   b) *The function* ℓ_r *is holomorphic on the open unit disc,* $D_{<1}$,
   c) *The function* e^{ℓ_r} *(resp.* $e^{-\ell_r}$*) is entire (resp. meromorphic), and admits a countable set of isolated zeroes (resp. poles) on the complex plane which is expressed as* $\biguplus_{\chi\in G_r}\chi\mathbb{Z}_{\leq -1}$.

20. see Examples 13 and 12, respectively.

Proof of independence over $\mathbb{C}$ of eulerian functions

1. Since $(\ell_r)_{r\geq 1}$ is triangular[21] then $(\ell_r)_{r\geq 1}$ is $\mathbb{C}$-linearly free. So is $(e^{\ell_r} - e^{\ell_r(0)})_{r\geq 1}$, being triangular, we get that $(e^{\ell_r})_{r\geq 1}$ is $\mathbb{C}$-lin. free and free from $1_{\mathcal{H}(\Omega)}$. Since $\{x^*\}_{x\in\mathcal{X}}$ are alg. free over $(\mathbb{C}\langle\mathcal{X}\rangle, ш, 1_{\mathcal{X}^*})$ then we get the next results.

2. To prove the $\mathbb{C}$-algebraic independence of $\{e^{\ell_r}\}_{r\geq 1}$ and $(\ell_r)_{r\geq 1}$, using the result of the first item, we apply BTT with u_{y_r} defined as in a) and b), respectively.

3.
   a) Since $\ell_r(0) = 0, \partial e^{\ell_r} = e^{\ell_r}\partial\ell_r$ then ℓ_r, e^{ℓ_r} are $\mathbb{C}$-alg. free.
   b) We have $e^{\ell_1(z)} = \Gamma^{-1}(1+z)$ which proves the claim for $r = 1$. For $r \geq 2$, note that $1 \leq \zeta(r) \leq \zeta(2)$ which implies that the radius of convergence of the exponent is 1 and means that ℓ_r is holomorphic on the open unit disc. This proves the claim.
   c) $e^{\ell_r(z)} = \Gamma_{y_r}^{-1}(1+z)$ (resp. $e^{-\ell_r(z)} = \Gamma_{y_r}(1+z)$) is entire (resp. meromorphic) as finite product of entire (resp. meromorphic) functions and Weierstrass factorization yields zeroes (resp. poles).

21. $(g_i)_{i\geq 1}$ is said to be *triangular* if the valuation of $g_i, \varpi(g_i)$, equals $i \geq 1$. It is easy to check that such a family is $\mathbb{C}$-lin. free and that is also the case of families s.t. $(g_i - g(0))_{i\geq 1}$ is triangular.

Independence of $\{e^{\ell_r}\}_{k\geq 1}$ over differential subalgebra

The algebra $\mathbb{C}[L]$ (resp. $\mathbb{C}[E]$) is generated freely by $(\ell_r)_{r\geq 1}$ (resp. $(e^{\ell_r})_{r\geq 1}$) which are holomorphic on $D_{<1}$ (resp. entire) functions. Moreover, any $f \in \mathbb{C}[L] \setminus \mathbb{C}.1_{\mathcal{H}(\Omega)}$ (resp. $g \in \mathbb{C}[E] \setminus \mathbb{C}.1_{\mathcal{H}(\Omega)}$) is holomorphic on $D_{<1}$ (resp. entire) and then $f \notin \mathbb{C}[E]$ (resp. $g \notin \mathbb{C}[L]$). Hence, $E \cap L = \{0\}$ and more generally, $\mathbb{C}[E] \cap \mathbb{C}[L] = \mathbb{C}.1_{\mathcal{H}(\Omega)}$.

Let $\mathcal{L} := \mathbb{C}\{\{(\ell_r^{\pm 1})_{r\geq 1}\}\} = \mathbb{C}[\{\ell_r^{\pm 1}, \partial^i \ell_r\}_{r,i\geq 1}]$ and $\mathcal{E} := \mathbb{C}\{\{(e^{\pm \ell_r})_{r\geq 1}\}\}$. Let $\mathcal{L}^+ := \mathbb{C}[\{\partial^i \ell_r\}_{r,i\geq 1}]$, being integral domain generated by holomorphic functions, and then $\mathrm{Frac}(\mathcal{L}^+)$ is generated by meromorphic functions. Since there is $0 \neq q_{i,l,k} \in \mathcal{L}^+$ s.t. $(\partial^i e^{\pm \ell_k})^l = q_{i,l,k} e^{\pm l \ell_k}$, $i, l, k \geq 1$ then

