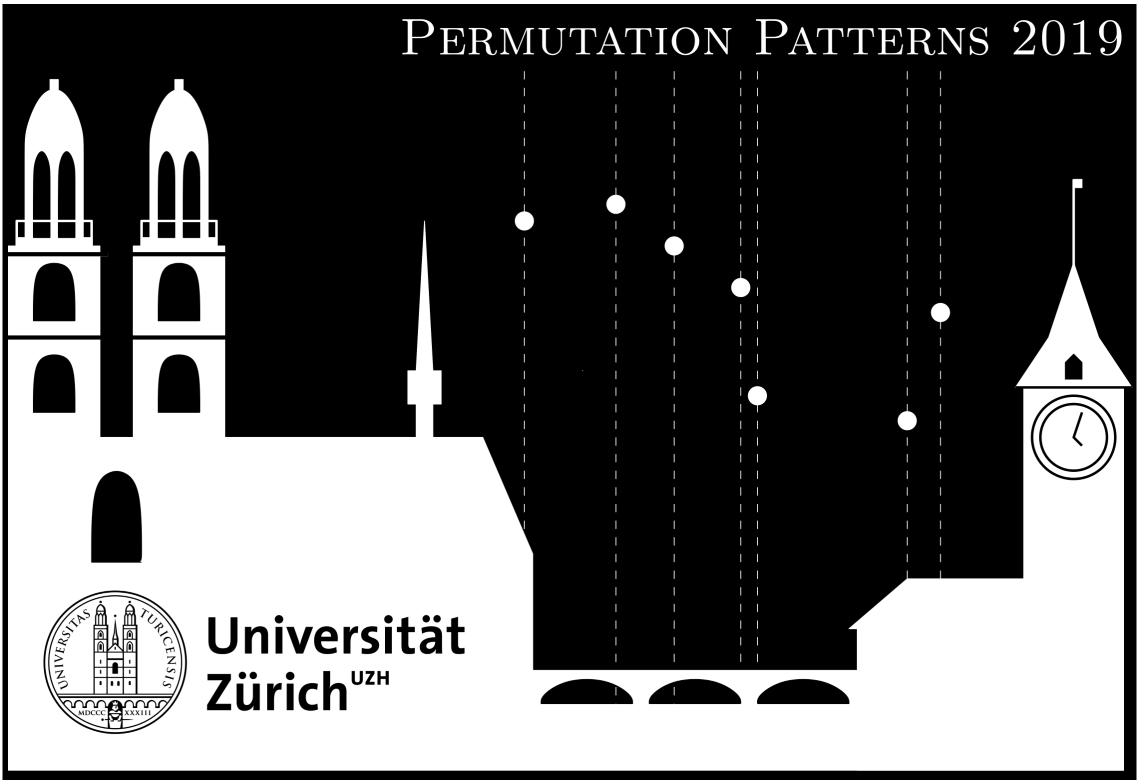
Permutation Patterns 2019

University of Zürich

June 17-21, 2019



Welcome to Permutation Patterns 2019 in Zurich!

It is a great pleasure for us to host the 2019 edition of the conference Permutation Patterns in Zurich. We hope that you can enjoy the conference, the city of Zurich, and possibly the surroundings. The section *Activities* lists a few highlights of Zurich, and suggests some excursions around Zurich that you can do in a day or a half-day.

In case you need assistance during your stay, please get in touch with one of the local members of the organizing committee: Jacopo Borga, Mathilde Bouvel, Valentin Féray, Raúl Penaguião, Grit Schütze or Benedikt Stufler.

Organizing a conference, even the relatively small size of PP, is always some challenge. Many people have participated in putting the conference together, and their help has been precious along the past year. In particular, I am grateful to Jacopo Borga, Valentin Féray, Lucas Gerin, Raúl Penaguião, Lara Pudwell, Grit Schütze, Erik Slivken, Benedikt Stufler and Katya Vassilieva for helping with many organizational tasks. And my apologies to those who did help but I forgot to list here.

Working with Kassie Archer, Robert Brignall and Luca Ferrari in the program committee has been incredibly easy, and I want to thank them for their availabilty and efficiency. Assuming (and hoping!) that both the organizing and the program committees did their job well, the conference can only be a success thanks to all speakers, whose abstracts promise a selection of exciting talks on various fields related to permutations and their patterns.

And of course, talks (as interesting as they may be) in front of an empty class room cannot be considered a success. So, thank you for coming to Zurich and attending Permutation Patterns 2019!

Sincerely yours,
Mathilde Bouvel,
for the organizing committee.

Sponsors

Permutation Patterns 2019 is supported by fundings of the University of Zurich (Einrichtungskredit no. P-71117), the Swiss National Science Foundation (projects no. 200020_172515 and 200021_172536), the NSF (grant no. DMS-1901853), the Dipartimento di Matematica e Informatica of the University of Florence, and a GRC grant of the University of Zurich for the pre-conference workshop (grant no. 2018_Q3_G_016).

Special Issue of DMTCS for Permutation Patterns 2019: Call for papers

We are pleased to announce that DMTCS will be publishing a special issue devoted to permutation patterns. Full papers in any topic of permutation patterns, broadly interpreted, are welcome to be submitted for consideration. While the special issue is linked to the conference Permutation Patterns 2019, please note that submission is not restricted to results presented at the conference or to researchers who attended the conference.

Submissions will be accepted starting on June 17, 2019, the first day of Permutation Patterns 2019 at the University of Zurich.

Submissions will close on December 31, 2019.

DMTCS is a member of the Free Journal Network, which is open only to journals that are controlled by the scholarly community and have no financial barriers to readers or authors. DMTCS articles are fully indexed in both MathSciNet (MR) and ZentralBlatt (zbMath).

To submit an article:

1. First upload your article to the ArXiv, HAL, or CWI, and wait for the ID to be generated (for ArXiv, this takes until the next business day).
2. Once you have the paper ID, go to http://dmtcs.episciences.org/, login, select "submit new article", choose "Permutation Patterns 2019" as the volume, and then choose "No Section" for the section.
3. Follow the instructions on that page.

All submissions will be refereed in accordance with the usual refereeing standards of DMTCS. Further information on this peer review process can be found at `https://dmtcs.episciences.org/page/policies`

The guest editors for this special issue are:
Miklos Bona (University of Florida)
Mathilde Bouvel (University of Zurich)
Lara Pudwell (Valparaiso University)
Vince Vatter (University of Florida)

Local Information

In case of emergency... and hoping that this section won't be useful. The emergency number in Switerzland is 112. It should be used for medical emergencies, in case of fire, threat to your security, or in all emergency situations.

University Mensas The city center campus of the University of Zurich has several cafeterias/dining facilities (called "mensas") where you can have lunch (and dinner for one of them). All mensas offer vegetarian options, and one of them (located at Rämistrasse 59) is entirely vegan. The list of mensas (with locations and menus) can be found on-line at `https://www.mensa.uzh.ch/en/standorte.html`.

Participants of the conference has received three mensa coupons, which are meant to be used on Monday, Tuesday and Thursday. They include a main dish and a drink. Extras can of course be purchased. Please note that you need to pay cash (in Swiss francs) or only contactless with your credit/debit card in the university mensas.

ATM The ATM closest to the conference venue (a *Postomat*) is located in the ETH building, below the polyterrasse.

Wifi Eduroam is available in all facilities of the University (and the ETH). For participants who cannot use Eduroam, we have 60 guest accounts for the *public* network of the university. You can obtain the login information with your registration package, or asking one of the organizers during the week.

Restaurants in Zurich

Going down from the conference place and walking in the narrow streets of Zurich's city center, you will find an important choice of restaurants. A non-exhaustive list of recommendations from the organizers: Zeughauskeller (nice a bit touristy Swiss Restaurant near Paradeplatz), Tibits/Hiltl (vegeterian buffet – you help yourself and pay up to weight –, perfect for a quick but tasty meal), Globus am Bellevue (fine pizzas, salad bar and asian dishes), Le Cèdre (Lebanese Mezze), many Italian restaurants incl. Santa Lucia (at Kunsthaus or in Markgasse) and Vapiano (Bellevue), an outside grill on the lake side near the opera house (Bellevue/Opernhaus tram stop), and many more...

In general restaurants in Zurich are good but expensive (at least 20,- CHF to 25,- CHF for a main dish). If you're looking for a cheap option, note that one of the university cafeterias is open in the evening (Mercato, aka untere Mensa, dinner until 7.30pm, a main dish costs 10.50CHF for externals).

Zoological museum: you might have noticed that the ground floor of our conference building is dedicated to the zoological museum. For the reasonable price of nothing, you'll get to see a variety of "not moving animals" (as Oscar would say, opposed to the "museum of moving animals", i.e. the zoo). Interesting and a must-do with kids on rainy days.

Polyterasse: did you like the picture on the web site? You can take it yourself! The Polyterasse is 2 minutes walk from the conference venue and offers nice views on the city. Please boycott the nice bar, called Bequem, down the stairs: it is affiliated with ETH and there is a friendly rivalry with the university ☺

Walking in the center: both sides of the river, the Limmat, have many little streets, which are worth to explore. In addition to the Polyterasse, you can enjoy nice views of the city (and play chess) from Lindenhof. The lake side starting at Bellevue provides a very nice place to walk and/or have some grilled sausages and beer (possibility to come back by boat; from the Casino, the trip can be done with a standard day ticket, without extra).

Aha Shop: this is an amazing little shop of mathematical objects, if you are looking for a gift to yourself or a mathematician friend. Location: Spiegelgasse 14. Open Wednesday-Friday 1.00 pm to 6.30 pm. Saturday 11.00 am to 4.00 pm.

Marc Chagall's Church Windows: Marc Chagall created five magnificent windows for the Fraumünster church in Zurich. Location: Fraumünster Stadthausquai 19. Open every day 10.00 am - 6.00 pm.

Museums: If June is as cold as May was, visiting a museum might be the best option. The Kunsthaus (modern art museum) is within walking distance (less than 10 minutes) from the conference venue and the permanent collection is free on Wednesday. Other nice options are the Swiss National Museum (near the main train station) and the Rietberg Museum (presenting art from all over the world, and located in a nice park).

Taking a boat tour: if you missed the opportunity to come all the way from the US by ship, you have a second chance to get on a boat. Short trips on Zurich river, the Limmat, are included in a tram day card. Longer trips on the lake are a nice way to see the mountains in the background. The longest roumdtrip go all the way to Rapperswil (almost 2 hours each way), where you can stop: this city has a nice old center culminating with a castle and a nice lake side with many Italian restaurants (going and/or coming back by regional train - S Bahn - is also an option). There is also the possibility to dine on the boat (which I would recommend on a nice summer day).

Swimming in the river/lake: the water may be a bit cold at that time of the year, but swimming in the river or the lake is a must do in Zurich. In the lake it is authorized to swim anywhere (e.g. near the Chinese garden, on Bellevue side); there are places with some extra installation (changing rooms, slides, diving, standup paddle to rent, ...; 8CHF per person). In the river please use the free installations at Unteren Letten or Oberen Letten.

Ütliberg: this is the highest hill surrounding Zurich. Beautiful views of the lake and (on nice days) of the mountains in the background await you on the top (as well as grilled sausages and beer). There are trains every 20 minutes from the main station to Ütliberg (S10; circa 20 minutes train and 10 minutes walk to go to the viewpoint), but people missing the PP traditional hike will prefer to walk all the way up from the final station of tram line 13, Albisguetli (though this is not advised with a buggy if the path is snowy, believe me; well, you should be safe in June).

Rhine Falls: this is a standard day or half-a-day excursion from Zurich. You need circa 1 hour by train to go there, and then the site can be explored by walking around or taking a boat. Possibility to have some grilled sausages and beer (as everywhere in Switzerland, basically).

Rigi: if you want to have a hike in the mountain, I would recommend the Rigi area, closed to the Lake of Lucerne. You need 1h30 from Zurich by train, to go to the Rigi-Kulm station where you can start walking. Please visit the following page for details (the duration of the train ride to the Rigi is underestimated there) and alternate suggestions: https://www.zuerich.com/en/visit/hiking

Poster Session

Monday, 5:00pm–7:00pm, Building KOL-D-49
The poster session on Monday late afternoon will be held in the *Lichthof* (atrium) of the main building of the university (KOL-D-49). It will be accompanied by fingerfood and refreshments. The following posters will be presented:

Bell numbers, Stirling numbers and set partitions,
Walaa Asakly

Enumeration of isolated vertices in permutation graphs,
Charles Burnette

The number of separators, a new parameter for the symmetric group,
Estrella Eisenberg and Moria Sigron

A Generalization of Dyck Paths and Catalan numbers,
Young-Yoon Lee

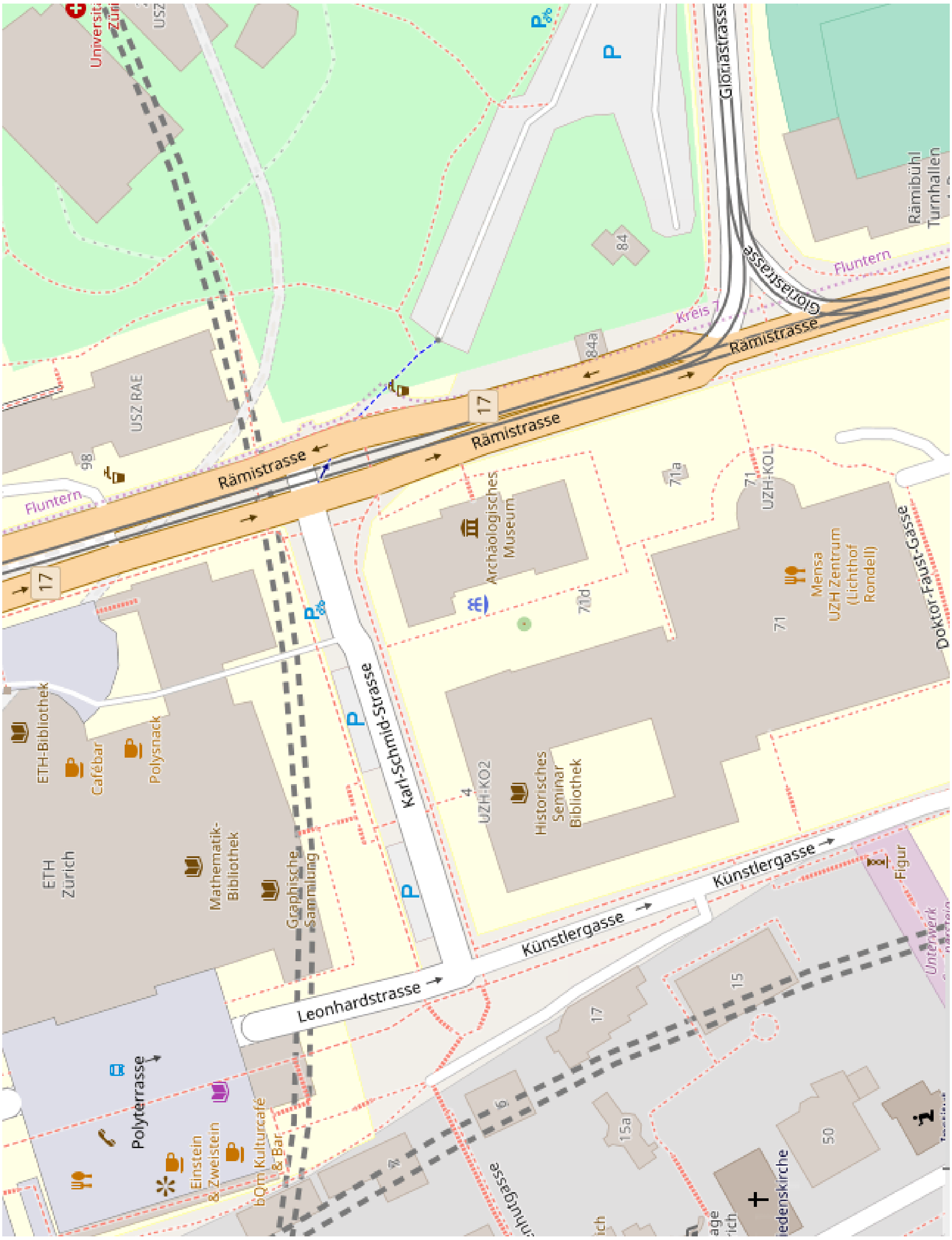
k**-partial permutations and the center of the wreath product** $\mathcal{S}_k \wr \mathcal{S}_n$ **algebra**,
Omar Tout

Conference banquet

Tuesday, 6:45pm
Our conference dinner will be held on Tuesday, June 18, starting at 6:45pm, in the restaurant Sento. The restaurant is located Plattenstrasse 26, 8032 Zürich (entrance on the Zurichbergstrasse), about 10 minutes walk from the conference venue. The symbol on your nametag indicates the main dish that you have chosen at your registration.

Wednesday afternoon

There are plenty of options to spend a pleasant afternoon in Zurich. There is no official excursion planned, but many suggestions of things to do can be found in this booklet.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ETH-Bibliothek
Cafébar
Polysnack
ETH Zürich
Mathematik-Bibliothek
Graphische Sammlung
Polyterrasse
Einstein & Zweistein
bQm Kulturcafé & Bar
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Monday, June 17

All talks take place in room F-152 of building KO2

9:00–9:30	*Registration / Welcome*
9:30–9:55	**Pattern-Avoiding Fillings of Skew Shapes**, *Vít Jelínek*
10:00–10:25	**Widdershins permutations and well-quasi-order**, *Michael Engen*
10:30–11:00	*Coffee break*
11:00–11:25	**Countable universal and existentially closed permutations in geometric grid classes** , *Samuel Braunfeld*
11:30–11:55	**Enumeration of Permutation Classes by Inflation of Independent Sets of Graphs**, *Émile Nadeau*
12:00–2:00	*Lunch break*
2:00–2:25	**Cyclic Schur-positive permutation sets**, *Yuval Roichman*
2:30–2:55	**Hopping from Chebyshev polynomials to permutation statistics**, *Jordan Tirrell*
3:00–3:30	*Coffee Break*
3:30–3:55	**Catalan words avoiding pairs of length three patterns**, *Carine Khalil*
4:00–4:25	**Exhaustive generation of pattern-avoiding permutations**, *Hung P. Hoang*
5:00–7:00	*Poster Session, with fingerfood and refreshments (*Lichthof *of Building KOL-D-49), see page 6*

Tuesday, June 18

All talks take place in room F-152 of building KO2

9:00–9:25	**The growth of the Möbius function on the permutation poset**, *David Marchant*
9:30–9:55	**Scaling limits of permutation classes with a finite specification: a dichotomy**, *Adeline Pierrot*
10:00–10:25	**Pattern Hopf algebras on marked permutations and enriched set species**, *Raúl Penaguião*
10:30–11:00	*Coffee Break*
11:00–11:55	**(Invited talk) Endless Pattern-Avoiding Permutations**, *Neal Madras*
12:00–2:00	*Lunch break*
2:00–2:25	**Classes of Sum-Decomposable Affine Permutations**, *Justin M. Troyka*
2:30–2:55	**On the poset of king-non-attacking permutations**, *Esterella Eisenberg and Moriah Sigron*
3:00–3:30	*Coffee Break*
3:30–3:55	**On extremal cases of pop-stack sorting**, *Andrei Asinowski*
4:00–4:25	**Sorting Permutations with Pattern-Avoiding Stacks**, *Giulio Cerbai*
4:30–5:00	*Problem Session*
6:45–	*Banquet (see page 6 for details)*

Wednesday, June 19

All talks take place in room F-152 of building KO2

9:00–9:25	**Avoiding Baxter-like patterns**, *Simone Rinaldi*
9:30–10:25	**On pattern-avoiding Fishburn permutations**, *Juan B. Gil*
10:00–10:25	**Automatic Discovery of Polynomial Time Enumerations**, *Unnar Freyr Erlendsson*
10:30–11:00	*Coffee Break*
11:00–11:25	**On the equidistribution of** MAJ **and** BAST, *Shishuo Fu*
11:30–11:55	**Pattern avoidance in permutations and their squares**, *Rebecca Smith*
12:00–	*Free Afternoon*

Thursday, June 20

All talks take place in room F-152 of building KO2

9:00–9:25	**How many chord diagrams have no short chords?**, *Jay Pantone*
9:30–9:55	**Consecutive permutation patterns in trees and mappings**, *Alois Panholzer*
10:00–10:25	**Permutation patterns: gamma-positivity and (-1)-phenomenon**, *Bin Han*
10:30–11:00	*Refreshments*
11:00–11:55	**(Invited talk) Patterns by accident**, *Bridget Tenner*
12:00–2:00	*Lunch break*
2:00–2:25	**Square permutations and convex permutominoes**, *Enrica Duchi*
2:30–2:55	**Square permutations are typically rectangular**, *Jacopo Borga*
3:00–3:30	*Coffee break*
3:30–3:55	**On a New Parameter of Permutations Arising in a Context of Testing for Forbidden Patterns**, *Gil Laufer*
4:00–4:25	**Packing patterns in restricted permutations**, *Lara Pudwell*
4:30–4:55	**Applications of a q-analog for Riordan arrays to various combinatorial objects**, *Gi-Sang Cheon*
5:00–5:25	*Announcements: next Permutation Patterns*

Friday, July 13

All talks take place in room F-152 of building KO2

9:00–9:25	**The odd behaviour of the permutation displacement ratio**, *David Bevan*
9:30–9:55	**Enumerative combinatorics of intervals in the Dyck pattern poset**, *Matteo Cervetti*
10:00–10:25	**Substitution decomposition for permutation classes with infinitely many simple permutations**, *Arnar Bjarni Arnarson*
10:30–11:00	*Coffee break*
11:00–11:25	**Solving hard problems effectively on permutations of small grid-width**, *Michal Opler*
11:30–11:55	**Two families of Wilf-equivalences**, *Michael Albert*

Abstracts

David Bevan The Open University

(This text is a brief presentation of basic definitions and notation used in permutation patterns research. Is was initially produced by D.Bevan for Permutation Patters 2015.*)*

Permutations, containment and avoidance

A permutation is considered to be simply an arrangement of the numbers $1, 2, \ldots, n$ for some positive n. The length of permutation σ is denoted $|\sigma|$, and S_n or $\mathfrak{S}_n$ is used for the set of all permutations of length n.

It is common to consider permutations graphically. Given a permutation $\sigma = \sigma(1)\ldots\sigma(n)$, its *plot* consists of the the points $(i, \sigma(i))$ in the Euclidean plane, for $i = 1, \ldots, n$.

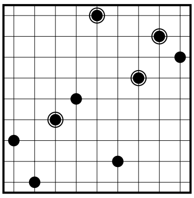

Figure 1: The plot of permutation **314592687** with a **1423** subpermutation marked

A permutation, or *pattern*, π is said to be *contained* in, or to be a *subpermutation* of, another permutation σ, written $\pi \leqslant \sigma$ or $\pi \preccurlyeq \sigma$, if σ has a (not necessarily contiguous) subsequence whose terms are order isomorphic to (i.e. have the same relative ordering as) π. From the graphical perspective, σ contains π if the plot of π results from erasing zero or more points from the plot of σ and then rescaling the axes appropriately. For example, **314592687** contains **1423** because the subsequence **4968** (among others) is ordered in the same way as **1423** (see Figure 1).

If σ does not contain π, we say that σ *avoids* π. For example, **314592687** avoids **3241** since it has no subsequence ordered in the same way as **3241**.

If λ is a list of distinct integers, the *reduction* or *reduced form* of λ, denoted $\mathrm{red}(\lambda)$, is the permutation obtained from λ by replacing its i-th smallest entry with i. For example, we have $\mathrm{red}(\mathbf{4968}) = \mathbf{1423}$. Thus, $\pi \leqslant \sigma$ if there is a subsequence λ of σ such that $\mathrm{red}(\lambda) = \pi$.

Permutation structure

Given two permutations σ and τ with lengths k and ℓ respectively, their *direct sum* $\sigma \oplus \tau$ is the permutation of length $k+\ell$ consisting of σ followed by a shifted copy of τ:

$$(\sigma \oplus \tau)(i) \;=\; \begin{cases} \sigma(i) & \text{if } i \leqslant k, \\ k+\tau(i-k) & \text{if } k+1 \leqslant i \leqslant k+\ell. \end{cases}$$

The *skew sum* $\sigma \ominus \tau$ is defined analogously. See Figure 2 for an illustration.

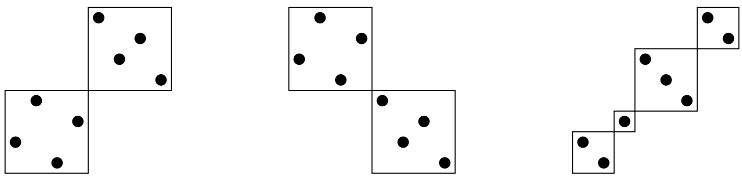

Figure 2: The direct sum $\mathbf{2413} \oplus \mathbf{4231}$, the skew sum $\mathbf{2413} \ominus \mathbf{4231}$, and the layered permutation $\mathbf{21} \oplus \mathbf{1} \oplus \mathbf{321} \oplus \mathbf{21}$

A permutation is called *sum indecomposable* if it cannot be expressed as the direct sum of two shorter permutations. A permutation is *skew indecomposable* if it cannot be expressed as the skew sum of two shorter permutations. Every permutation has a unique representation as the direct sum of a sequence of sum indecomposable permutations, and also as the skew sum of a sequence of skew indecomposable permutations. If a permutation is the direct sum of a sequence of *decreasing* permutations, then we say that the permutation is *layered*. See Figure 2 for an example.

An *interval* of a permutation σ corresponds to a contiguous sequence of indices $a, a+1, \ldots, b$ such that the set of values $\{\sigma(i) : a \leqslant i \leqslant b\}$ is also contiguous. Graphically, an interval in a permutation is a square "box" that is not cut horizontally or vertically by any point not in it. Every permutation of length n has intervals of lengths 0, 1 and n. If a permutation σ has no other intervals, then σ is said to be *simple*.

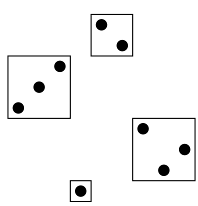

Figure 3: The inflation $\mathbf{3142}[\mathbf{123}, \mathbf{1}, \mathbf{21}, \mathbf{312}] = \mathbf{567198423}$

Given a permutation $\sigma \in S_m$ and nonempty permutations $\tau_1, \ldots, \tau_m$, the *inflation* of σ by $\tau_1, \ldots, \tau_m$, denoted $\sigma[\tau_1, \ldots, \tau_m]$, is the permutation obtained by replacing each entry $\sigma(i)$ of σ with an interval that is order isomorphic to τ_i. See Figure 3 for an illustration.

A simple permutation is thus a permutation that cannot be expressed as the inflation of a shorter permutation of length greater than 1. Conversely, every permutation except **1** is the inflation of a unique simple permutation of length at least 2.

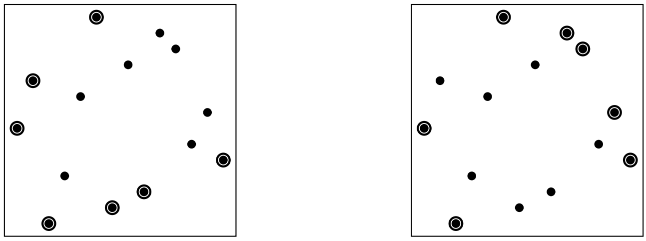

Figure 4: The three left-to-right maxima and four right-to-left minima, and the two left-to-right minima and five right-to-left maxima, of a permutation

Sometimes we want to refer to the extremal points in a permutation. A value in a permutation is called a *left-to-right maximum* if it is larger than all the values to its left. *Left-to-right minima*, *right-to-left maxima* and *right-to-left minima* are defined analogously. See Figure 4 for an illustration.

Permutation statistics

An *ascent* in a permutation σ is a position i such that $\sigma(i) < \sigma(i+1)$. Similarly, a *descent* is a position i such that $\sigma(i) > \sigma(i+1)$. A pair of terms in a permutation σ such that $i < j$ and $\sigma(i) > \sigma(j)$ is called an *inversion*.

A permutation *statistic* is simply a map from the set of permutations to the non-negative integers. Classical statistics include the following:

- the number of descents
  $\mathsf{des}(\sigma) = |\{i : \sigma(i) > \sigma(i+1)\}|$
- the number of inversions
  $\mathsf{inv}(\sigma) = |\{(i,j) : i < j \text{ and } \sigma(i) > \sigma(j)\}|$
- the number of *excedances*
  $\mathsf{exc}(\sigma) = |\{i : \sigma(i) > i\}|$
- the *major index*[1], the sum of the positions of the descents
  $\mathsf{maj}(\sigma) = \sum_{\sigma(i)>\sigma(i+1)} i$

The statistics des and exc are equidistributed. That is, for all n and k, the number of permutations of length n with k descents is the same as the number of permutations of length n with k excedances. Furthermore, inv and maj also have the same distribution. Any permutation statistic that is distributed like des is said to be *Eulerian*, and a statistic that is distributed like inv is said to be *Mahonian*[2].

[1]Named after Major Percy Alexander MacMahon.
[2]See footnote 1.

Classical permutation classes

The subpermutation relation is a partial order on the set of all permutations. A classical *permutation class*, sometimes called a *pattern class*, is a set of permutations closed downwards (a down-set) under this partial order. Thus, if σ is a member of a permutation class $\mathcal{C}$ and τ is contained in σ, then it must be the case that τ is also a member of $\mathcal{C}$. From a graphical perspective, this means that erasing points from the plot of a permutation in $\mathcal{C}$ always results in the plot of another permutation in $\mathcal{C}$ when the axes are rescaled appropriately. It is common in the study of classical permutation classes to reserve the word "class" for sets of permutations closed under taking subpermutations.

It is natural to define a classical permutation class "negatively" by stating the minimal set of permutations that it avoids. This minimal forbidden set of patterns is known as the *basis* of the class. The class with basis B is denoted $\mathsf{Av}(B)$, and $\mathsf{Av}_n(B)$ or $S_n(B)$ is used for the set of permutations of length n in $\mathsf{Av}(B)$. As a trivial example, $\mathsf{Av}(\mathbf{21})$ is the class of increasing permutations (i.e. the identity permutation of each length). As another simple example, the class of **123**-avoiders, $\mathsf{Av}(\mathbf{123})$, consists of those permutations that can be partitioned into two decreasing subsequences.

The basis of a permutation class is an antichain (a set of pairwise incomparable elements) under the containment order, and may be infinite. Classes for which the basis is finite are called *finitely based*, and those whose basis consists of a single permutation are called *principal* classes.

Non-classical patterns

Permutation patterns have been generalised in a variety of ways.

A *barred* pattern is specified by a permutation with some entries barred ($\mathbf{5\bar{3}2\bar{1}4}$, for example). If $\hat{\pi}$ is a barred pattern, let π be the permutation obtained by removing all the bars in $\hat{\pi}$ (**53214** in the example), and let π' be the permutation that is order isomorphic to the non-barred entries in $\hat{\pi}$ (**312** in the example). An occurrence of barred pattern $\hat{\pi}$ in a permutation σ is then an occurrence of π' in σ that is not part of an occurrence of π in σ. Conversely, for σ to avoid $\hat{\pi}$, every occurrence in σ of π' must feature as part of an occurrence of π.

A *vincular* or *generalised* pattern specifies adjacency conditions. Two different notations are used. Traditionally, a vincular pattern is written as a permutation with dashes inserted between terms that need not be adjacent and no dashes between terms that must be adjacent. Alternatively, and perhaps preferably, terms that must be adjacent are underlined. For example, **314265** contains two occurrences of $\mathbf{2\underline{31}4}$ (or **2–31–4**) and a single occurrence of $\mathbf{2\underline{314}}$ (**2–314**), but avoids $\mathbf{\underline{23}\,\underline{14}}$ (**23–14**).

A vincular pattern in which *all* the terms must occur contiguously is known as a *consecutive pattern*.

In a *bivincular* pattern, conditions are also placed on which terms must take adjacent *values*.

Classical, vincular and bivincular patterns are all example of the more general family of *mesh* patterns. Formally, a mesh pattern of length k is a pair (π, R) with $\pi \in S_k$ and $R \subseteq [0,k] \times [0,k]$ a set of pairs of integers. The elements of R identify the lower left corners of unit squares in the plot of π, which specify forbidden regions. Mesh pattern (π, R) is depicted by a figure consisting of the plot of π with the forbidden regions shaded. See Figure 5 for an example.

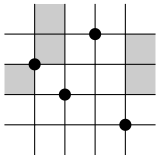

Figure 5: Mesh pattern $(\mathbf{3241}, \{(0,2),(1,3),(1,4),(4,2),(4,3)\})$

An occurrence of mesh pattern (π, R) in a permutation σ consists of an occurrence of the classical pattern π in σ such that no elements of σ occur in the shaded regions of the figure. A vincular pattern is thus a mesh pattern in which complete columns shaded.

Sets of permutations defined by avoiding barred, vincular, bivincular or mesh patterns that are not closed under taking subpermutations are known as *non-classical* permutation classes.

Growth rates

Given a permutation class $\mathcal{C}$, we use $\mathcal{C}_n$ to denote the permutations of length n in $\mathcal{C}$. It is natural to ask how quickly the sequence $(|\mathcal{C}_n|)_{n=1}^{\infty}$ grows.

In proving the Stanley–Wilf Conjecture, Marcos and Tardos established that the growth of every classical permutation class except the class of all permutations is at most exponential. Hence, the *upper growth rate* and *lower growth rate* of a class $\mathcal{C}$ are defined to be

$$\overline{\mathrm{gr}}(\mathcal{C}) = \limsup_{n\to\infty} |\mathcal{C}_n|^{1/n} \qquad \text{and} \qquad \underline{\mathrm{gr}}(\mathcal{C}) = \liminf_{n\to\infty} |\mathcal{C}_n|^{1/n}.$$

The theorem of Marcos and Tardos states that $\overline{\mathrm{gr}}(\mathcal{C})$ and $\underline{\mathrm{gr}}(\mathcal{C})$ are both finite.

When $\overline{\mathrm{gr}}(\mathcal{C}) = \underline{\mathrm{gr}}(\mathcal{C})$, this quantity is called the *proper growth rate* (or just the *growth rate*) of $\mathcal{C}$ and denoted $\mathrm{gr}(\mathcal{C})$. Principal classes, those of the form $\mathsf{Av}(\pi)$, are known to have proper growth rates. The growth rate of $\mathsf{Av}(\pi)$ is sometimes known as the *Stanley–Wilf limit* of π and denoted $L(\pi)$. It is widely believed, though not yet proven, that every classical permutation class has a proper growth rate.

Wilf equivalence

Given two classes, $\mathcal{C}$ and $\mathcal{D}$, one natural question is to determine whether they are equinumerous, i.e. $|\mathcal{C}_n| = |\mathcal{D}_n|$ for every n. Two permutation classes that are equinumerous are said to be *Wilf equivalent* and the equivalence classes are called *Wilf classes*. If principal classes $\mathsf{Av}(\sigma)$ and $\mathsf{Av}(\tau)$ are Wilf equivalent, we simply say that σ and τ are Wilf equivalent.

From the graphical perspective, it is clear that classes related by symmetries of the square are Wilf equivalent. Thus, for example, $\mathsf{Av}(\mathbf{132})$, $\mathsf{Av}(\mathbf{231})$, $\mathsf{Av}(\mathbf{213})$ and $\mathsf{Av}(\mathbf{312})$ are equinumerous. However, not all Wilf equivalences are a result of these symmetries. Indeed, as is well known, both $\mathsf{Av}(\mathbf{123})$ and $\mathsf{Av}(\mathbf{132})$ are counted by the Catalan numbers, so all permutations of length three are in the same Wilf class.

Generating functions

The ordinary *generating function* of a permutation class $\mathcal{C}$ is defined to be the formal power series

$$C(z) \;=\; \sum_{n\geqslant 0} |\mathcal{C}_n| z^n \;=\; \sum_{\sigma\in\mathcal{C}} z^{|\sigma|}.$$

Thus, each permutation $\sigma \in \mathcal{C}$ makes a contribution of $z^{|\sigma|}$, the result being that, for each n, the coefficient of z^n is the number of permutations of length n. Clearly, two classes are Wilf-equivalent if their generating functions are identical.

A generating function is *rational* if it is the ratio of two polynomials. A generating function $F(z)$ is *algebraic* if it can be defined as the root of a polynomial equation. That is, there exists a bivariate polynomial $P(z, y)$ such that $P(z, F(z)) = 0$.

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Endless Pattern-Avoiding Permutations

Neal Madras York University

This talk is based on joint work with Justin Troyka

The plots of large randomly generated pattern-avoiding permutations offer visual representations of the structure of these permutations. A particularly tantalizing case is that of **4231**-avoidance, where a canoe-like shape appears (see Figure 1). Visually, the middle part of the canoe looks roughly like two parallel lines that are joined by distinct perpendicular segments (as a canoe's gunwales are joined by thwarts). The parallelism of the "gunwales" are distorted near the canoe's ends, since the plot must fit into a square. Following a suggestion of Nathan Clisby that uses intuition borrowed from statistical physics, perhaps the ends of our **4231**-avoiding permutation are a "boundary effect" that we can try to separate from the main part of the permutation, far from the ends. Is there a way to make rigorous sense of this idea, particularly at the level of asymptotics when N tends to infinity?

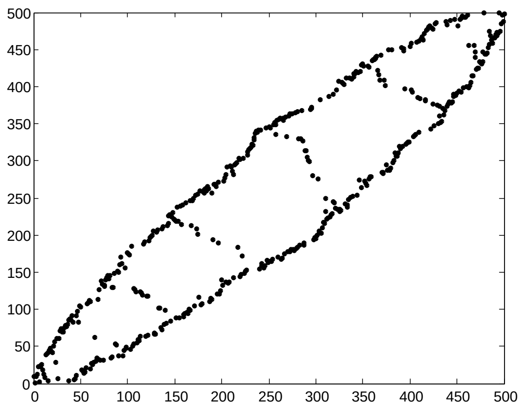

Figure 1: A random **4231**-avoiding permutation of length $N = 500$.

One traditional way to eliminate boundary effects of a system in a large square region is to put the system on a torus, i.e. to extend it periodically. To do this for permutations, we are led to the study of affine permutations, which we augment with a *boundedness* condition, defined as follows.

Definition 1. (a) An *affine permutation of period* N is a bijection $\omega : \mathbb{Z} \to \mathbb{Z}$ such that

$$\omega(i+N) \;=\; \omega(i) + N \quad \text{for every } i \in \mathbb{Z}$$

and

$$\sum_{i=1}^{N} \omega(i) \;=\; \sum_{i=1}^{N} i \qquad \text{(a "centering" condition).}$$

(b) An affine permutation ω of period N is said to be *bounded* if

$$|\omega(i) - i| \;<\; N \quad \text{for every } i \in \mathbb{Z}.$$

We write BA_N for the set of bounded affine permutations of period N.

Observe that for any (ordinary) permutation $\sigma \in S_N$, the periodic extension of σ to a doubly infinite direct sum

$$\cdots \oplus \sigma \oplus \sigma \oplus \sigma \oplus \cdots \tag{1}$$

is in BA_N.

The number of bounded affine permutations is given asymptotically by the following theorem.

Theorem 2.

$$|\mathsf{BA}_N| \;\sim\; N! \times 2^N N^{-1/2}\sqrt{\frac{3}{2\pi e}} \quad \textit{as } N \to \infty.$$

For a given pattern $\tau \in S_m$, we let $\mathsf{AvBA}_N(\tau)$ be the set of all bounded affine permutations of period N that avoid the pattern τ. This will be our model of "endless" pattern-avoiding permutations. We shall assume that $\tau(1) > \tau(m)$, so that if σ is an ordinary τ-avoiding permutation then the infinite direct sum of Equation (1) is necessarily in $\mathsf{AvBA}_N(\tau)$.

One important initial question is whether the growth rate of our endless pattern-avoiding permutations is the same as for as the corresponding ordinary pattern-avoiding permutations; that is, does

$$\lim_{N\to\infty} |\mathsf{AvBA}_N(\tau)|^{1/N}$$

exist and equal the Stanley-Wilf limit of τ? We don't know the general answer, but we do know that the answer is yes for decreasing permutations and various other patterns. In particular, we have the following asymptotic result for the pattern **321**.