$$\begin{aligned}
\mathcal{E}^+ &:= \mathrm{span}_{\mathbb{C}}\{(\partial^{i_1} e^{\pm \ell_{r_1}})^{l_1} \ldots (\partial^{i_k} e^{\pm \ell_{r_k}})^{l_k}\}_{(i_1,l_1,r_1),\ldots,(i_k,l_k,r_k)\in(\mathbb{N}_{\geq 1})^3, k\geq 1} \\
&= \mathrm{span}_{\mathbb{C}}\{q_{i_1,l_1,r_1} \ldots q_{i_k,l_k,r_k} e^{l_1\ell_{r_1}+\ldots+l_k\ell_{r_k}}\}_{(i_1,l_1,r_1),\ldots,(i_k,l_k,r_k)\in \mathbb{N}_{\geq 1}\times\mathbb{Z}_{\neq 0}\times\mathbb{N}_{\geq 1}, k\geq 1} \\
&\subset \mathrm{span}_{\mathcal{L}^+}\{e^{l_1\ell_{r_1}+\ldots+l_k\ell_{r_k}}\}_{(l_1,r_1),\ldots,(l_k,r_k)\in\mathbb{Z}^*\times\mathbb{N}_{\geq 1}, k\geq 1} =: \mathcal{C}.
\end{aligned}$$

Note that $\mathcal{E}^+ \cap E = \{0\}$ and $\mathcal{C}$ is a differential subring[22] of $\mathcal{A} = \mathcal{H}(\Omega)$.

Theorem 16

1. *The algebras $\mathbb{C}[E]$ and $\mathbb{C}[L]$ are alg. disjoint, within $\mathcal{A}$.*
2. *The family $(e^{\ell_r})_{r\geq 1}$ (resp. $(\ell_r)_{r\geq 1}$) is alg. free over $\mathcal{E}^+$ (resp. $\mathcal{L}^+$).*

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22. $\mathrm{Frac}(\mathcal{C})$ is a differential subfield of $\mathrm{Frac}(\mathcal{A})$.

Proof of independence of eulerian functions

Considering the Chen series of the differential forms $\{u_{y_r}dz\}_{r\geq 1}$, with $u_{y_r} = e^{\ell_r}\partial\ell_r$, let $Q \in \mathrm{Frac}(\mathcal{L})$ (resp. $\mathrm{Frac}(\mathcal{C})$) and let $\{c_y\}_{y\in Y} \in \mathbb{C}^{(Y)}$ be a sequence of complex numbers, non simultaneously vanishing, s.t.

$$\partial Q = \sum_{y\in Y} c_y u_y = \sum_{r\geq 1} c_{y_r} e^{\ell_r}\partial\ell_r.$$

If $\partial Q \neq 0$ then, integrating, $Q \in E$ and then $E \supset \mathrm{Frac}(\mathcal{L}) \supset \mathcal{L} \supset \mathbb{C}[L]$ (resp. $E \supset \mathrm{Frac}(\mathcal{C}) \supset \mathcal{C} \supset \mathcal{E}^+$) contradicting with $E \cap \mathbb{C}[L] = \{0\}$ (resp. $E \cap \mathcal{E}^+ = \{0\}$). It remains that $\partial Q = 0$.

Since $\{e^{\ell_r}\}_{r\geq 1}$ and then $\{\partial e^{\ell_r}\}_{r\geq 1}$ are $\mathbb{C}$-lin. free, then $c_{y_r} = 0$ $(r \geq 1)$.

By BTT, $\{\alpha_0^z(S_l)\}_{l\in\mathcal{L}ynY}$ and then $\{\alpha_0^z(S_y)\}_{y\in Y}$ are, respectively,

1. $\mathcal{L}$-alg. free yielding the $\mathbb{C}[L]$-alg. independence of $(e^{\ell_r})_{r\geq 1}$. It follows that $\mathbb{C}[E]$ and $\mathbb{C}[L]$ are alg. disjoint[23], within $\mathcal{H}(\Omega)$.
2. $\mathcal{C}$-alg. free yielding the alg. independence of $(e^{\ell_r})_{r\geq 1}$ over $\mathcal{E}^+$.
3. Now, suppose there is an algebraic relation among $(\ell_r)_{r\geq 1}$ over $\mathcal{L}^+$. By differentiating and substituting $\partial\ell_r$ by $e^{-\ell_r}\partial e^{\ell_r}$ in this relation, we obtain an algebraic relation among $\{e^{\ell_r}\}_{r\geq 1}$ over $\mathbb{C}[L]$ and $\mathcal{E}^+$ contradicting with two first items. Hence, $(\ell_r)_{r\geq 1}$ is $\mathcal{L}^+$-alg. free.