Theorem 3.

$$|\mathsf{AvBA}_N(\mathbf{321})| \;\sim\; \frac{4^N N^{1/2}}{2\sqrt{\pi}} \quad \textit{as } N \to \infty.$$

For comparison, since we know that the cardinality of $\mathsf{Av}_N(\mathbf{321})$ is the N^{th} Catalan number, we see that

$$|\mathsf{AvBA}_N(\mathbf{321})| \;\sim\; \frac{N^2}{2}\,|\mathsf{Av}_N(\mathbf{321})| \quad \text{as } N \to \infty.$$

In this talk I shall present the above results, including some interesting ingredients of the proof of Theorem 3. Along the way, I shall describe a few insights that can be obtained by thinking probabilistically. Also, at a more informal level, I shall indicate some (occasionally subjective) connections between pattern-avoiding permutations and models in lattice statistical mechanics, particularly self-avoiding walks.

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Patterns by accident

Bridget Tenner DePaul University

This talk is based on joint work with Kyle Peterson and Kári Ragnarsson

The goal was to untangle the structure of the Bruhat order on the symmetric group. Of course, this was rather ambitious, but an attractive detour appeared – a detour that led to permutation patterns!

We will discuss how avoidance of two particular patterns dictates an enormous amount of algebraic structure in the symmetric group. This includes features of the Bruhat order, equality of complexity when represented by different sets of generators, and aspects of the cycle structure of a permutation. Avoidance of these two patterns also leads to a cell complex with beautiful topology. The permutations that avoid these two patterns can be enumerated by a familiar sequence, recovering (and uniting) older results; moreover, this enumeration can be refined by length.

We can push this analysis further – and more finely than the usual "avoid" versus "contain" – to see the impact of the total number of occurrences of these two patterns, whatever that number may be. We can also extend it to signed analogues (involving signed patterns) in the Coxeter groups of types B and D.

Two families of Wilf-equivalences

Michael Albert University of Otago

This talk is based on joint work with Vít Jelínek, Michal Opler

Two permutations π and σ belonging to a permutation class $\mathcal{C}$ are *Wilf-equivalent relative to* $\mathcal{C}$ if the two classes: $\mathcal{C} \cap \mathrm{Av}(\pi)$ and $\mathcal{C} \cap \mathrm{Av}(\sigma)$ are Wilf-equivalent, i.e., have the same number of permutations of each size. Let c_n be the number of permutations in $\mathcal{C}$ if size n, and w_n be the number of equivalence classes of this relation on permutations of size n. The class $\mathcal{C}$ exhibits a *Wilf-collapse* if $w_n = o(c_n)$. If in fact $w_n = o(r^n c_n)$ for some $r < 1$ then we say that $\mathcal{C}$ exhibits an *exponential Wilf-collapes*.

Recent work [2, 3, 4] has highlighted the prevalence of this phenomenon particularly in classes where there is a greedy algorithm for detecting patterns and/or a convenient representation of the permutations in the class as words over an ordered alphabet – the two seem to go hand in hand. In many such contexts, as well as demonstrating a Wilf-collapse the proofs provide bijections between the corresponding classes (though often somewhat implicitly or recursively).

We present two further examples that support this thesis.

Subpermutations of the increasing oscillation

The *infinite increasing oscillation* can be represented as:

$$2, 4, 1, 6, 3, 8, 5, 10, 7, \ldots$$

The finite permutations that can be found inside it form a sum-closed class, $\mathcal{SIO}$. The class $\mathcal{SIO}$ also plays a central role in the study of growth rates of permutation classes providing a fundamental building block for the construction of intervals in the set of achievable growth rates and other threshold phenomena [5, 6, 7].

Wilf-collapse in sum-closed classes is one of the main themes of [4] but some conditions that are required in that work are not found in $\mathcal{SIO}$. Nevertheless we can prove:

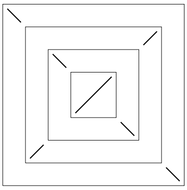
Theorem 1. *The class $\mathcal{SIO}$ exhibits an exponential Wilf-collapse.*

The class $\mathcal{X}$

The class $\mathcal{X}$ can be defined as the closure of the single permutation 1 under the operations $\pi \mapsto 1 \oplus \pi$, $\pi \mapsto 1 \ominus \pi$, $\pi \mapsto \pi \oplus 1$, and $\pi \mapsto \pi \ominus 1$. An equivalent recursive formulation is that all permutations in $\mathcal{X}$ begin or end with their maximum or minimum element (and because $\mathcal{X}$ is a class this is true when that element is removed etc.) In terms of a basis $\mathcal{X} = \mathrm{Av}(2143, 2413, 3142, 3412)$. In the literature $\mathcal{X}$

arose in [1] as an ingredient in the characterisation of which subclasses of the separable permutations have rational generating functions.

Because of the recursive representation, permutations in $\mathcal{X}$ having at least two elements can be described uniquely by a single integer of absolute value greater than 1 followed by a sequence of pairs of non-negative integers (at least one of which is positive). The initial term represents the monotone core of the permutation (increasing if positive, decreasing if negative) and the remaining pairs represent the whiskers that must be added to the core to form the permutation as shown below (for a positively sloping, i.e., increasing, core):



For instance the permutation corresponding to the sequence: -3, (1,2), (2,4), (0,1) is:

$$0 \oplus (2 \ominus (1 \oplus (-2) \oplus 2) \ominus 3) \oplus 1 = (10)\,9\,4\,6\,5\,7\,8\,3\,2\,1\,(11)$$

where numbers stand for monotone intervals of the corresponding length (and appropriate direction).

Of course $\mathcal{X}$ is far from being a sum-closed class but the obvious greedy algorithm for checking whether one permutation is involved in another in $\mathcal{X}$ is correct. This yields:

Theorem 2. *If two permutations in $\mathcal{X}$ have the same sized core and the same multiset of whiskers considered as* unordered *pairs then they are Wilf-equivalent.*

For instance the permutation shown above is Wilf-equivalent in $\mathcal{X}$ to:

$$2 \ominus (2 \oplus (1 \ominus (2) \ominus 0) \oplus 1) \ominus 3 = (11)\,(10)\,4\,5\,8\,6\,7\,9\,3\,2\,1.$$

That the sign of the core is not relevant should not be surprising as this can be changed by one of the symmetries that leaves $\mathcal{X}$ invariant – but the fact that the Wilf-class of a permutation is insensitive to the order of its whiskers or to the order of the sides of each whisker independently is not immediately evident, but follows relatively easily from the correctness of the greedy algorithm mentioned above.

The fact that reordering is allowed means that the total number of Wilf-classes behaves "like" (in a very rough sense) the number of partitions of n and in particular grows sub-exponentially, whereas the growth rate of the $\mathcal{X}$ class itself is $2 + \sqrt{2}$.

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Substitution decomposition for permutation classes with infinitely many simple permutations

Arnar Bjarni Arnarson Reykjavik University

This talk is based on joint work with Christian Bean, Jay Pantone, Henning Ulfarsson

We present an automatic method, using the substitution decomposition [1, 4] and the CombSpecSearcher [5] algorithm, to enumerate classes containing infinitely many simple permutations. To accomplish this we use tilings, but allow ourselves to make assumptions about certain cells. The method for enumerating the sum and skew decomposable permutations in the class remains the same as in previous work. We introduce new strategies to find a combinatorial specification for the (possibly infinite) set of simple permutations in a permutation class. Our approach simultaneously performs the inflations. This method has been able to enumerate many permutation classes with infinitely many simple permutations, for example $\mathrm{Av}(1324, 2431)$ and $\mathrm{Av}(1324, 2413)$.

We use the CombSpecSearcher to build the structure of the simple permutations, by placing the bottom-most maximal interval into rows, representing a point in the underlying simple permutation. We can then factor out intervals that are no longer required to preserve simplicity. These two strategies, alongside others, enable the CombSpecSearcher to find combinatorial specifications for permutation classes. Instead of only placing bottom-most maximal intervals, we also allow left, right, and top-most, allowing us to find specifications for different structures.

Inflations of simple permutations in $\mathrm{Av}(1324, 2413, 2431)$

In order to illustrate our strategies, we will use the substitution decomposition to find a combinatorial specification for $\mathrm{Av}(1324, 2413, 2431)$. This class contains one simple permutation for each length $n \geq 4$. The first step is to separate the permutations into three cases: sum-decomposable permutations, skew-decomposable permutations and those that are the inflation of a simple permutation of length at least 4.

If a permutation is sum-decomposable then it can be written as the sum of two permutations. To preserve uniqueness we assume the left operand is sum-indecomposable. After some case analysis, we can recursively find a specification for the subclasses, where some are sum-indecomposable. The skew-decomposable case is similar.

In order to handle the inflation of simple permutations, we must define some cells in the tiling as maximal intervals, meaning that there is no larger proper interval that contains it. If all cells are maximal intervals, the tiling represents inflations of a specific simple permutation in the class.

In Figure 1, we see the structure of the inflations of the simple permutations in $\mathrm{Av}(1324, 2413, 2431)$, which we will describe in more detail.

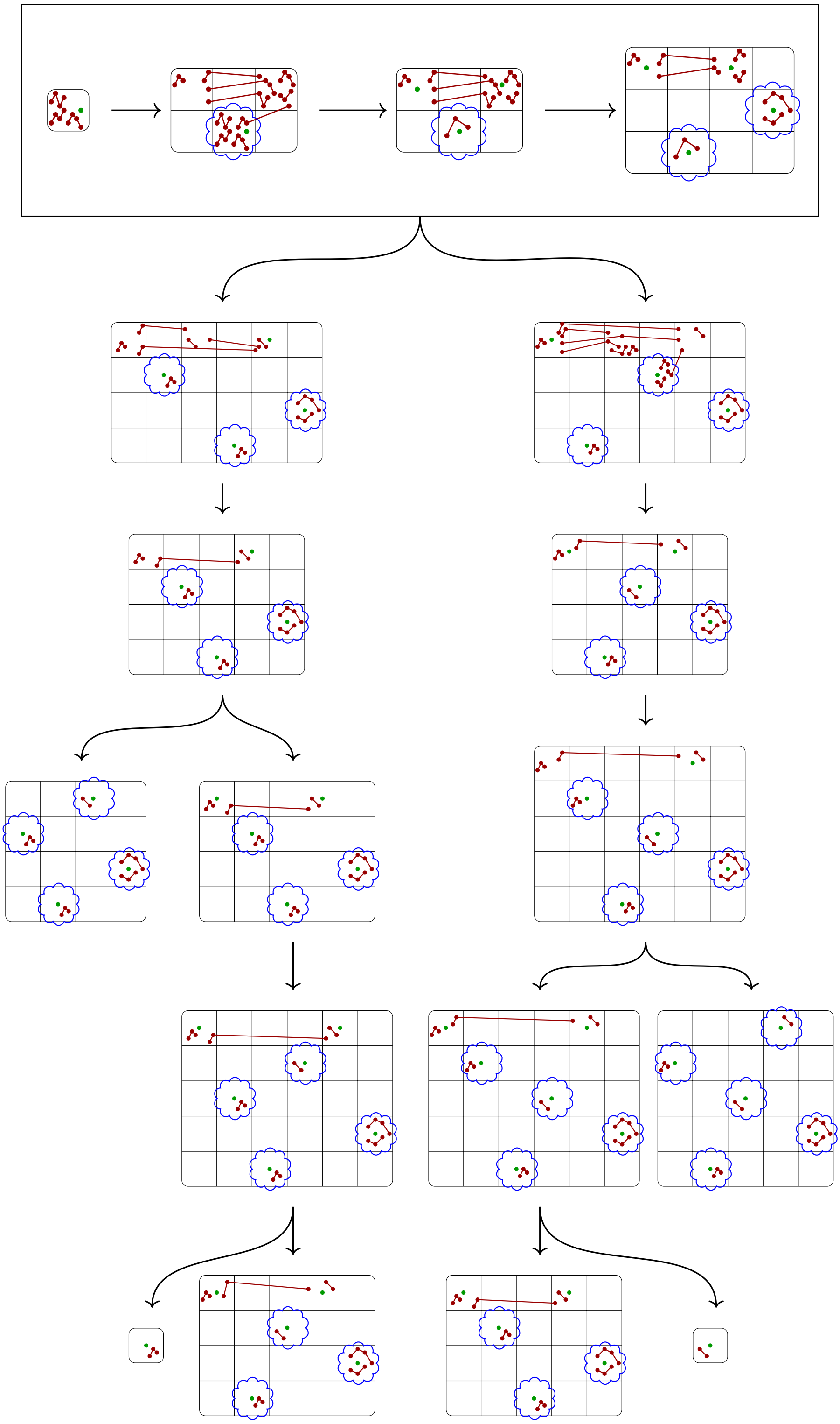

Figure 1: The inflations of the simple permutations in Av$(1324, 2413, 2431)$.

We first place the bottom-most interval, marked by a blue cloud, and in order for this to be an inflation of a simple permutation of length at least 4, the other two cells must each contain a point, otherwise the permutations will be sum or skew decomposable. We again place the bottom-most maximal interval which must be on the right since we avoid 2413. The cell above and to the right of the new interval must also contain a point. The next interval can go in either of the two cells in the top row. We will explain the steps taken in the left branch, the right is similar.

The middle cell in the top row must be empty because it cannot contain a maximal interval without breaking the simplicity condition. There are two possibilities for the top left cell, either it is empty or it contains a point. If it is empty then every cell is a maximal interval and we have a tiling representing inflations of the simple permutation 3142. Otherwise, we can place the bottom-most maximal interval in the top row, which must be in the right cell because we avoid 2413. We can factor out the maximal interval in the third row since it is no longer required to preserve simplicity. We then see a tiling which appears in the right branch which can be described with similar methods. From this specification we automatically derive the enumeration.

Future work

When a permutation class has a regular insertion encoding there is an algorithm which finds the rational generating function for the class [6]. We conjecture that a similar argument can be made for permutation classes whose simple permutations have a regular insertion encoding. In the literature, there is work on inflating the simple permutations that can be geometrically gridded. These all have a rational generating function and their inflations are algebraic [2, 3]. We conjecture that our algorithm will be able to enumerate all such permutation classes.

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Bell numbers, Stirling numbers and set partitions

Walaa Asakly University of Haifa

In this talk, I will present an explicit formula for the total number of sum weighted records over set partitions of $[n]$ according to the statistic sum of weighted records in terms of Stirling numbers and terms of Bell numbers.

Short introduction

Let us start with short introduction about set partitions:

Definition 1. A *partition* Π of set $[n]$ of size k is a collection $\{B_1, B_2, \dots, B_k\}$ of non empty disjoint subsets of $[n]$, called *blocks*, whose union is equal to $[n]$; we can also say that Π is a *partition of* $[n]$ *with exactly* k *blocks*. We assume that blocks are listed in increasing order of their minimal elements, that is, $minB_1 < minB_2 < \cdots < minB_k$.

Definition 2. We denote the set of all partitions of $[n]$ with exactly k blocks to be $P_{n,k}$ and we denote the set of all partitions of $[n]$ to be P_n.

Example 3. The partitions of [3] with 2 blocks are: $\{\{1\},\{2,3\}\}, \{\{1,2\},\{3\}\}$, $\{\{1,3\},\{2\}\}$.

Definition 4. The number of set partitions of $[n]$ with k blocks is denoted by $S_{n,k}$ and called the *Stirling* number.

Remark 5. Note that by definition,

$$\mid P_n \mid = \sum_{k=1}^{n} S_{n,k} = B_n$$

which are known as Bell numbers.

Definition 6. Any partition Π can be written as $\pi_1\pi_2\cdots\pi_n$, where $i \in B_{\pi_i}$ for all i, and this form is called the *canonical sequential form*.

Example 7. The canonical sequential form of $\{\{1\},\{2,3\}\}$ is 122.

For more details about set partitions we suggest [TM]. Researchers studied several statistics on set partitions. Let us consider the following one.

Definition 8. Let $\pi = a_1a_2\cdots a_m$ be any permutation of length m. An element a_i in π is a *record* if $a_i > a_j$ for all $j = 1, 2, \cdots, i-1$. Moreover, the index i of the record a_i is called the *position* of this record.

Example 9. The permutation $\pi = 1324$ has 3 records; first one is 1 of position 1, second one is 3 of position 2 and the last one is 4 of position 4.

For more details about records see the following articles: [AT], [ATS]. In this talk I will further define the following statistic on set partitions:

Definition 10. We denote the statistic *swrec*, where $\mathrm{swrec}(\pi)$ is the sum over all the records in P_n of the position of a record in π multiplied by the value of the record.

Example 11. Let $\{\{1\},\{2,3\}\}$ be a partition of [3] with the canonical form 122. We have two records: in position 1 with value 1 and in position 2 with value 2, so, $\mathrm{swrec}(122) = 1\cdot 1 + 2\cdot 2 = 3$.

The ordinary generating function

Let $P_k(x,q)$ be the ordinary generating function for the number of partitions of $[n]$ with exactly k blocks according to the statistic swrec, that is

$$P_k(x,q) = \sum_{n\geq k}\sum_{\pi\in P_{n,k}} x^n q^{\mathrm{swrec}(\pi)}.$$

Now, we can state our first result:

Theorem 12. *The ordinary generating function for the number of partitions of $[n]$ with exactly k blocks according to the statistic* swrec *is given by*

$$P_k(x,q) = \prod_{i=1}^{k} \frac{xq^i q^{(k+1-i)(k-i)}}{1-ix\prod_{j=i+1}^{k} q^j}. \tag{1}$$

Unfortunately, we do not have enough space for a complete proof.

The total number of swrec taken over all set partitions of $P_{n,k}$

Now we can present an important result

Theorem 13. *The total number of* swrec *taken over all set partitions of $P_{n,k}$, is given by*

$$S_{n,k}\left(\binom{k+1}{2}+2\binom{k+1}{3}\right)+\sum_{j=1}^{n-k} S_{n-j,k}\sum_{i=1}^{k}\frac{i^j(k+i+1)(k-i)}{2}.$$

Proof. The main idea of the proof is to differentiate (1) with respect to variable q, then substituting $q=1$ in the partial derivative, and the last step is to find the coefficients $[x^n]$ in the partial derivative. □

The total number of swrec taken over all set partitions of P_n

Finally, we can show the main result

Theorem 14. *The total number of* swrec *taken over all set partitions of* $[n]$, *is given by*

$$\frac{3}{4}(B_{n+3} - B_{n+2}) - (n + \frac{7}{4})B_{n+1} - \frac{1}{2}(n+1)B_n.$$

Proof. The main idea of the proof is to pass from the ordinary generating function for the number of partitions of $[n]$ with exactly k blocks according to the statistic swrec, to the exponential generating function for the number of partitions of $[n]$ with exactly k blocks according to the statistic swrec. And finally to find the coefficients $[x^n]$ in the partial derivative. □

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On extremal cases of pop-stack sorting

Andrei Asinowski University of Klagenfurt

This talk is based on joint work with Cyril Banderier (University of Paris North) and Benjamin Hackl (University of Klagenfurt) [3].

Pop-stack sorting is a natural sorting procedure and a fascinating process to analyse. It finds its roots in the seminal work of Knuth on sorting algorithms and permutation patterns [3]. We present several results on permutations that need few (resp. many) iterations of this procedure to be sorted. In particular, we represent the "2-pop-stack sortable permutations" by lattice paths to prove conjectures raised by Pudwell and Smith, and we characterize some families of permutations related to the image of the pop-stack sorting and to its "worst case".

Introduction and definitions

Each permutation can be split uniquely into *runs* – the maximal ascending strings, and into *falls* – the maximal descending strings. For example, the permutation 413625 is split into runs as 4|136|25, and into falls as 41|3|62|5.

One iteration of pop-stack sorting is defined as the transformation T that reverses all the falls. For example, $T(41|3|62|5) = 143265$. If, given a permutation π of size n, one applies T successively sufficiently many times (thus obtaining $T(\pi)$, $T^2(\pi)$, etc.), one eventually reaches the identity permutation Id. Ungar proved [6] that each each permutation of size n needs at most $n-1$ iterations of T to be sorted by pop-stack [4]. Equivalently: for each permutation of size n we have $T^{n-1}(\pi) = \text{Id}$). This bound is tight: there are permutations that need n, but not fewer, iterations of T to be sorted. Thus, we refer to this situation as the "worst case".