23. $\mathbb{C}[E] = \mathbb{C}[\{e^{\ell_r}\}_{r\geq 1}]$ and $\mathbb{C}[L] = \mathbb{C}[\{\ell_r\}_{r\geq 1}]$ are free and since $\{e^{\ell_r}\}_{r\geq 1}$ (resp. $\{\ell_r\}_{r\geq 1}$) is alg. free over $\mathbb{C}[L]$ (resp. $\mathbb{C}[E]$) then $\mathbb{C}[E + L]$ is freely generated by $\{e^{\ell_r}, \ell_r\}_{r\geq 1}$ and $\mathbb{C}[E] \cap \mathbb{C}[L] = \mathbb{C}.1_{\mathcal{H}(\Omega)}$.

$\mathrm{Dom}(\mathrm{Li}_\bullet)$ AND $\mathrm{Dom}(\mathrm{H}_\bullet)$

Chen series of $\omega_0(z) = z^{-1}dz$ and $\omega_1(z) = (1-z)^{-1}dz$

Let $\gamma_0(\varepsilon)$ and $\gamma_1(\varepsilon)$ be the circular paths of radius ε encircling 0 and 1 clockwise, respectively. In particular, letting $\beta = \beta_1 - \beta_0$, one considers

$$\begin{array}{rcccl} \gamma_0(\varepsilon,\beta) & = & \varepsilon e^{\mathrm{i}\beta_0} \rightsquigarrow \varepsilon e^{\mathrm{i}\beta_1} & \subset & \gamma_0(\varepsilon), \\ \gamma_1(\varepsilon,\beta) & = & 1-\varepsilon e^{\mathrm{i}\beta_0} \rightsquigarrow 1-\varepsilon e^{\mathrm{i}\beta_1} & \subset & \gamma_1(\varepsilon). \end{array}$$

On the one hand, one has, for any $i = 0$ or 1 and $w \in X^+$,

$$|\langle C_{\gamma_i(\varepsilon,\beta)}|w\rangle| \leq \varepsilon^{|w|_{x_i}} \beta^{|w|} |w|!^{-1}.$$

It follows then

$$C_{\gamma_i(\varepsilon,\beta)} = e^{\mathrm{i}\beta x_i} + o(\varepsilon) \quad \text{and} \quad C_{\gamma_i(\varepsilon)} = e^{2\mathrm{i}\pi x_i} + o(\varepsilon).$$

Hence[24], for $R \in \mathbb{C}^{\mathrm{rat}}\langle\langle X \rangle\rangle$ of minimal representation (λ, μ, η), one has

$$\langle R \| C_{\gamma_i(\varepsilon,\beta)} \rangle = \lambda \Big(\prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\alpha_{\gamma_i(\varepsilon,\beta)}(S_l)\mu(P_l)} \Big) \eta,$$

$$\langle R \| C_{\gamma_i(\varepsilon)} \rangle = \lambda \Big(\prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\alpha_{\gamma_0(\varepsilon)}(S_l)\mu(P_l)} \Big) \eta.$$

24. Recall that the map $\alpha_{z_0}^z : \mathbb{C}^{\mathrm{rat}}\langle\langle X \rangle\rangle \to \mathcal{H}(\Omega)$ is not injective. For example, $\alpha_{z_0}^z(z_0 x_0^* + (1-z_0)(-x_1)^* - 1_{X^*}) = 0$.

$\mathrm{Dom}(\mathrm{Li}_\bullet)$, $\mathrm{Dom}_R(\mathrm{Li}_\bullet)$ and $\mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}_\bullet)$

Let $\mathcal{C} := \mathbb{C}[z^a,(1-z)^b]_{a,b\in\mathbb{C}}$ and $\Omega := \mathbb{C}\setminus(]-\infty,0]\cup[1,+\infty[)$.