A permutation is *k-pop-stack-sortable* (*kPS*) Avis and Newborn showed that if $T^k(\pi) = \text{Id}$. 1PS-permutations are precisely the layered permutations [2]. Pudwell and Smith [4] found a structural characterization of 2PS-permutations and showed that their generating function is rational. Claesson and Guðmundsson [5] generalized the latter result showing that for each fixed k, the generating function for kPS-permutations is rational. The pop-stack sorting process offers many fascinating open questions (the main one being the average cost analysis of the corresponding algorithm). In this article, we offer new links with other combinatorial objects to derive further results.

[3] Research of Andrei Asinowski and Benjamin Hackl was supported by the project *Analytic Combinatorics: Digits, Automata and Trees* (P 28466) funded by the Austrian Science Fund (FWF).

[4] Ungar proved this result as a lemma for solving a geometric problem concerning the number of directions determined by a planar set of points.

Results concerning 2-pop-stack sortable permutations

First, let us concentrate on permutations which need few iterations of T to be sorted. Specifically, we prove two conjectures on 2PS-permutations by Pudwell and Smith, and reprove one of their theorems in a more combinatorial way which allows us to keep track of additional parameters.

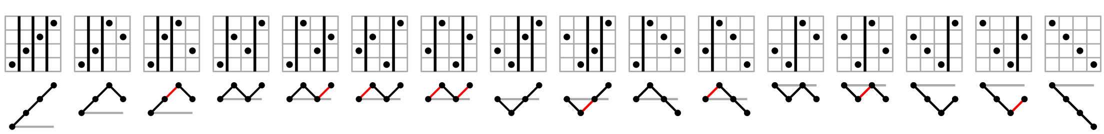
Theorem 1 ([4] Thm. 2)**.** *The generating function of 2-pop-stack-sortable permutations is* $A(x,y) = \sum a_{n,k} x^n y^k = x(1+x^2y)/(1-x-xy-x^2y-2x^3y^2)$, *where* $a_{n,k}$ *is the number of 2PS permutations of size n with exactly k ascents.*

Proof (sketch). We count 2PS-permutations taking descents rather than ascents as the second parameter: let $\mathcal{C}_{n,k}$ be the set of 2PS-permutations of size n with k descents, $c_{n,k} = |\mathcal{C}_{n,k}|$, $C(x,y) = \sum c_{n,k} x^n y^k$. We have $a_{n,k} = c_{n,n-1-k}$ and $A(x,y) = \frac{1}{y} C(xy, 1/y)$.

For fixed k, let F_k be the generating function for 2PS-permutations with k descents: $F_k(x) = \sum_{n \geq 0} c_{n,k} x^n$. We show $F_0(x) = \frac{x}{1-x}$ and, for $k \geq 1$, $F_k(x) = \frac{x^{k+1}(1+x)^2(1+x+2x^2)^{k-1}}{(1-x)^{k+1}}$. The cases of $F_0(x)$ and $F_1(x)$ are easily seen directly. We show that, for $k \geq 2$, we have $F_k(x)/(xF_{k-1}(x)) = (1+x+2x^2)/(1-x) = 1 + 2x + 4x^2 + 4x^3 + 4x^4 + \ldots$ We introduce a third parameter: let $\mathcal{C}_{n,k,d}$ be the set of those permutations in $\mathcal{C}_{n,k}$, in which the distance between the two rightmost descents is d. We construct a mapping which is a union of a 1:1 bijection between $\mathcal{C}_{n,k,1}$ and $\mathcal{C}_{n-1,k-1}$; a 2:1 bijection between $\mathcal{C}_{n,k,2}$ and $\mathcal{C}_{n-2,k-1}$; and a 4:1 bijection between $\mathcal{C}_{n,k,d\geq 3}$ and $\mathcal{C}_{n-d,k-1}$. This proves the result for $F_k(x)$. Now, $C(x,y)$ is obtained as the sum of geometric series $\sum_{k \geq 0} y^k F_k(x)$. □

Theorem 2 ([4] Conj. 2)**.** *The generating function for* $(a_{2n+1,n})_{n \geq 0}$ *is* $\sqrt{(1+x)/(1-7x)}$, *and the numbers are* $a_{2n+1,n} = \sum_{i=0}^{n-1} (-1)^i 2^{n-i} \binom{2(n-i)}{n-i} \binom{n-1}{i}$.

Proof[5] *(sketch).* As shown in [4], a 2PS-permutation is determined by positions of ascents / descents and indicating, for each ascent, whether the maximum of the run to its left is smaller (by 1) or larger (by 1) than the minimum of the run to its right. The second option is only possible when at least one of the adjacent runs has length > 1. Therefore, 2PS-permutations of size n are in bijection with Dyck walks (U $= 1$ for ascents, D $= -1$ for descents) of size $n-1$ where U's that have a D neighbor can be colored black or red, and U's that have no D neighbor can be colored only black.



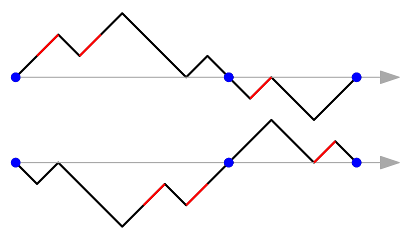
The paths that correspond to the diagonal values $a_{2n+1,n}$ are precisely the *bridges* – the walks that terminate at altitude 0. We first consider *excursions* – bridges that never go below the x-axis, and forget the colors for the time being, and get the

[5] It is possible to obtain the generating function by residue calculus, but we give a structural proof.

generating function $E = E(x, y)$ for such excursions, where x is the variable for the semi-length, and y for the number of U's that have at least one adjacent D. We compose bridges from excursions and their reflections, and get their generating function $B(x, y) = E/(1-(E-1)) = \sqrt{(1-x+xy)/(1-x-3xy)}$. Since each non-colored bridge with k regular U's generates 2^k colored bridges, their generating function is $B(x, 2) = \sqrt{(1+x)/(1-7x)} = 1/\sqrt{1-\ 8x/(1+x)} = (1/\sqrt{1-4t})|_{t=2x/(1+x)}$, and the coefficients of $1/\sqrt{1-4t}$ are well known to be central binomial coefficients. □

Additionally, we adjust the structure (Dyck walks with fixed final altitude) to get generating functions for any array of coefficients of $A(x, y)$ parallel to the diagonal, at distance m. Their shapes are $\sqrt{(1+x)/(1-7x)}((1-x-\sqrt{(1+x)(1-7x)})/(2x))^m$ or $\sqrt{(1+x)/(1-7x)}((1-x-\sqrt{(1+x)(1-7x)})/((2x)(1+2x)))^m$ (depending on the side). Another explicit formula for $a_{2n+1,n}$ is $\sum_{k\geq 0}\binom{n}{2k}\binom{2k}{k}2^{2k+1}3^{n-2k-1}\left(2-\frac{k}{n}\right)$.

Theorem 3 ([4] Conj. 3)**.** *Let $\mathcal{B}_{n,k}$ be the set of permutations in $\mathcal{A}_{n,k}$ (2PS, size n, k ascents) whose last fall has size* 1*, and let $b_{n,k} = |\mathcal{B}_{n,k}|$. Then we have $a_{2n+1,n} = 2b_{2n+1,n}$.*



Proof (sketch). The last fall has size 1 if and only if we have an ascent at $2n$. Thus we need to prove that precisely one half of $\mathcal{A}_{2n+1,n}$ ends with an ascent. As above, we represent the permutations from $\mathcal{A}_{2n+1,n}$ by colored Dyck bridges. For such a bridge, split it into maximal excursions and anti-excursions and rotate each such string by 180°. Then the bridges with last step U are mapped bijectively to the bridges with last step D. This yields an autobijection in $\mathcal{A}_{2n+1,n}$ such that "ascent at $2n \leftrightarrow$ descent at $2n$". □

Results concerning the image of T

The image of the pop-stack sorting transformation has the following characterization.

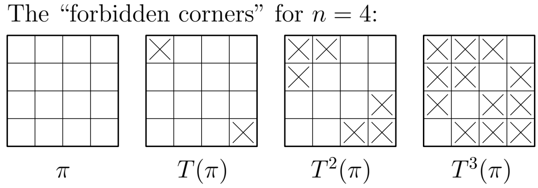
Theorem 4. *A permutation belongs to* $\mathrm{Im}(T)$ *if and only if its adjacent runs overlap.*

Enumerative aspects concerning the image of T can be found in our paper [1]. In the present work we study the structural and the enumerative aspects of $\mathrm{Im}(T^m)$.

Theorem 5. *If $\tau = a_1 a_2 \dots a_n \in \mathrm{Im}(T^m)$ (where $0 \leq m \leq n-1$), then for each i ($1 \leq i \leq n$), we have $|a_i - i| \leq n - m - 1$.*

Proof (sketch). Our proof generalizes Ungar's argument [6]. It uses a poset structure associated to a "projection" of the successive images of the permutation π, and it is conveniently visualized by "forbidden corners" in the diagrams of these images. □

The "forbidden corners" for $n = 4$:



The case $m = n - 1$ of Theorem 5 gives Ungar's result: each permutation of size n is sorted by at most $n - 1$ iterations of T. Our main result is the characterization and enumeration of $\mathrm{Im}(T^{n-2})$, the pre-image of Id in the longest chains $\pi \to T(\pi) \to T^2(\pi) \to \cdots \to T^{n-1}(\pi) = \mathrm{Id}$.

Theorem 6. *A permutation $\tau = a_1 a_2 \dots a_n$ belongs to $\mathrm{Im}(T^{n-2})$ if and only if it is thin[6] and has no inner runs of odd size. This implies $|\mathrm{Im}(T^{n-2})| = 2^{n/2-1} + 2^{n/2} - 1$ for even n, $2^{(n+1)/2} - 1$ for odd n (OEIS A052955).*

Proof (sketch). Consider a thin permutation $\tau \neq \mathrm{Id}$ without odd inner runs. The runs of τ (listed left to right) are of lengths $(r_1, r_2, \dots, r_s)$, where $s \geq 2$ and $r_2, \dots, r_{s-1}$ are even. Now, let π be *the* skew layered permutation[7] with runs of lengths $(r_s, r_{s-1}, \dots, r_1)$. It can then be checked that $T^{n-2}(\pi) = \tau$. Let us now prove the reciprocal. If one considers π and τ such that $T^{n-2}(\pi) = \tau$, then τ must be thin by Theorem 5. If we assume that τ has an odd inner run, then analysing the successive images $T^m(\pi)$, $1 \leq m \leq n-2$, leads to a contradiction. Namely, it can be shown that in this case all the letters in τ before (or after) this odd run are already sorted. This contradicts the fact that an inner run starts and ends with a descent.
For the enumeration, as the run lengths determine a thin permutation uniquely, we just need to choose the even/odd positions for the borders between runs. □

In particular, this proof shows that each skew-layered permutation of size n without odd inner runs needs exactly $n - 1$ iterations of T to be sorted. We conclude with the following conjectured (and supported by computer experiments) complete classification of skew-layered permutations with respect to the number of iterations of T needed to sort them. Denote by $\sigma(\pi)$ the smallest number m such that $T^m(\pi) = \mathrm{Id}$.

Conjecture 7. *Let π be a skew-layered permutation of size n such that π is neither the identity nor the anti-identity permutation. If n is even, then $\sigma(\pi) = n - 1$. For odd n, $\sigma(\pi)$ depends on the structure of π around the central letter (that is, $(n+1)/2$) as follows: if this letter is the middle of a run/fall of size ≥ 3, then $\sigma(\pi) = n - 2$; otherwise, $\sigma(\pi) = n - 1$. From the enumerative point of view: apart for the identity and the anti-identity, for even n we have $2^{n-1} - 2$ skew-layered permutations with $\sigma(\pi) = n - 1$; for odd n we have $(2^{n-2} - 2)/3$ (OEIS A020988) skew-layered permutations with $\sigma(\pi) = n - 2$, and $(5 \cdot 2^{n-2} - 4)/3$ (OEIS A080675) skew-layered permutations with $\sigma(\pi) = n - 1$.*

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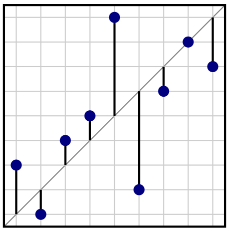
David Bevan University of Strathclyde, Glasgow

This talk is based on joint work with Pete Winkler

The *total displacement* [2] of a permutation $\sigma = \sigma_1 \ldots \sigma_n$ is $\mathsf{td}(\sigma) = \sum_i |\sigma_i - i|$.



$$\begin{aligned} \mathsf{td}(314592687) &= 2+1+1+1+4+4+1+0+2 = 16 \\ \mathsf{inv}(314592687) &= 0+1+0+0+0+4+1+1+2 = 9 \end{aligned}$$

Both $\mathsf{td}(\sigma)$ and $\mathsf{inv}(\sigma)$ are natural measures of how close σ is to the identity, and take a similar range of values:

$$0 \leqslant \mathsf{td}(\sigma), \mathsf{inv}(\sigma) \leqslant n^2/2$$

If $\mathsf{inv}(\sigma) > 0$, the *displacement ratio* of σ is the ratio of the total displacement to the number of inversions, $R(\sigma) = \mathsf{td}(\sigma)/\mathsf{inv}(\sigma)$. It is known [1] that $R(\sigma)$ lies in the half-open interval $(1, 2]$.

Let $\boldsymbol{\pi}_{n,m}$ denote a permutation chosen uniformly at random from the set of all permutations of length n with exactly m inversions. In this talk, we consider the behaviour of the expected asymptotic displacement ratio $R[m(n)] = \lim\limits_{n\to\infty} \mathbb{E}\big[R(\boldsymbol{\pi}_{n,m(n)})\big]$.

How does $R[m(n)]$ behave as $m(n)$ increases from 1 to $\binom{n}{2}$? We can consider the evolution of the random permutation to pass through four *epochs*:

very sparse	sublinear	$m(n) = n^{1/2}$, for example
sparse	linear	$m(n) = \alpha n$
semi-sparse	superlinear but subquadratic	$m(n) = n^{3/2}$, for example
dense	quadratic	$m(n) = \rho\binom{n}{2}$

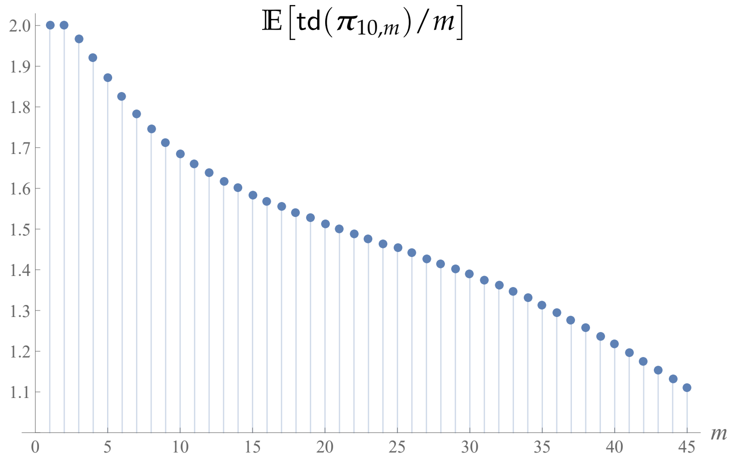
If σ is close to being maximally sparse, then $R(\sigma) = 2$.

If $\mathsf{inv}(\sigma) = 1$, then $\mathsf{td}(\sigma) = 2$, so $R[1] = 2$.
If $\mathsf{inv}(\sigma) = 2$, then $\mathsf{td}(\sigma) = 4$, so $R[2] = 2$.

And at the maximally dense end, we have $\lim\limits_{n\to\infty} R(n \ldots 21) = 1$:

$$\mathsf{inv}(n \ldots 21) = \binom{n}{2} \quad \text{and} \quad \mathsf{td}(n \ldots 21) = \lfloor n^2/2 \rfloor.$$

These observations suggest that $R[m]$ might decrease as m increases, an idea which is supported by the data.



Conjecture 1. $\mathbb{E}\left[\mathsf{td}(\pi_{n,m})/m\right]$ *decreases strictly as* m *increases from* 2 *to* $\binom{n}{2}$.

We have not been able to establish this, but assuming it is true, how does $R[m]$ decrease? How does it behave in the various epochs for $m(n)$?

The answers we do have are somewhat surprising:

very sparse	$m \ll n$	$R[m] = 2$
sparse	$m = \alpha n$	$2 > R[m] > 2\log 2 \approx 1.3863$
semi-sparse	$n \ll m \ll n^2$	$R[m] = 2\log 2$
dense	$m = \rho\binom{n}{2}$	$2\log 2 > R[m] \geqslant 1$

We describe how these results were determined using permutons and inversion sequences. We also suggest how this odd behaviour might be explained in terms of the different effects that local and global constraints have on $\pi_{n,m}$.

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Jacopo Borga University of Zurich

This talk is based on joint work with Erik Slivken [4]

We establish permuton convergence and local convergence for large uniform random square permutations. First we describe the global behavior by showing that these permutations have a permuton limit which can be characterized as a random rectangle. We also explore fluctuations about this random rectangle, which we can describe through coupled Brownian motions. Second, we consider the limiting behavior of the neighborhood of a point in the permutation through local limits. As a byproduct, we also determine the random limiting distribution of the proportion of occurrences and consecutive occurrences of any given pattern in a uniform random square permutation.

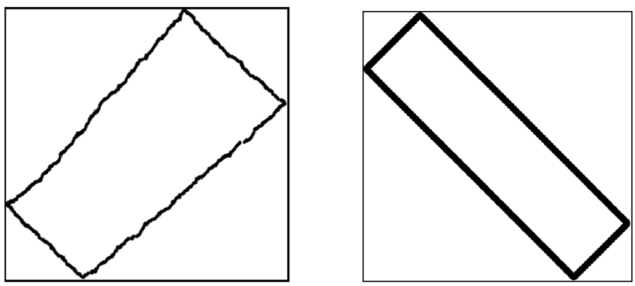

Figure 1: The diagram of two typical square permutations of size 1 000 and 1 000 000.

Square permutations

Square permutations are permutations where every point is a record, *i.e.*, a maximum or minimum, either from the left or from the right. Square permutations can be also described as a pattern-avoiding class, where the avoided patterns are all 16 patterns of length five with a point that is not a record. Mansour and Severini [7] determine the enumeration of the class proving that there are $2(n+2)4^{n-3} - 4(2n-5)\binom{2n-6}{n-3}$ square permutations of size n. This permutation class was later discussed in [5, 6, 1] (in this last paper the authors refer to square permutations as *convex permutations*). In this talk we focus on the shape of square permutations.

Sampling asymptotically uniform square permutations

The starting point for all our results is the sampling procedure described in this section. We define a projection from the set of square permutations to the set of *anchored pairs of sequences* of labels, *i.e.*, triples $(X, Y, z_0) \in \{U, D\}^n \times \{L, R\}^n \times [n]$. For every square permutation σ, the labels of (X, Y) are determined by the record types (the sequence X records if a point is a maximum (U) or a minimum (D) and the sequence Y records if a point is a left-to-right record (L) or a right-to-left record (R)) and the anchor z_0 is determined by the value $\sigma^{-1}(1)$ (see Fig. 2 for an example).

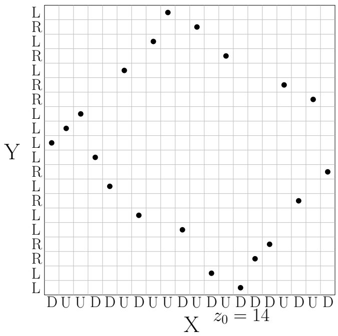

Figure 2: A square permutation σ with the associated anchored pair of sequences (X, Y, z_0). The sequence X is reported under the diagram of the permutation and the sequence Y on the left.

This projection map is not surjective, but we can identify subsets of anchored pairs of sequences (called *regular*) and of square permutations where the projection map is a bijection. We then construct a simple algorithm to produce a square permutation from regular anchored pairs of sequences. We show that asymptotically almost all square permutations can be constructed from regular anchored pairs of sequences, thus a permutation sampled uniformly from the set of regular anchored pairs of sequences will produce, asymptotically, a uniform square permutation.

Permuton limits, fluctuations and local limits

The first result we proved is the existence of the permuton limit for uniform square permutations. A permuton is a probability measure on the square $[0,1]^2$ with uniform marginals. Every permutation can be associated with the permuton induced by the sum of Dirac measures on points of the diagram of the permutation scaled to fit within $[0,1]^2$. We show that for a large square permutation σ that projects to a regular anchored pair of sequences, the permuton associated with σ is close to a permuton given by a rectangle (see for instance Fig. 1) embedded in $[0,1]^2$ with sides of slope ± 1 and bottom corner at $(\sigma^{-1}(1)/n, 0)$. This allows us to show that the permuton limit of uniform square permutations is a rectangle embedded in $[0,1]^2$ with sides of slope ± 1 and bottom corner at $(\boldsymbol{z}, 0)$, where $\boldsymbol{z}$ is a uniform point in the interval $[0,1]$ (we denote random quantities using **bold** characters).

The second result deals with fluctuations about the lines of the rectangle of the permuton limit. We show that they can be described by certain coupled Brownian motions. The latter arises naturally from the projection map described above, namely, the fluctuations around the bottom left edge of the rectangle (see Fig. 2) are determined by the distribution of the D's in the left part of the sequence X and of the L's in the lower part of the sequence Y. The coupling between Brownian motions comes from the fact that the total number of labels of each type on a given interval (either horizontal or vertical) sums up to the size of the interval.

The third result is a local limit theorem for square permutations. Our result is stated in terms of the local topology introduced by the speaker in [2]. We look at the neighborhood of a random element of a uniform square permutation and we study, for all $h \in \mathbb{N}$, the consecutive pattern induced by the h elements on the right and on the left of the chosen element, showing that, when the size of the whole permutation tends to infinity, this consecutive pattern converges in distribution to a random limiting pattern. Square permutations are the first natural but non-trivial model were the local limiting object is random (we recall that this is not the case for uniform random permutations avoiding a pattern of length three [2] or for uniform permutations in substitution-closed classes [3], where the local limiting objects are deterministic).

Our first and third results, *i.e.*, the permuton and local limits, can be interpreted in terms of the convergence of the proportion of occurrences and consecutive occurrences of any given pattern in a uniform random square permutation. We denote with $\widetilde{\text{occ}}(\pi, \sigma)$ (resp. $(\widetilde{\text{c-occ}}(\pi, \sigma))$ the proportion of occurrences (resp. consecutive occurrences) of a pattern π in σ, and with $\mathcal{S}$ the set of permutations. We can deduce that if $\boldsymbol{\sigma}_n$ is a uniform random square permutation of size n, then the following convergences (w.r.t. the product topology) hold:

$$(\widetilde{\text{occ}}(\pi, \boldsymbol{\sigma}_n))_{\pi\in\mathcal{S}} \xrightarrow{d} (\boldsymbol{\Lambda}_\pi)_{\pi\in\mathcal{S}} \quad \text{and} \quad \big(\widetilde{\text{c-occ}}(\pi, \boldsymbol{\sigma}_n)\big)_{\pi\in\mathcal{S}} \xrightarrow{d} (\boldsymbol{\Delta}_\pi)_{\pi\in\mathcal{S}},$$

where $(\boldsymbol{\Lambda}_\pi)_{\pi\in\mathcal{S}}$ and $(\boldsymbol{\Delta}_\pi)_{\pi\in\mathcal{S}}$ are random vectors that can be described in terms of the permuton and local limits of square permutations.

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Countable universal and existentially closed permutations in geometric grid classes

Samuel Braunfeld University of Maryland, College Park

Inspired by the study of the corresponding question for graph classes [2], we investigate when a permutation class admits a *countable universal permutation*, i.e. a countable permutation avoiding the specified patterns into which all other such countable permutations embed. This may be seen as a strengthening of atomicity, which is equivalent to the existence of a countable permutation avoiding the specified patterns into which all other such finite permutations embed.

For graph classes, we may consider forbidding either induced or non-induced subgraphs. In the case of induced subgraphs, the existence of a countable universal graph is undecidable. In the case of non-induced subgraphs, decidability is open, but the question seems tractable by instead considering existentially closed graphs. Existential closedness is a largeness condition, and when a class admits a unique countable existentially closed graph, it serves as a canonical countable universal graph. (A structure M avoiding certain forbidden substructures is existentially closed if given finite $A \subset M$ and a finite configuration of points B with the isomorphism type of $A \cup B$ specified, if the configuration can be added to M without creating any forbidden substructures, then the configuration is already realized in M.)

In permutation classes, we are in the case of forbidding induced substructures, and so might expect chaos. However, the proof for the induced subgraph case seems difficult to adapt, and the case of forbidding permutations of length 3 suggests similarities to the non-induced subgraph case [3].

We show that geometric grid classes of permutations [1] admit countable universal permutations and have natural candidates for unique countable existentially closed permutations.

In the course of pursuing these questions, we arrive at more traditional-seeming questions concerning geometric grid classes. These include which grids represent the same permutation class and when a permutation class specified by forbidden patterns is a geometric grid class (possibly using an infinite grid).

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Enumeration of isolated vertices in permutation graphs

Charles Burnette Saint Louis University

In this talk, we will describe both exact and limiting distributions for the number of isolated vertices in a uniform random permutation graph.

Introduction

Let us recall what a permutation graph is.

Definition 1. For a given permutation $\pi \in \mathfrak{S}_n$, the permutation graph G_π associated with π is the labelled graph with vertex set $[n]$ formed by drawing an edge between two vertices i and j precisely when $(i-j)(\pi(i)-\pi(j)) < 0$.

Let $\text{isol}_n(\pi)$ be the number of isolated vertices in G_π. Isolated vertices of G_π are also called strong fixed points of π. More specifically, node k is an isolated vertex of G_π if

$$\pi(k) = \max\{\pi(1), \pi(2), \ldots, \pi(k)\} = \min\{\pi(k), \pi(k+1), \ldots, \pi(n)\}.$$

Note in particular, then, that $\pi(k) = k$.

Certain enumerative aspects of strong fixed points of permutations have been studied fairly extensively. (See, for instance, sequence A052186 on [4], [1], and chapter 1, problem 128(b) in [5].) They are also relevant to the analysis of the quicksort algorithm. (Read section 2.2 of [6] where Wilf affectionately calls them "splitters.") Here, however, we will examine strong fixed points of permutations purely in the context of permutation graphs. Our main contribution is the following:

Theorem 2. *The scaled statistic $n \cdot \text{isol}_n$ weakly converges to a gamma-distributed random variable with shape parameter 2 and scale parameter 1.*

The Exact Distribution of isol_n

The exact distribution of isol_n can be described by way of an interesting recurrence relation for the number a_n of permutation graphs on $[n]$ having no isolated vertices at all. (Note that $a_0 = 1$ since the empty graph is vacuously free of isolated vertices.) Below are some pertinent facts about a_n.

Theorem 3. *For all nonnegative integers n,*

$$a_{n+1} = na_n + \sum_{k=1}^{n} k(n-k)!a_{k-1}. \tag{1}$$

Theorem 4. *For all nonnegative integers n,*

$$a_n + \sum_{k=1}^{n} (n-k)!a_{k-1} = n!. \tag{2}$$

Proofs of Theorems 3 and 4 are omitted for the sake of space. We can say, however, that the reasoning behind these recurrences is largely inspired by the combinatorics of connected permutation graphs presented in [2].

Throughout the rest of this abstract, we let $\mathbb{P}_n$ and $\mathbb{E}_n$ denote the uniform probability measure on $\mathfrak{S}_n$ and its associated expectation operator, respectively. Also let $\lambda \vdash n$ denote that λ is an integer partition of n, and let λ_k be the number of parts of size k that λ has, and let $|\lambda| = \sum \lambda_k$ be the total number of parts that λ has. We are now ready to state the probability mass function of isol_n.

Theorem 5. *For all nonnegative integers m and n with $m \le n$,*

$$\mathbb{P}_n(\mathrm{isol}_n(\pi) = m) = \frac{(m+1)!}{n!} \sum_{\substack{\lambda \vdash (n+1), \\ \lambda \text{ has } m+1 \text{ parts}}} \left(\prod_{j=1}^{n+1} \frac{a_{j-1}^{\lambda_j}}{(\lambda_j)!} \right). \tag{3}$$

One can also use the principle of inclusion-exclusion to count the number of permutation graphs on $[n]$ having no isolated vertices.

Theorem 6. *For all nonnegative integers n,*

$$a_n = \sum_{\lambda \vdash (n+1)} (-1)^{|\lambda|-1} |\lambda|! \left(\prod_{j=1}^{n+1} \frac{((j-1)!)^{\lambda_j}}{(\lambda_j)!} \right) \tag{4}$$

The major advantage of (2), however, is that it makes it more apparent that asymptotically almost all permutation graphs on $[n]$ have no isolated vertices. Indeed,

$$\begin{aligned}
\mathbb{P}_n(\mathrm{isol}_n(\pi) \neq 0) &= \frac{n! - a_n}{n!} \\
&= \frac{1}{n!} \sum_{k=1}^{n} (n-k)! a_{k-1} \\
&< \frac{1}{n!} \sum_{k=1}^{n} (n-k)!(k-1)! \\
&= \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{\binom{n-1}{k-1}} = \frac{2}{n}(1 + o(1))
\end{aligned}$$

as $n \to \infty$. This will become even more apparent in the next section though.

The Limiting Distribution of isol_n

For each $\pi \in \mathfrak{S}_n$ and each integer $k \in [n]$, define set $I_k(\pi) = 1$ if node k is an isolated vertex of G_π and set $I_k(\pi) = 0$ otherwise. Notice that then

$$\mathrm{isol}_n(\pi) = \sum_{k=1}^{n} I_k(\pi). \tag{5}$$

As a result, for each integer $s \in [n]$.

$$\mathrm{isol}_n(\pi)(\mathrm{isol}_n(\pi)-1)(\mathrm{isol}_n(\pi)-2)\cdots(\mathrm{isol}_n(\pi)-s+1) = \sum I_{k_1}(\pi)I_{k_2}(\pi)\cdots I_{k_s}(\pi)$$

where the sum is over all s-permutations of integers taken from $[n]$. This provides a very convenient way to calculate the factorial moments of isol_n.

Theorem 7. *For all nonnegative integers n and s with $s \leq n$,*

$$\mathbb{E}_n((\mathrm{isol}_n)_s) = \frac{(s+1)!}{n!} \sum_{\substack{\lambda \vdash (n+1), \\ \lambda \text{ has } s+1 \text{ parts}}} |\lambda|! \left(\prod_{j=1}^{n+1} \frac{((j-1)!)^{\lambda_j}}{(\lambda_j)!} \right) = \frac{(s+1)!}{n^s}(1+o(1)) \quad (6)$$

as $n \to \infty$.

Combining the method of factorial moments with a generalization of Lemma 2 from [3] yields our main result.

Theorem 8. *For all real x*

$$\lim_{n\to\infty} \mathbb{P}_n(n \cdot \mathrm{isol}_n(\pi) \leq x) = \begin{cases} 1 - \dfrac{x+1}{e^x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} .$$

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Sorting Permutations with Pattern-Avoiding Stacks

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This talk is based on joint work with Anders Claesson, Luca Ferrari and Einar Steingrimsson

The problem of sorting a permutation using a stack was proposed by Knuth in the 1960s. As it is well known, sortable permutations can be characterized in terms of pattern avoidance and their enumeration is given by the Catalan numbers. Unfortunately, adding just another stack in series makes the problem extremely hard. Hoping to gain a better understanding of the general 2-stacksort problem, we start the analysis of a new sorting device, where some restrictions on the stacks are given in terms of pattern avoidance. We will use a *right-greedy* procedure, in analogy with [W]. Here we provide the first general results in this new framework and propose some open problems.

Counting Classes and Non-Classes

Let γ be a permutation. We consider a sorting device consisting in two stack in series S_1 and S_2, with the following right-greedy algorithm:

1. push an element from the input into S_1, unless it creates an occurrence of γ in S_1, reading from top to bottom;
2. otherwise, push the top of S_1 into S_2, unless it is bigger than the top of S_2.
3. otherwise, push the top of S_2 into the output.

If the output is the identity, the input permutation is said to be γ-*sortable*. Denote with $\mathcal{S}(\gamma)$ the set of the γ-sortable permutations and let $\mathcal{S}_n(\gamma) = \mathcal{S}(\gamma) \bigcap \mathfrak{S}_n$.

Theorem 1. *Let $\gamma = \gamma_1\gamma_2\gamma_3 \cdots \gamma_k$ a permutation and let $\hat{\gamma} = \gamma_2\gamma_1\gamma_3 \cdots \gamma_k$. Then $\mathcal{S}(\gamma)$ is a permutation class if and only if $\hat{\gamma} \geq 231$. In this case, we have $\mathcal{S}(\gamma) = Av(132, \gamma^R)$, where $\gamma^R = \gamma_k \cdots \gamma_1$ is the reverse of γ.*

Corollary 2. *The number of permutations $\gamma \in \mathfrak{S}_n$ such that $\mathcal{S}(\gamma)$ is not a permutation class is the n-th Catalan numbers $\mathfrak{c}_n$.* *(sequence A000108 in [S])*

Theorem 1 guarantees that if $\mathcal{S}(\gamma)$ is a class, then its basis is either $\{132\}$ or $\{132, \gamma^R\}$. We are able to count the patterns γ such that the basis of $\mathcal{S}(\gamma)$ has cardinality 2.

Theorem 3. *Let f_n be the number of permutations γ such that $\mathcal{S}(\gamma)$ is a permutation class and γ^R avoids 132. Then $f_n = \mathfrak{c}_n - 2\mathfrak{c}_{n-1}$.* *(sequence A002057 in [S])*

123-Sortable Permutations

In the following we focus on specific patterns γ. The hidden combinatorial structure in most of the cases seems to be surprisingly deep, making the analysis quite challenging already for patterns of length 3. We start with the pattern 123; notice that, as a consequence of Theorem 1, $\mathcal{S}(123)$ is not a permutation class.

Theorem 4. $\mathcal{S}(12) = Av(213)$.

Theorem 5. *Given $\pi \in \mathcal{S}(123)$ and $k \geq 1$, let π' be obtained from π by k-inflating* [8] *the first element of π. Then π is sortable if and only if π' is sortable.*

Theorem 6. *Let $\pi = \pi_1 \cdots \pi_n$ a permutation with $\pi_1 = n$. Then π is 123-sortable if and only if π avoids 213.*

Theorem 7. *For $n, k \geq 1$, denote with $\mathcal{S}_n^{des}(123)\ [k]$ the set of 123-sortable permutations of length n that start with a descent and have k left-to-right maxima. Then there exists a bijection between $\mathcal{S}_n^{des}(123)\ [k]$ and $\mathcal{S}_{n+1}^{des}(123)\ [k+1]$.*

What we have proved so far completely determine the structure of 123-sortable permutations. Indeed, any $\pi \in \mathcal{S}_n(123)$ (except for the identity) can be uniquely constructed by choosing $\alpha = \alpha_1\alpha_2 \cdots \alpha_k \in Av_k(213)$, with $\alpha_1 = k$, then adding h new maxima, according to the bijection of Theorem 7, and finally adding $n-k-h$ consecutive ascents at the beginning, by inflating the first element.

Exploiting this structural description of $\mathcal{S}(123)$, we are able to find a bijection between 123-sortable permutations of length n and `UHD`-avoding Schröder paths of semilength $n-1$, enumerated in [CF]. Thus, we have the following result.

Corollary 8. *For all $n \geq 1$, $|\mathcal{S}_n(123)| = 1 + \sum_{h=1}^{n-1} (n-h)\mathfrak{c}_h$.* *(sequence A294790 in [S])*

132-Sortable Permutations

Although $\mathcal{S}(132)$ is not a permutation class, due to Theorem 1, we show a useful characterization of the 132-sortable permutations in terms of barred patterns.

Theorem 9. $\mathcal{S}(132) = Av(2314, \bar{3}142, 14\bar{2}3, 24\bar{1}3)$.

The previous result offers a very precise and geometrical description of $\mathcal{S}(132)$, which enables to find a connection with restricted growth functions of certain set partitions, whose enumeration can be found in [JM].

Definition 10. Given the set partition $B = B_1|B_2|\ldots|B_k$ of $[n]$, where $min(B_1) < min(B_2) < \cdots < min(B_k)$, the *Restricted Growth Function* (briefly RGF)of B is the word $RGF(B) = r_1 \cdots r_n$, where $r_i = j$ if $i \in B_j$.

[8] i.e., replacing the element with an increasing sequence of consecutive elements and suitably rescaling the other entries.

Theorem 11. *For each $n \geq 1$, there is a bijection between $\mathcal{S}_n(132)$ and the set partitions of $[n]$ whose RGF avoids* 2231*. Moreover, the bijection maps the number of left-to-right minima of a sortable permutation into the number of blocks of the partition.*

Corollary 12. $|\mathcal{S}_n(132)| = \sum_{k=0}^{n-1} \binom{n-1}{k} \mathfrak{c}_k.$ *(sequence A*007317 *in [S])*

Being the above formula so neat, it is quite natural to ask if it can be read off directly from sortable permutations. We report also another conjecture, verified computationally for small values of n, that seems to suggest a deeper link with some Catalan-type combinatorial objects.

Question 13. *Prove that the number of* 132*-sortable permutations of length $n+1$ with $k+1$ left-to-right minima is $\sum_{i=k}^{n} \binom{n}{i} \mathfrak{n}_{i,k}$, where $\mathfrak{n}_{i,k}$ is the (i,k)-th Narayana number.*

Open Problems for Future Work

By Theorem 1, we have $\mathcal{S}(321) = Av(123, 132)$, meaning that there are 3 more patterns of length 3 whose enumeration remains to be solved.

Question 14. *Characterize and enumerate γ-sortable permutations for the remaining patterns γ of length* 3*, namely* 213, 231, 312*.*

Concerning longer patterns, it would be interesting to classify our sorting machines in terms of the number of permutations they sort. This gives rise to a notion of Wilf-equivalence on sorting machines, which seems to be particularly interesting when the set of sortable permutations constitute a class. For instance, a thorough case by case analysis shows that there are two Wilf-classes for patterns of length 4 (Catalan numbers and odd-indexed Fibonacci numbers, sequence *A*001519 in [S]), five Wilf-classes for patterns of length 5 and 10 for patterns of length 6.

Question 15. *Enumerate the Wilf-classes for devices whose corresponding set of sortable permutations is a class.*

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Enumerative combinatorics of intervals in the Dyck pattern poset

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This talk is based on joint work with Antonio Bernini and Luca Ferrari

The *Dyck pattern poset* has been first introduced in [BFPW] and further studied in [BBFGPW]. A *Dyck path* is a lattice path starting from the origin of a fixed Cartesian coordinate system, ending on the x-axis, never falling below the x-axis and using only two types of steps, namely up steps $U = (1, 1)$ and down steps $D = (1, -1)$. The sequence of up and down steps of a Dyck path is a word on the alphabet $\{U, D\}$ such that each prefix has at least as many U's as D's and the total number of U's and D's is the same. Such words are commonly called *Dyck words*. Given two Dyck paths P and Q, we say that P is a *pattern* of Q and we write $P \leq Q$, when P is a subword of Q (i.e. there exists a subset of the letters of Q which, read from left to right, are equal to P). Any subword of Q which is equal to P is called an *occurrence* of P in Q. So, for instance, $UUDD \leq UDUDUD$, whereas $UUDDUD$ and $UUDUUUDDDD$ are incomparable. The Dyck pattern poset has a minimum, which is the path UD, and has no maximum; moreover, it is graded, the rank of an element being its semilength.

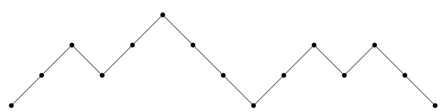

Figure 1: The Dyck path UUDUUDDDUUDUDD.