Let $[S]_n = \sum_{w\in X^*,|w|=n} \langle S|w\rangle w$ denotes the homogeneous components of S (of degree n). Then $\mathrm{Dom}(\mathrm{Li}_\bullet)$ is the set of $S = \sum_{n\geq 0}[S]_n$ s.t. $\sum_{n\geq 0}\mathrm{Li}_{[S]_n}$ is unconditionally convergent for the standard topology on $\mathcal{H}(\Omega)$.

Denoting the open disk by $D_{<R}$ $(0<R\leq 1)$, let

$$\begin{aligned}\mathrm{Dom}_R(\mathrm{Li}_\bullet) &:= \{S\in\mathbb{C}\langle\langle X\rangle\rangle x_1\oplus\mathbb{C}1_{X^*} \mid \sum_{n\geq 0}\mathrm{Li}_{[S]_n} \text{ is unconditionally} \\ &\qquad \text{convergent for the standard topology on } \mathcal{H}(D_{<R})\}.\\ \mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}_\bullet) &:= \bigcup_{0<R\leq 1}\mathrm{Dom}_R(\mathrm{Li}_\bullet).\end{aligned}$$

Proposition 7 ($\mathrm{L}(z) = C_{z_0\rightsquigarrow z}\mathrm{L}(z_0)$)

Let $\rho := \langle R\|\mathrm{L}\rangle$ ($R\in\mathrm{Dom}(\mathrm{Li}_\bullet)$). Then $\partial^n\rho = \langle R\|\mathbf{d}^n\mathrm{L}\rangle$ and $\mathbf{d}^n\mathrm{L} = p_n\mathrm{L}$, where $\{p_n\}_{n\geq 0}$ are given previously, using

$$\tau_r(x_0) = -r!(-z)^{-(r+1)}x_0 \text{ and } \tau_r(x_1) = r!(1-z)^{-(r+1)}x_1.$$

The following assertions are equivalent :

1. *ρ satisfies a differential equation with coefficients in $(\mathcal{C},\partial)$.*
2. *There exists $P\in\mathcal{C}\langle X\rangle$ such that $\langle R\|P\mathrm{L}\rangle = \langle R\triangleleft P\|\mathrm{L}\rangle = 0$.*

Proposition 8

1. $\mathrm{Dom}(\mathrm{Li}_\bullet)$, *containing* $\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle \sqcup\!\sqcup\, \mathbb{C}\langle X\rangle$, *is closed by shuffle and then* $\mathrm{Li}_{S \sqcup\!\sqcup\, T} = \mathrm{Li}_S\,\mathrm{Li}_T$, *for* $S, T \in \mathrm{Dom}(\mathrm{Li}_\bullet)$.

2. *Let* $S \in \mathbb{C}\langle\langle X\rangle\rangle x_1 \oplus \mathbb{C}1_{X^*}$ *and* $0 < R \leq 1$ *s.t.* $\sum_{n\geq 0} \mathrm{Li}_{[S]_n}$ *is unconditionally convergent, for the standard topology, on* $\mathcal{H}(D_{<R})$. *Then* $\sum_{N\geq 0} a_N z^N = \frac{1}{1-z}\sum_{n\geq 0} \mathrm{Li}_{[S]_n}(z)$ *is unconditionally convergent in the same domain and* $a_N = \sum_{n\geq 0} \mathrm{H}_{\pi_Y([S]_n)}(N)$.

3. *If* $S \in \mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}_\bullet)$ *then* $\mathrm{H}_{\pi_Y(S)} \in \mathrm{Dom}(\mathrm{H}_\bullet) := \pi_Y\mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}_\bullet)$.

4. $S \sqcup\!\sqcup\, T \in \mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}_\bullet)$ *and* $\pi_X(\pi_Y(S) \uplus \pi_Y(T)) \in \mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}_\bullet)$, *for* $S, T \in \mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}_\bullet)$. *Moreover,*

$$\begin{aligned}
\mathrm{Li}_{S \sqcup\!\sqcup\, T} &= \mathrm{Li}_S\,\mathrm{Li}_T\,. \\
\mathrm{H}_{\pi_Y(S) \uplus \pi_Y(T)}(N) &= \mathrm{H}_{\pi_Y(S)}(N)\mathrm{H}_{\pi_Y(T)}(N), \qquad N \geq 0. \\
\frac{\mathrm{Li}_S(z)}{1-z} \odot \frac{\mathrm{Li}_T(z)}{1-z} &= \frac{\mathrm{Li}_{\pi_X(\pi_Y(S) \uplus \pi_Y(T))}(z)}{1-z}.
\end{aligned}$$

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