For a graded poset $\mathcal{P}$ and nonnegative integers ℓ and k, denote by $s_\ell(\mathcal{P})$ the number of saturated chains of length ℓ in $\mathcal{P}$ and denote by $s_\ell^{(k)}(\mathcal{P})$ the number of saturated chains in $\mathcal{P}$ having length ℓ and top element of rank k. In particular, $s_0(\mathcal{P})$ is just the number of elements of $\mathcal{P}$, $s_0^{(k)}(\mathcal{P})$ is the number of elements of $\mathcal{P}$ with rank k and $s_1(\mathcal{P})$ is the number of edges of the Hasse diagram of $\mathcal{P}$. In the above mentioned papers some enumerative properties of the Dyck pattern poset have been investigated, mainly focusing on pattern avoidance questions. In this work, we start to deal with the enumerative combinatorics of intervals in the Dyck pattern poset by computing $s_\ell^{(k)}([UD, P])$, for some choices of the integers ℓ, k and of the paths P.

The interval $[UD, (UD)^n]$

Our first result is an explicit formula for the number of elements in the interval $[UD, (UD)^n]$, for $n \in \mathbb{N}$. For this purpose, we characterize the Dyck paths in the interval $[UD, (UD)^n]$ in terms of the number of their ascents. Given a Dyck path P, an *ascent* of P is a maximal consecutive substring of P of the form U^m, for some $m > 0$, and the number of ascents of P will be denoted by $\mathrm{asc}(P)$. The Dyck paths in the interval $[UD, (UD)^n]$ can now be characterized as follows.

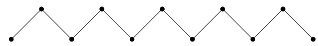

Figure 2: The Dyck path $(UD)^5$ =UDUDUDUDUD.

Lemma 1. *Let $n > 0$, $k \in \{1, ..., n\}$ and P be a Dyck path of semilength k. Then $P \leq (UD)^n$ if and only if $\mathsf{asc}(P) \geq 2k - n$.*

It is well known that, given $n, k \in \mathbb{N}$, Dyck paths of semilength n having k ascents are counted by the *Narayana number* $N_{n,k}$, where $N_{0,0} = 1$, $N_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ for $n, k \geq 1$ and $N_{n,k} = 0$ in the remaining cases (sequence A001263 in [S]). As an immediate consequence, we deduce the following explicit formulas.

Proposition 2. *Let $n > 0$ and $k \in \{1, ..., n\}$, then*

(i) $s_0^{(k)}([UD, (UD)^n]) = \sum_{m=\max\{1,2k-n\}}^{k} N_{k,m}$

(ii) $s_0([UD, (UD)^n]) = \sum_{k=1}^{n} \sum_{m=\max\{1,2k-n\}}^{k} N_{k,m}$.

The sequence $(s_0([UD, (UD)^n]))_{n\geq 0}$ of the sizes of the interval $[UD, (UD)^n]$ starts 1,2,4,8,16,33,70,152,337 and is not recorded is [S]; however, it is the sequence of the partial sums of A004148 of [S], called “generalized Catalan numbers" and counting, among other things, peak-less Motzkin paths with respect to the length. The sequence $(s_0^{(k)}([UD, (UD)^n]))_{n\geq k\geq 0}$ appears as A137940 in [S].

The interval $[UD, U^{a+h}D^aU^bD^{b+h}]$

Let a, b, h be positive integers (and assume, w.l.o.g., that $b \geq a \geq 1$) and $Q_{a,b}^{(h)} = U^{a+h}D^aU^bD^{b+h}$, that is a generic Dyck path with two peaks. We provide an explicit formula for the number of elements in $[UD, Q_{a,b}^{(h)}]$, depending on a, b and h.

Proposition 3. *Denote by $\varphi_h(a, b)$ the number of Dyck paths having two peaks in the interval $[UD, Q_{a,b}^{(h)}]$. Then:*

(i) $\varphi_0(a, b) = \frac{a(a+1)(3b-a+1)}{6}$*;*

(ii) $\varphi_h(a, b) = \varphi_0(a, b) + hab$*;*

(iii) $s_0([UD, Q_{a,b}^{(0)}]) = \varphi_h(a, b) + b + h$*.*

The numbers $(\varphi_0(a, b))_{a,b\geq 0}$ match the entries of the triangular matrix recorded as sequence A082652 in OEIS, which counts the number of squares that can be found in a rectangular $a \times b$ grid. A bijection between such squares and Dyck paths with two peaks contained in $Q_{a,b}^{(0)}$ can be succinctly described as follows: each path of the form

$U^{k+i}D^iU^jD^{j+k}$ with two peaks contained in $Q^{(0)}_{a,b}$ corresponds to the square whose side has length $h+1$ and whose topmost and leftmost vertex is the topmost and leftmost vertex of the unit square in the i-th row (from top to bottom) and j-th column (from left to right) of the grid.

As a further result, we also provide the following explicit formulas for the number of elements in the interval $[UD, Q^{(0)}_{a,b}]$ with rank k.

Proposition 4. *Let* $1 \leq k \leq a+b$, *then*

$$s_0^{(k)}([UD, Q^{(0)}_{a,b}]) = \begin{cases} \binom{k}{2}+1 & 1 \leq k \leq a \\ \binom{a+1}{2}+1 & a < b,\ a+1 \leq k \leq b \\ \binom{a+b-k+2}{2} & b+1 \leq k \leq a+b\ . \end{cases}$$

Finally, we provide an explicit formula for the number of edges in the Hasse diagram of the interval $[UD, Q^{(0)}_{a,b}]$. To this aim, we will compute $s_1([UD, Q^{(0)}_{a,b}])$ using the following lemma.

Lemma 5. *Let* i, j, k *be non negative integers and* $\Delta Q^{(k)}_{i,j}$ *be the number of Dyck paths covered by* $Q^{(k)}_{i,j}$, *then*

$$\Delta Q^{(k)}_{i,j} = \begin{cases} 0 & (i,j,k) \in \{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\} \\ 1 & (i,j,k) \in \{(1,1,0),(i,0,k),(0,j,k) :\ i+k, j+k \geq 2\} \\ 2 & (i,j,k) \in \{(i,1,0),(1,j,0),(1,1,k) :\ i,j \geq 2,\ k \geq 1\} \\ 3 & (i,j,k) \in \{(i,j,0),(i,1,k),(1,j,k) :\ i,j \geq 2,\ k \geq 1\} \\ 4 & (i,j,k) \in \{(i,j,k) :\ i,j \geq 2,\ k \geq 1\} \end{cases}$$

Now, after having observed that

$$s_1([UD, Q^{(0)}_{a,b}]) = \sum_{n \geq 0} n \cdot \Delta_n([UD, Q^{(0)}_{a,b}]),$$

we get:

$$s_1([UD, Q^{(h)}_{a,b}]) = -\frac{1}{3}(2a^3 - 6a^2b + a - 3b + 3). \tag{1}$$

The above number triangle does not appear in [S].

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Applications of a q-analog for Riordan arrays to various combinatorial objects

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This talk is based on joint work with Ji-Hwan Jung, AORC in Sungkyunkwan University

One of the fundamental concepts in combinatorics is that of enumeration, and one of the basic techniques for dealing with problems of enumeration is that of generating functions. Heuristically, a generating function f is a representation of a counting function $N : \mathbb{N} \to \mathbb{N}$ as an element $f(N)$ of some algebra. There are several types of generating functions which have actually arisen in specific enumeration problems.

In this talk, we are interested to a q-analog of exponential generating function $\sum_{n\geq 0} a_n \frac{z^n}{n!}$ for a sequence $(a_n)_{n\geq 0}$. It is called the *Eulerian generating function* defined by

$$\sum_{n\geq 0} a_n \frac{z^n}{[n]_q!}$$

where $[n]_q! = \prod_{k=1}^{n} [k]_q$, $[0]_q! = 1$, and $[k]_q = \frac{q^k-1}{q-1}$. This q-analog arises in several combinatorial applications such as finite vector spaces, partitions and counting permutations by inversions. For example, while $n!$ counts the number of permutations of length n, $[n]_q!$ counts permutations while keeping track of the number of inversions.

For $n \in \mathbb{N}_0$, let $\mathcal{E}_q(n)$ be the set of the Eulerian generating functions of the form

$$a_n \frac{z^n}{[n]_q!} + a_{n+1} \frac{z^{n+1}}{[n+1]_q!} + \cdots, \quad a_n = 1.$$

With a pair of functions $g \in \mathcal{E}_q(0)$ and $f \in \mathcal{E}_q(1)$, a *q-Riordan array* [4] written $(g, f)_q = (\ell_{n,k})_{n,k\in\mathbb{N}_0}$ is defined by

$$\sum_{n\geq k} \ell_{n,k} \frac{z^n}{[n]_q!} = g(z) \frac{f^{[k]}(z)}{[k]_q!},$$

where $f^{[k]}(z)$ is the kth symbolic power of $f(z)$. Since $\ell_{n,k} = 0$ for $n < k$ and $\ell_{n,n} = 1$ for $n \in \mathbb{N}_0$, every q-Riordan array is an infinite lower triangular matrix with unit diagonal elements. If $q = 0$ and $q = 1$, then $(g, f)_q$ reduces to the usual Riordan array (g, f) and the exponential Riordan array $\langle g, f \rangle$, respectively. Thus the q-Riordan array is a q-analog for a Riordan array.

Interestingly, various combinatorial objects arising in enumeration problem can be expressed as q-Riordan arrays, see [1, 2, 3, 4]. Consider a k-partition $\pi = \{B_1, \ldots, B_k\}$ of the n-set $[n]$, and we denote the set of such partitions by $\Pi_{n,k}$. The following theorem asserts that if G and F are the Eulerian generating functions whose coefficients are associated to the counting functions $\mathbb{N}_0 \to \mathbb{C}[[q]]$, then the q-Riordan array $(G, F)_q$ can be applied to the enumeration problem of set partitions by block inversions.

Theorem 1. *[4] Let $g, f : \mathbb{N}_0 \to \mathbb{C}[[q]]$ be counting functions with $g(0) = 1$, $f(0) = 0$ and $f(1) = 1$. If $h_k : \mathbb{N}_0 \to \mathbb{C}[[q]]$ for fixed k is defined by*

$$h_k(n) = \sum_{\pi=\{B_1,\ldots,B_{k+1}\}\in\Pi_{n+1,k+1}} g(|B_1|-1)f(|B_2|)\cdots f(|B_{k+1}|)q^{\mathrm{inv}(\pi)},$$

then the array $(a_{n,k})_{n,k\in\mathbb{N}_0}$ where $a_{n,k} = h_k(n)$, may be expressed as the q-Riordan array given by $(G, F)_q$.

In this talk, we will see how this notion can be applied to q-analogs of combinatorial objects from the set partitions. As an example, it will be introduced by the q-analog of Laguerre polynomials that include a nice combinatorial description. Indeed, the coefficients of the q-Laguerre polynomial can be expressed in terms of the q-rook numbers $r_k(B, q)$ for a Ferrers board B which were introduced by Garsia and Remmel [6]:

$$r_k(B, q) = \sum_{C\in\mathcal{C}_k(B)} q^{\mathrm{inv}(C)}$$

where $\mathcal{C}_k(B)$ is the collection of all placements of k non-attacking rooks on B and $\mathrm{inv}(C)$ is defined as follows: first cross out all squares which either contain a rook, or are below or to the left of any rook. Then we count the remaining squares. Furthermore, the q-Laguerre polynomials are orthogonal and satisfy the three-term recurrence relation. As a matter of fact, Gessel [8, p.174] suggested finding a rook polynomial interpretation for the orthogonality, in the usual sense, of q-Laguerre polynomials. We hope that our q-version of Laguerre polynomials shall provide clues to this problem.

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Square permutations and convex permutominoes

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In this talk we consider square permutations, a natural subclass of permutations defined in terms of geometric conditions, that can also be described in terms of pattern avoiding permutations, and convex permutoninoes, a related subclass of polyominoes.

We propose a common approach to the enumeration of these two classes of objets that allows us to explain the known common form of their generating functions, and to derive new refined formulas and linear time random generation algorithms for these objects.

Introduction and results

Square permutations and convex permutominoes are natural subclasses of permutations and polyominoes that were introduced independently in the last ten years and have been shown to enjoy remarkably simple and similar enumerative formulas: their respective generating functions $Sq(t)$ and $Cp(t)$ are

$$Sq(t) = \frac{t^2}{1-4t}\left(2+\frac{2t}{1-4t}\right) - \frac{4t^3}{(1-4t)^{3/2}} \tag{1}$$

$$Cp(t) = \frac{t^2}{1-4t}\left(2+\frac{2t}{1-4t}\right) - \frac{t^2}{(1-4t)^{3/2}} \tag{2}$$

Equivalently, the number Sq_n of square permutations with n points and the number Cp_n of convex permutominoes with size n are respectively

$$(n+2)\,2^{2n-5} \;-\; 4(2n-5)\binom{2n-6}{n-3}$$

$$(n+2)\,2^{2n-5} \;-\; (2n-3)\binom{2n-4}{n-2}$$

These results were first obtained by Mansour *et al* [8], Duchi *et al* [7] and Albert *et al* [1] for $Sq(t)$, and by Boldi *et al* [3] and Disanto *et al* [6] for $Cp(t)$. Known proofs of these formulas rely on writing recursive decompositions resulting into linear equations with one catalytic variable that can be easily solved via the kernel method. Link with pattern avoiding permutations are described in [9]. An explicit connection between the two classes of objects was obtained by Bernini *et al* [2], resulting in a composition relation of the form $Cp(t) = ISq(t,2)$ where $ISq(t,u)$ is the generating function of certain indecomposable square permutations counted by their size and number of free fixed points.

While relatively simple, none of these known proofs explain, as far as we know, the common shape of the formulas, nor its particular form as a difference between an asymptotically dominant rational term and a simple subdominant algebraic term.

We give here a common proof of the two formulas that explains their form and allows to generalize them to take into account natural parameters extending the Naranaya refinement for Catalan numbers. In order to state our result let us consider the following rational sets of words: Let $\mathcal{W} = \mathcal{A}^*$ denote the set of (bi)words on the alphabet $\mathcal{A} = \{U, D\} \times \{L, R\}$, and let $\mathcal{M}$ denote the set of marked words (w, m) consisting of

- a word $w = (u_1, v_1) \cdots (u_n, v_n) \in \{(X, Y)\} \cdot \mathcal{W} \cdot \{(X, Y)\}$
- and a mark m with $1 \leq m \leq n$ and $v_m \in \{L, Y\}$.

Let moreover $M(t; x, y)$ and $W(t; x, y)$ be respectively the generating functions of $\mathcal{M}$ and $\mathcal{W}$ with respect to the length (var. t), number of U and X (var. x) and number of L and Y (var. y). Then

$$W(t; x, y) = \frac{1}{(1 - (1 + x)(1 + y)t)}$$

and, upon dealing separately with the case $m \in \{1, n\}$ where the mark is on a letter Y from the case $1 < m < n$ where the mark is on a letter L, we have that

$$M(t; x, y) = 2 \cdot txy \cdot W(t; x, y) \cdot txy + (txy) \cdot W(t; x, y) \cdot t(1 + x)y \cdot W(t; x, y) \cdot txy.$$

In particular $M(t, 1, 1)$ gives an *a priori unrelated* combinatorial interpretation of the dominant rational term in Formula (1) and (2) since

$$W(t; 1, 1) = \frac{1}{1 - 4t} \quad \text{and} \quad M(t; 1, 1) = \frac{t^2}{1 - 4t}\left(2 + \frac{2t}{1 - 4t}\right).$$

By defining the *horizontal/vertical encoding* of square permutations and of convex permutominoes by words of $\mathcal{M}$ we prove the following theorem:

Theorem 1. *The horizontal/vertical encoding defines bijections*

$$\mathcal{S}q \equiv \mathcal{M} \;-\; \mathcal{T}^{\swarrow} \cdot \{(D, L), (D, R)\} \cdot \mathcal{W} \cdot \{(X, Y)\} \;-\; \mathcal{T}^{\nwarrow} \cdot \{(U, R), (D, L)\} \cdot \mathcal{W} \cdot \{(X, Y)\}$$

$$\mathcal{C}p \equiv \mathcal{M} \;-\; \mathcal{D}^{\swarrow} \cdot \{(D, L)\} \cdot \mathcal{W} \cdot \{(X, Y)\} \;-\; \mathcal{D}^{\nwarrow+} \cdot \mathcal{W} \cdot \{(X, Y)\}$$

where $\mathcal{T}^{\swarrow}$ and $\mathcal{T}^{\nwarrow}$ are languages encoding particular classes of triangular permutations*, while $\mathcal{D}^{\swarrow}$ and $\mathcal{D}^{\nwarrow+}$ are languages encoding particular classes of* directed convex permutominoes.

From Theorem 1 it already appears that the two sets of *non-coding words* for $\mathcal{S}q$ and for $\mathcal{C}p$ have a similar structure. In fact the analogy between the two results goes further since the languages $\mathcal{T}^{\swarrow}$ and $\mathcal{T}^{\nwarrow}$, $\mathcal{D}^{\swarrow}$, $\mathcal{D}^{\nwarrow+}$ have essentially the same univariate generating functions:

$$T^{\swarrow}(t) = T^{\nwarrow}(t) = (2D^{\swarrow}(t) + 1)t = 2D^{\nwarrow+}(t) - t = \frac{t}{\sqrt{1 - 4t}} \tag{3}$$

and these results together with Theorem 1 immediately imply Formulas (1) and (2).

The evaluations (3) can be obtained from algebraic decompositions of the respective classes of objects, but we choose also to obtained them by re-using the same horizontal/vertical encoding. Moreover, from this analysis we obtain the refinements of Formulas (1) and (2) according the number of upper points and the number of left points, which involve refinements of Narayana numbers.

Finally, from the vertical/horizontal encoding for square permutations we obtain the following:

Corollary 2. *There is a random sampling algorithm to generate uniform random square permutations with n points or uniform random convex permutominoes of size n in expected time linear in n.*

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Michael Engen University of Florida

This talk is based on joint work with Robert Brignall and Vincent Vatter [1]

In this talk, we will discuss widdershins permutations and some properties of their corresponding permutation and graph classes.

The widdershins class $\mathcal{W}$

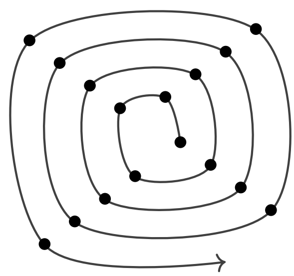

Figure 1: A representative widdershins permutation.

In his thesis [3], Murphy defined *widdershins* permutations, which we have drawn an example of in Figure 1 (widdershins being a Lower Scots word meaning "to go anti-clockwise"). The downward closure $\mathcal{W}$ of the widdershins permutations turns out to be a relatively well-behaved permutation class. We show that $\mathcal{W}$ has the rational generating function

$$\frac{1-4x+3x^2}{1-5x+6x^2-2x^3-x^4-3x^5}.$$

and has the finite basis which is the rotational closure of

$$\{2143, 2413, 314562, 412563, 415632\}.$$

Of particular interest to us is that $\mathcal{W}$ is well-quasi-ordered.

Well-quasi-ordered graph classes

The motivation for studying this permutation class lies in graph theory. Under the induced subgraph order, the collection of all graphs contains infinite antichains (e.g. the set of all cycles $\{C_3, C_4, C_5, \dots\}$. A vibrant area of research in graph theory is to characterize the graph classes which do not contain infinite antichains, which are called *well-quasi-ordered*, and determine the consequences of a class being well-quasi-ordered. In [2], Korpelainen, Lozin, and Razgon conjectured that any finitely-based well-quasi-ordered graph class is also labeled-well-quasi-ordered.

We provide a counterexample to the conjecture by proving that the class of graphs of widdershins permutations meet the hypotheses of the conjecture while not being labeled-well-quasi-ordered.

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On the poset of king-non-attacking permutations

Esterella Eisenberg and Moriah Sigron Jerusalem College of Technology, Jerusalem

This talk is based on joint work with Eli Bagno and Shulamit Reches

In this article we deal with the poset of king non-attacking permutations under the relation of containment. We present some structural results and bring information of its Möbious function.

Introduction

The Hertzsprung's problem is to find the number of ways to arrange n non-attacking kings on an $n \times n$ chess board such that each row and each column contains exactly one king. Let S_n be the symmetric group on n elements. By identifying a permutation $\sigma = [\sigma_1, \ldots, \sigma_n] \in S_n$ with its plot, i.e. the set of all lattice points of the form (i, σ_i) where $1 \le i \le n$, this problem reduces to finding the number of permutations $\sigma \in S_n$ such that for each $1 < i \le n$, $|\sigma_i - \sigma_{i-1}| > 1$. This set is counted in OEIS A002464. In the squeal, we switch between some notations for permutations. Occasionally, we omit the commas in the writing of $\sigma = [\sigma_1, \ldots, \sigma_n] \in S_n$ as in $[3142] \in S_4$.

Let K_n be the set of all such permutations in S_n. In this paper we call them simply *king permutations* or just *kings*. For example: $K_1 = S_1, K_2 = K_3 = \emptyset$, $K_4 = \{[3142], [2413]\}$. Observe that K_n is closed to the reverse and inverse actions.

The set of all permutations $\cup_{n\in\mathbb{N}} S_n$ is a poset under the partial order given by containment.

We are interested in the sub-poset $\cup_{n\in\mathbb{N}} K_n$ containing only the king permutations.

In order to analyse properties of the posets we are dealing with, one can use the Manhattan or taxicab metric $d_\sigma(i,j)$. The *breadth* of an element $\sigma \in S_n$ is defined in [2] to be:

$$br(\sigma) = min_{i,j\in[n], i\neq j} d_\sigma(i,j)$$

It is easy to observe that for $n > 1$ we have $\sigma \in K_n$ if and only if $br(\sigma) \geq 3$.

In a paper by Bevan, Homborger and Tenner, [2], the authors define the notion of a *k-prolific permutation*. A permutation $\sigma \in S_n$ is called k-prolific if each subset of the letters of $\sigma = [\sigma_1, \ldots, \sigma_n]$ of order $n - k$ forms a unique pattern.

It is shown there that σ is k-prolific if and only if $br(\sigma) \geq k + 2$. Hence, $\sigma \in K_n$ if and only if it is 1- prolific.

Main results

The first main result in this paper claims that the permutations [2413] and [3142] serve as building blocks of the poset of king permutations.

Theorem 1. *For every $\sigma \in K_n$ ($n \geq 4$), either $[2413] \preceq \sigma$ or $[3142] \preceq \sigma$.*

The following corollary adds some more information about the structure of the posets of king permutations.

Theorem 2. *Let $n > 4$. For each $\sigma \in K_n$ there exist $\pi \in K_5$ s.t. $\pi \preceq \sigma$.*

The following result is a basic ingredient in a series of theorems which explore the structure of the poset K_n.

Theorem 3. *Let $\sigma \in K_n$ with $n \geq 4$ and let $\pi \in K_{n-2}$ be such that $\pi \prec \sigma$. Then there exists $\tau \in K_{n-1}$ such that $\pi \prec \tau \prec \sigma$.*

Theorem 4. *For each two king permutations $\sigma \prec \pi$ there exists a chain of king permutations $\sigma = \sigma^0 \prec \sigma^1 \cdots \prec \sigma^k = \pi$ such that $|\sigma^i| - |\sigma^{i-1}| \in \{1,3\}$.*

We observe that the chains in the poset K_n might contain holes. In order to characterize the permutations of which this phenomenon occurs we define $\sigma \in K_{n-1}$ to be a *regent* of $\pi \in K_n$ if $\sigma \prec \pi$. We give a complete description of all the permutations which have no regents in the following:

Theorem 5. *The following conditions are equivalent for each $\pi \in K_n$ with $n \geq 4$.*

1. *There are $\alpha_1, \ldots, \alpha_k \in \{[3142],[2413]\}$ and $\sigma \in S_k$ such that $\pi = \sigma[\alpha_1, \ldots, \alpha_k]$, where $\sigma[\alpha_1, \ldots, \alpha_k]$ is the inflation of permutation σ by the permutations $\alpha_1, \ldots, \alpha_k$*

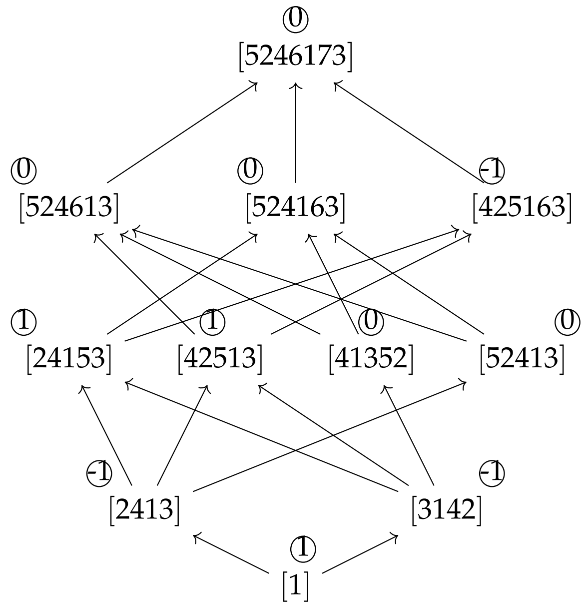
2. *For each $i \in \{1, \ldots, n\}$, by removing i from π, we get a block of length 3.*

3. *π has no regents.*

Corollary 6. *The number of permutations in K_n which have no regents is:*

$$\begin{cases} 2^k k! & n = 4k \\ 0 & O.W. \end{cases}$$

We present now some results regarding the Möbious function of K_n. We start with an example depicting the poset of the downward set of the king permutation [5246173]. The circled numbers next to each permutation π is the value of $\mu(\pi)$.



Our main result on the vanishing of the Möbious function on $\bigcup_{n\in\mathbb{N}} K_n$ is:

Theorem 7. *Let $\pi \in K_n$, with $n > 4$. If $[2413] \not\prec \pi$ or $[3142] \not\prec \pi$ then $\mu(\pi) = 0$.*

Definition 8. Another vanishing result is the following:

Let

$$\mathbb{G} = \{[24153], [35142], [42513], [31524]\}.$$

It is easy to see that $\mathbb{G}$ consists of all the elements of K_5 which contain both [2413] and [3142].

Note that in K_5, $\mu(\pi) = 1$ if and only if $\pi \in \mathbb{G}$ (otherwise $\mu(\pi) = 0$).

Theorem 9. *Let $\pi \in K_n$ with $n > 5$ such that there is only one $\sigma \prec \pi$ such that $\sigma \in \mathbb{G}$ and for each $\sigma' \prec \pi$ such that $\sigma \not\prec \sigma'$ we have either σ' avoids [3142], or σ' avoids [2413]. Then in the poset of king permutations $\mu(\pi) = 0$.*

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The number of separators, a new parameter for the symmetric group

Esterella Eisenberg and Moriah Sigron Jerusalem College of Technology, Israel

This talk is based on joint work with Eli Bagno and Shulamit Reches

Introduction

Let S_n be the Symmetric group of n elements. A $2-$*block* or a *bond* in a permutation $\pi = [\pi_1, \ldots, \pi_n] \in S_n$ is a consecutive sub-sequence of π of the form π_i, π_{i+1} such that $|\pi_i - \pi_{i+1}| = 1$. For example, the permutation $\pi = [34521]$ has three bonds: 34 ,45 and 21.

Permutations of S_n which have no bonds are connected to the problem of placing n non-attacking kings in an $n \times n$ chess board. These permutations were counted in [3], and the structure of their containment poset is discussed in a recent paper by the authors of this note [1]. The set of such permutations will be denoted by K_n.

A digit of a permutation, a removal of which produces a **new** bond, will be called a *separator*. (see the formal definition below).

Example 1. In $\pi = [567139482]$ we can omit $\pi_7 = 4$ and after standardizing we get the permutation $[45613872]$ which has the new 2-block 87, so 4 is a separator. the digit 2 is also a separator, since the removal of it creates the permutation $[45612837]$ containing the bond 12. Note that if we remove $\pi_2 = 6$, we get the permutation $[56138472]$ which contains the bond 56 that already exists in π, so 6 is not a separator in π.

In this work we introduce a new parameter: the number of separators, we provide some general information on this parameter, as well as a generating function exhibiting its distribution.

We start with the formal definition:

Definition 2. For $\sigma = [\sigma_1, \ldots, \sigma_n] \in S_n$, we say that σ_i *separates* σ_{j_1} from σ_{j_2} in σ if by omitting σ_i from σ we get a **new** $2-$block. This happens if and only if one of the following cases holds:

1. Vertical separator:j_1, i, j_2 are subsequent numbers and $|\sigma_{j_1} - \sigma_{j_2}| = 1$, i.e.,

$$\sigma = [\ldots, \mathbf{a}, \mathbf{b}, \mathbf{a} \pm \mathbf{1}, \ldots.]$$

2. Horizontal separator: $\sigma_{j_1}, \sigma_i, \sigma_{j_2}$ are subsequent numbers and $|j_1 - j_2| = 1$, i.e.,

$$\sigma = [\ldots, \mathbf{a}, \ldots, \mathbf{a} \pm \mathbf{1}, \mathbf{a} \mp \mathbf{1}, \ldots]$$

or

$$\sigma = [\ldots, \mathbf{a} \pm \mathbf{1}, \mathbf{a} \mp \mathbf{1}, \ldots, \mathbf{a}, \ldots].$$

Definition 3. Let $Sep_I(\pi)$ and $Sep_{II}(\pi)$ be the sets of vertical and horizontal separators π respectively. Let $Sep(\pi) = Sep_I(\pi) \cup Sep_{II}(\pi)$ and $sep(\pi) = |Sep(\pi)|$

Example 4. Let $\sigma = [132465879]$. Then $Sep_I(\sigma) = \{3,2,6,7\}$, and $Sep_{II}(\sigma) = \{3,2,5,8\}$. Thus $Sep(\pi) = \{3,2,5,6,7,8\}$ and $sep(\sigma) = 6$. Note that 7 is a vertical separator, even though 7 is a part of a $2-$block: 87, since by omitting 7 from σ we get a **new** $2-$block: 78.

Remark 5. Several comments are now in order for a permutation $\sigma \in S_n$:

1. Notice the significance of the word 'new' in Definition 2. For example, the identity permutation has plenty of 2 blocks even though it has no separators.

2. The numbers 1 and n can only be vertical separators, while σ_1 and σ_n can only be horizontal separators.

3. If σ_i is a vertical separator in σ then i is a horizontal separator in σ^{-1}. Hence $Sep_I(\sigma) = Sep_{II}(\sigma^{-1})$

4. We have: $Sep_I(\sigma) = Sep_I(\sigma^r)$ and $Sep_{II}(\sigma) = Sep_{II}(\sigma^r)$ where σ^r is the reverse of σ.

Main results

Our first results are related to permutations in K_n.

Theorem 6. *Let $\sigma \in K_n$. Assume that σ_i separates σ_j from some digit and that σ_j separates σ_i from some digit. Then σ_i and σ_j are separators of the same type.*

Theorem 7. *Let $\pi \in K_n$. Then the number of $n-1$ patterns of π which are elements of K_{n-1} is $n - sep(\pi)$ where $sep(\pi)$ is the number of separators in π.*

The number of permutations in S_n that have no separators of any type, is the sequence A137774 from OEIS which counts the number of ways to place n non-attacking empresses on an $n \times n$ chess board.

Our next result is about the opposite case, i.e., the number of permutations, all the digits of which are separators.

Theorem 8. *The number of permutations in S_n which have n different separators,(i.e. every digit is a separator) is:*

$$\begin{cases} 2^k k! & n = 4k \\ 0 & O.W. \end{cases}$$

A generating function for the number of vertical separators

For each $n, k \in \mathbb{N}$ let $s_{n,k}$ be the number of permutations $\pi \in S_n$ with exactly k vertical separators. We want to calculate the generating function: $h(z,u) = \sum_{n\geq 3}\sum_{k=0}^{n-2} s_{n,k} z^n u^k$.

For a permutation $\pi \in S_n$ we denote by π_{odd} and π_{even} the sequences of the digits located in the odd indices and the even indices respectively. Note that (t,s) is a bond of π_{odd} if and only if the element of π_{even} which lies between t and s in π is a vertical separator. Similarly, define separators for bonds in π_{even}.

In order to calculate $h(z,u)$ we used the generating function which counts the number of permutations having a specific number of bonds, which appears in [2], using the inclusion-exclusion principle.

In order to compute the generating function $h(z,u)$, we will use the well known *Hadamar product* of polynomials which is not more than entry-wise multiplication.

Definition 9. Let R be a ring and let $f(x) = \sum_{n\in\mathbb{N}} a_n x^n$, $g(x) = \sum_{n\in\mathbb{N}} b_n x^n \in R[[x]]$ be two power series in x. The *Hadamar product* of $f(x)$ and $g(x)$ is

$$f(x) * g(x) = \sum_{n=0}^{\infty} a_n b_n x^n.$$

Example 10. $(2+3x-4x^2)*(5+x+7x^2) = 10+3x-28x^2$.

Now we claim:

Theorem 11. *Denote $p(z,u) = z^2 + 2z^4(u-1)^2 + 2z^6(u-1)^2 + 2z^8(u-1)^3 + \cdots$. Then the generating function of the vertical separators is*

$$\begin{aligned} h(z,u) &= \sum_{m_o,m_e=0}^{\infty} (m_o+m_e)!(p(z,u))^{m_o} * (p(z,u))^{m_e} \\ &+ \sum_{m_o,m_e=0}^{\infty} (m_o+m_e)!(p(z,u))^{m_o}\tfrac{1}{z} * (p(z,u))^{m_e} z \end{aligned}$$

where $*$ is the Hadamard product in $\mathbb{Q}[[u]][[z]]$.

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Automatic Discovery of Polynomial Time Enumerations

Unnar Freyr Erlendsson Reykjavik University

This talk is based on joint work with Christian Bean, Jay Pantone, Henning Ulfarsson

The TileScope [2] algorithm has proven very successful in enumerating permutation classes with patterns of length 4 and smaller. It relies on finding structure in the permutation class and using that structure to write down a system of equations that can be solved to find the generating function. The systems are typically well behaved, however one can write strategies with more complicated operators such as the *fusion* strategy. This operator adds an extra variable to the system of equations and makes it hard to solve the system.

Using the fusion strategy we demonstrate how to write down recurrence relations from the structure discovered by TileScope resulting in a polynomial time algorithm. This means that TileScope can find polynomial time algorithms for all permutation classes whose basis consists of length 4 patterns, except five principal classes and $\mathrm{Av}(1432, 2143)$.

Although Wilf [3] states that polynomial time algorithms can be viewed as sufficient answers to enumeration problems, a preferable answer is the generating function. An experimental method still under development for solving systems of equations is the *guess-and-check* [1] method. A requirement for this is to have many initial terms for every unknown in the system. Our polynomial time algorithms can often compute upwards of 500 terms.

Structural description to recurrence relation

In Figure 1, we see the structural description obtained by TileScope for $\mathrm{Av}(123)$. We will explain the steps taken to obtain the structural description and then how to translate the description to a recurrence relation.

We start by placing the topmost point in a_0. Either there is no topmost point in which case it is empty (a_1) or it contains a topmost point in which case it is a_2. This corresponds to a disjoint union. From a_2 we place the topmost point in the bottom row, either it is empty (a_5), the topmost point is in first column (a_3), or the topmost point is in the third column (a_4). We can factor out the point from a_3 to obtain a recursion, this corresponds to a Cartesian product on the set level. We can also factor out a point from a_4 but we do not immediately get a recursion as we have not seen this object before. For a_7 we need to *fuse* the first two columns together to obtain a recursion. This can be done because the first two columns correspond to a decreasing sequence of points, but in order to obtain the enumeration afterwards we need to track the number of points in the blue region. Since it is recursing we will need to track the region from the object it is recursing to all throughout the tree.

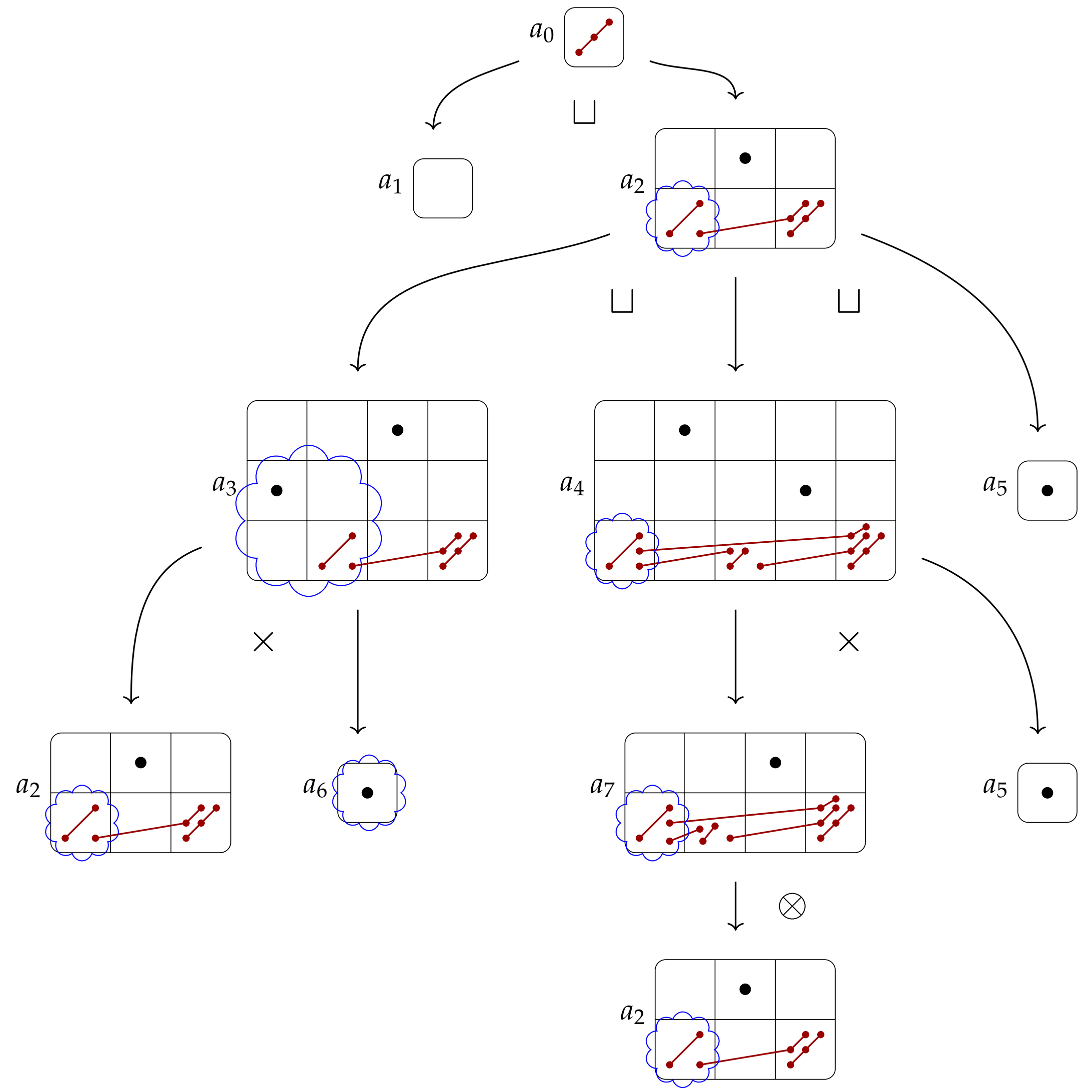

Figure 1: Structural description of Av(123) obtained with the TileScope algorithm. Blue clouds represent regions that need to be tracked because of fusion.

There are three types of recurrence relations in this tree, the first being disjoint union. This is represented as a sum in our recurrence relation. For a_0 we are allowed to have any number of points in the blue region of a_2 so we get

$$a_0(n) = a_1(n) + \sum_{i=0}^{n} a_2(n,i)$$

where i tracks the number of points inside that region. The recurrence relation for a_2 is simply

$$a_2(n,i) = a_3(n,i) + a_4(n,i) + a_5(n,i).$$

The second operator is the Cartesian product. For this operator we need to decide the number of points to place into each object as well as the number of points being placed into each region that is being split. The recurrence relation for a_3 is therefore

$$a_3(n,i) = \sum_{m=0}^{n} \sum_{j=0}^{i} a_2(m,j) \cdot a_6(n-m, i-j)$$

and a_4 is

$$a_4(n,i) = \sum_{m=0}^{n} \sum_{j=0}^{i} a_7(m,j) \cdot a_5(n-m, i-j).$$

The final operator used is fusion, which we represent with $\otimes$. In order to enumerate a_7, we take a (gridded) permutation from a_7 which has i points in the blue region. This corresponds to a permutation in a_2 which has at least i points in its blue region. Therefore, the recurrence relation is

$$a_7(n,i) = \sum_{j=i}^{n} a_2(n,j).$$

Together with the base cases

$$a_1(n) = \begin{cases} 1 \texttt{ if } n = 0 \\ 0 \texttt{ otherwise} \end{cases}$$

$$a_5(n,i) = \begin{cases} 1 \texttt{ if } n = 1 \texttt{ and } i = 0 \\ 0 \texttt{ otherwise} \end{cases}$$

$$a_6(n,i) = \begin{cases} 1 \texttt{ if } n = 1 \texttt{ and } i = 1 \\ 0 \texttt{ otherwise} \end{cases}$$

this describes a recurrence relation for the number of avoiders of 123, namely the Catalan numbers.

This entire procedure for converting structural descriptions to recurrence relations using these operators has been automated. For more advanced strategies like fusion we might not be able to solve the systems of equations directly but we can at least get a polynomial time enumeration.

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On the equidistribution of MAJ and BAST

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This talk is based on joint work with Joanna N. Chen

In 2000, Babson and Steingrímsson [1] introduced the notion of "vincular pattern" and they showed that most of the Mahonian statistics in the literature can be expressed as the sum of vincular pattern functions. For example,

$$\begin{aligned}
\mathrm{INV} &= \underline{21} + 3\underline{12} + 3\underline{21} + 2\underline{31},\\
\mathrm{MAJ} &= \underline{21} + 1\underline{32} + 2\underline{31} + 3\underline{21},\\
\mathrm{STAT} &= \underline{21} + \underline{13}2 + \underline{21}3 + \underline{32}1,\\
\mathrm{BAST} &= \mathrm{STAT}^{\mathrm{rc}} = \underline{21} + 2\underline{13} + 1\underline{32} + 3\underline{21},
\end{aligned}$$

where rc stands for the function composition of the reversal r and the complement c. Given a permutation $p = p_1p_2\cdots p_n \in \mathfrak{S}_n$, recall that

$$\begin{aligned}
r(p_1p_2\cdots p_n) &= p_np_{n-1}\cdots p_1,\\
c(p_1p_2\cdots p_n) &= (n+1-p_1)(n+1-p_2)\cdots(n+1-p_n).
\end{aligned}$$

Moreover, let

$$\begin{aligned}
\mathrm{Des}(p) &= \{i : p_i > p_{i+1}\}, \quad \mathrm{des}(p) = \sum_{j\in\mathrm{Des}(p)} 1,\\
\mathrm{Db}(p) &= \{p_1\} \cup \{p_{i+1} : p_i > p_{i+1}, 1 \le i < n\},\\
\mathrm{Id}(p) &= \mathrm{Des}(p^{-1}), \quad \mathrm{ides}(p) = \sum_{j\in\mathrm{Id}(p)} 1.
\end{aligned}$$

Actually Babson and Steingrímsson have conjectured STAT to be Euler-Mahonian, when associated with "des". Soon later, Foata and Zeilberger [8] confirmed their conjecture. This result has been refined and generalized several times in the literature, making STAT one of the most prolific statistics introduced by Babson and Steingrímsson. For instance, Burstein [3], Chen and Li [6] independently derived joint distribution with other statistics and showed in particular the symmetric equidistribution $(\mathrm{MAJ},\mathrm{STAT}) \sim (\mathrm{STAT},\mathrm{MAJ})$. Kitaev and Vajnovszki [13] generalized this to words, while Fu, Hua and Vajnovszki [9] further generalized to any rearrangement class of words, jointly with set-valued statistics.

In this talk, we continue this theme and present several new equidistributions of MAJ and BAST (not STAT but closely related) over pattern-avoiding subsets, ordered set partitions, as well as ordered multiset partitions. In all three levels, the equidistributions are in fact jointly with other, sometimes set-valued statistics.

Equidistribution on $\mathfrak{S}_n(1\underline{32})$ and set partitions

As observed earlier by Claesson [7, Proposition 3], there is a natural one-to-one correspondence between permutations in $\mathfrak{S}_n(1\underline{32})$ with $k-1$ descents and set partitions of $[n]$ with k blocks. Our first three main results are

Theorem 1. *For $n \geq k \geq 1$, we have*

$$\sum_{\sigma\in\mathfrak{S}_n^{k-1}(1\underline{32})} q^{\mathrm{MAJ}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n^{k-1}(1\underline{32})} q^{\mathrm{BAST}(\sigma)} = S_q(n,k),$$

and

$$\sum_{\sigma\in\mathfrak{S}_n^{k-1}(1\underline{32})} q^{2\underline{13}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n^{k-1}(1\underline{32})} q^{2\underline{31}(\sigma)} = \widetilde{S}_q(n,k),$$

where $S_q(n,k)$ and $\widetilde{S}_q(n,k)$ are the two classic q-Stirling numbers of the second kind studied by Carlitz [4, 5] and Gould [10].

Theorem 2. *Statistics* $(\mathrm{Db}, \mathrm{Id}, \mathrm{MAJ})$ *and* $(\mathrm{Db}, \mathrm{Id}, \mathrm{BAST})$ *have the same joint distribution on $\mathfrak{S}_n(1\underline{32})$ for all $n \geq 1$. (See Table 1 for the case of $n = 4$.)*

Theorem 3. *Statistics* $(\mathrm{Db}, \mathrm{Id}, 2\underline{13})$ *and* $(\mathrm{Db}, \mathrm{Id}, 2\underline{31})$ *have the same joint distribution on $\mathfrak{S}_n(1\underline{32})$ for all $n \geq 1$.*

Equidistribution on ordered set partitions and ordered multiset partitions

We collect here further equidistribution results on ordered set partitions and ordered multiset partitions, undefined notions will be explained in the talk.

Theorem 4. *For $n \geq k \geq 1$, we have*

$$\sum_{w\in\mathcal{URG}(n,k)} q^{\mathrm{bmajMIL}(w)} = \sum_{w\in\mathcal{URG}(n,k)} q^{\mathrm{bmajBAST}(w)} = [k]_q! \cdot S_q(n,k).$$

The interests of considering permutation statistics on ordered multiset partitions stem from the Delta Conjecture (see for example [11, 14, 12, 2]), the Valley Version of which asserts the following combinatorial formula for the quasisymmetric function

$$\Delta'_{e_k} e_n = \mathrm{Val}_{n,k}(x;q,t) := \{z^{n-k-1}\} \left[\sum_{P\in\mathcal{LD}_n} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)} \prod_{i\in\mathrm{Val}(P)} (1 + z/q^{d_i(P)+1}) x^P \right]. \tag{1}$$

In this talk, we will not need the operator Δ'_f, the set $\mathcal{LD}_n$ of labeled Dyck paths, nor the other undefined notations appearing in (1); for details on them see [11]. We will extend both bmajMIL and bmajBAST to ordered multiset partitions and establish

Theorem 5. *For all $n > k \geq 0$, we have*

$$q^{\binom{k+1}{2}} \mathrm{Val}_{n,k}(x;q,0) = q^{\binom{k+1}{2}} \sum_{\beta\models_0 n} \sum_{\mu\in\mathcal{OP}_{\beta,k+1}} q^{\mathrm{inv}(\mu)} x^\beta \tag{2}$$

$$= \sum_{\beta\models_0 n} \sum_{\mu\in\mathcal{OP}_{\beta,k+1}} q^{\mathrm{bmajMIL}(\mu)} x^\beta = \sum_{\beta\models_0 n} \sum_{\mu\in\mathcal{OP}_{\beta,k+1}} q^{\mathrm{bmajBAST}(\mu)} x^\beta. \tag{3}$$

Finally, one key involution ψ we used to prove the above two theorems was originally constructed by us to reveal the following finer relation between BAST and STAT.

Theorem 6. *Statistics* $(\mathrm{F}, \mathrm{E}, \mathrm{des}, \mathrm{Id}, \mathrm{STAT}, \mathrm{BAST})$ *and* $(\mathrm{F}, \mathrm{E}, \mathrm{des}, \mathrm{Id}, \mathrm{BAST}, \mathrm{STAT})$ *have the same joint distribution on* $\mathfrak{S}_n$ *for all* $n \geq 1$.

Most of our proofs are explicit weight-preserving bijections, sometimes they are even involutions.

Table 1: The joint distribution of four statistics on $\mathfrak{S}_4(1\underline{32})$

$\mathfrak{S}_4(1\underline{32})$	Db	Id	MAJ	BAST
1234	{1}	∅	0	0
2134	{1,2}	{1}	1	2
2314	{1,2}	{1}	2	3
2341	{1,2}	{1}	3	1
2413	{1,2}	{1,3}	2	2
3124	{1,3}	{2}	1	2
3412	{1,3}	{2}	2	1
3214	{1,2,3}	{1,2}	3	5
3241	{1,2,3}	{1,2}	4	3
3421	{1,2,3}	{1,2}	5	4
4123	{1,4}	{3}	1	1
4213	{1,2,4}	{1,3}	3	4
4231	{1,2,4}	{1,3}	4	3
4312	{1,3,4}	{2,3}	3	3
4321	{1,2,3,4}	{1,2,3}	6	6

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On pattern-avoiding Fishburn permutations

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This talk is based on joint work with Michael D. Weiner

Motivated by a recent paper by G. Andrews and J. Sellers [1], we became interested in the Fishburn numbers $\xi(n)$, defined by the formal power series

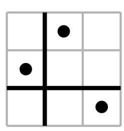
$$\sum_{n=0}^{\infty} \xi(n) q^n = 1 + \sum_{n=1}^{\infty} \prod_{j=1}^{n} (1 - (1-q)^j).$$

They are listed as sequence A022493 in [5] and have several combinatorial interpretations. For example, $\xi(n)$ gives the:

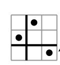
- ▷ number of linearized chord diagrams of degree n,
- ▷ number of unlabeled $(2+2)$-free posets on n elements,
- ▷ number of ascent sequences of length n,
- ▷ number of permutations in S_n that avoid the bivincular pattern $(231, \{1\}, \{1\})$.

For more on these interpretations, see [2] and the references there in.

We are primarily concerned with the aforementioned class of permutations. The fact that they are enumerated by the Fishburn numbers was proved in [2] by Bousquet-Mélou, Claesson, Dukes, and Kitaev, where the authors introduced bivincular patterns and gave a simple bijection to ascent sequences. For instance, the bivincular pattern $(231, \{1\}, \{1\})$ may be visualized by the plot



where bold lines indicate adjacent entries and gray lines indicate an elastic distance between the entries.

We let $\mathscr{F}_n$ denote the class of permutations in S_n that avoid the pattern , and since $|\mathscr{F}_n| = \xi(n)$ (see [2]), we call the elements of $\mathscr{F} = \bigcup_n \mathscr{F}_n$ *Fishburn permutations*. Further, $\mathscr{F}_n(\sigma)$ denotes the class of Fishburn permutations that avoid the pattern σ.

Our goal here is to study $|\mathscr{F}_n(\sigma)|$ for classical patterns of length 3 or 4. First we give a complete picture for Fishburn permutations that avoid a classical pattern of length 3. Then we discuss patterns of length 4, focusing on a Wilf equivalence class of Fishburn permutations that are enumerated by the Catalan numbers C_n. We also prove that $|\mathscr{F}_n(1342)| = \sum_{k=1}^{n} \binom{n-1}{k-1} C_{n-k}$, and conjecture two other equivalence classes.

Avoiding patterns of length 3

Clearly, $\mathsf{Av}_n(231) \subset \mathscr{F}_n$. Since every Fishburn permutation that avoids the classical pattern 231 is contained in the set of regular 231-avoiding permutations, we get

$$\mathscr{F}_n(231) = \mathsf{Av}_n(231), \text{ and so } |\mathscr{F}_n(231)| = C_n,$$

where C_n denotes the Catalan number $\frac{1}{n+1}\binom{2n}{n}$. Enumeration of the Fishburn permutations that avoid the other five classical patterns of length 3 is less obvious.

Theorem 1. *For $\sigma \in \{123, 132, 213, 312\}$, we have $|\mathscr{F}_n(\sigma)| = 2^{n-1}$.*

Theorem 2. *The set $\mathscr{F}_n(321)$ is in bijection with the set of Dyck paths of semilength n that avoid the subpath* UUDU. *Therefore, by [4, Prop. 5] we have*

$$|\mathscr{F}_n(321)| = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^j}{n-j}\binom{n-j}{j}\binom{2n-3j}{n-j+1}.$$

This is sequence [5, A105633].

This is a consequence of Krattenthaler's bijection between permutations in $\mathsf{Av}_n(321)$ and Dyck paths of semilength n, via the left-to-right maxima.

Summary:

Pattern σ	$\lvert\mathscr{F}_n(\sigma)\rvert$	OEIS
123, 132, 213, 312	1, 2, 4, 8, 16, 32, 64, 128, 256, 512, ...	A000079
231	1, 2, 5, 14, 42, 132, 429, 1430, 4862, ...	A000108
321	1, 2, 4, 9, 22, 57, 154, 429, 1223, 3550, ...	A105633

σ-avoiding Fishburn permutations.

Avoiding patterns of length 4

Regarding Fishburn permutations that avoid a pattern of length 4, there are at least 13 Wilf equivalence classes: 10 classes with a single pattern, a class with eight patterns enumerated by the Catalan numbers, and for the remaining 6 patterns, we have the following conjectures.

Conjecture 3. $\mathscr{F}_n(2413) \sim \mathscr{F}_n(2431) \sim \mathscr{F}_n(3241)$, *enumerated by* $1, 2, 5, 15, 52, 201, \ldots$.

Conjecture 4. $\mathscr{F}_n(3214) \sim \mathscr{F}_n(4132) \sim \mathscr{F}_n(4213)$, *enumerated by* $1, 2, 5, 14, 43, 143, \ldots$.

We can prove that $\mathscr{F}_n(1342)$ is enumerated by [5, A007317].

Theorem 5.

$$|\mathscr{F}_n(1342)| = \sum_{k=1}^{n}\binom{n-1}{k-1}C_{n-k}.$$

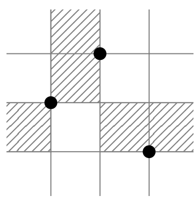
However, our focus will be on the enumeration of the Catalan equivalence class

Pattern σ	$\lvert\mathscr{F}_n(\sigma)\rvert$	OEIS
1234, 1243, 1324, 1423, 2134, 2143, 3124, 3142	1, 2, 5, 14, 42, 132, 429, 1430, 4862, ...	A000108

Theorem 6. *We have* $\mathscr{F}_n(3142) = \mathscr{F}_n(231)$, *hence* $\lvert\mathscr{F}_n(3142)\rvert = C_n$.

Proof. Since 3142 contains the pattern 231, we have $\mathscr{F}_n(231) \subseteq \mathscr{F}_n(3142)$.

To prove the reverse inclusion, suppose there exists $\pi \in \mathscr{F}_n(3142)$ such that π contains the pattern 231. Let $i < j < k$ be the positions of the most-left closest 231 pattern contained in π. In other words, assume the plot of π is of the form



where no elements of π may occur in the shaded regions. It follows that, if ℓ is the position of $\pi(k)+1$, then $i \leq \ell < j$. But this is not possible since, $\pi(\ell) < \pi(\ell+1)$ violates the Fishburn condition, and $\pi(\ell) > \pi(\ell+1)$ implies $\pi(k) > \pi(\ell+1)$ which forces the existence of a 3142 pattern. In conclusion, no permutation $\pi \in \mathscr{F}_n(3142)$ is allowed to contain a 231 pattern. Therefore, $\mathscr{F}_n(3142) \subseteq \mathscr{F}_n(231)$. $\square$

Theorem 7. $\mathscr{F}_n(1234) \sim \mathscr{F}_n(1243) \sim \mathscr{F}_n(2143) \sim \mathscr{F}_n(2134)$.

The first and third equivalence relations can be shown using a bijection

$$\phi : \mathsf{Av}_n(\tau \oplus 12) \to \mathsf{Av}_n(\tau \oplus 21)$$

given by West in [6]. A similar bijective map

$$\psi : \mathsf{Av}_n(12 \oplus \tau) \to \mathsf{Av}_n(21 \oplus \tau),$$

allows us to show that $\mathscr{F}_n(1234) \sim \mathscr{F}_n(2134)$ and $\mathscr{F}_n(1243) \sim \mathscr{F}_n(2143)$.

Finally, using several related bijections one can prove:

Theorem 8. $\mathscr{F}_n(1423) \sim \mathscr{F}_n(1243)$.

Theorem 9. $\mathscr{F}_n(3142) \sim \mathscr{F}_n(3124) \sim \mathscr{F}_n(1324)$.

Theorem 10. $\mathscr{F}_n(3142) \sim \mathscr{F}_n(2143)$.

Further remarks

Some classes of Fishburn permutations that avoid a pattern of length 4 appear to be in bijection with certain pattern avoiding ascent sequences. It would be interesting to provide explicit bijections.

Regarding *indecomposable* Fishburn permutations, we have results for avoiders of any pattern of length 3 (see table below) and for some patterns of length 4. For the latter, our limited preliminary data suggests the existence of 19 Wilf equivalence classes. We are particularly curious about $\mathscr{F}_n^{\mathsf{ind}}(1342)$ as it appears to be equinumerous with the set $\mathsf{Av}_{n-1}(2413, 3421)$, cf. [5, A165538].

Pattern σ	$\lvert\mathscr{F}_n^{\mathsf{ind}}(\sigma)\rvert$	OEIS
123	1, 1, 2, 5, 12, 27, 58, 121, 248, 503, ...	A000325
132, 213	1, 1, 2, 4, 8, 16, 32, 64, 128, 256, ...	A000079
231	1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, ...	A000108
312	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, ...	A000012
321	1, 1, 1, 2, 5, 13, 35, 97, 275, 794, ...	A082582

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Permutation patterns: gamma-positivity and (-1)-phenomenon

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This talk is based on joint work with Shishuo Fu, Dazhao Tang, Jiang Zeng

Introduction

A polynomial $h(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{R}[x]$ is called *palindromic* if $a_i = a_{n-i}$ for $0 \le i \le n$. As the vector space of all palindromic polynomials of degree no greater than n has the γ-basis $\{x^i(1+x)^{n-2i}\}_{i=0}^{\lfloor n/2 \rfloor}$, any palindromic polynomial $h(x)$ admits the *γ-expansion*

$$h(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^i (1+x)^{n-2i},$$

and the sequence $\{\gamma_k\}_{k=0}^{n/2}$ is called the *γ-vector* of h. Having a nonnegative γ-vector leads to the unimodality of the coefficient sequence $\{a_k\}_{k=0}^{n}$ directly. we say that h is *γ-positive* if the γ-vector of h is nonnegative.

Gamma positivity polynomials arise naturally in some combinatorial sequence. For example, the Narayana polynomials $N_n(t)$ are γ-positive (cf. [2, Section 4.3]) and have the following γ-expansion for the Narayana polynomial

$$N_n(t) := \sum_{\pi \in \mathfrak{S}_n(231)} t^{\mathsf{des}\,\pi} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}^{N} t^k (1+t)^{n-1-2k}. \tag{1}$$

where

$$\gamma_{n,k}^{N} = \#\{\pi \in \mathfrak{S}_n(231) : \mathsf{dd}\,\pi = 0, \mathsf{des}\,\pi = k\}. \tag{2}$$

By taking $t = -1$ in (1), we can recover the following (-1)*-phenomenon* involving the Catalan number C_n:

$$\sum_{\pi \in \mathfrak{S}_n(231)} (-1)^{\mathsf{des}\,\pi} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-1)^{\frac{n-1}{2}} C_{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases} \tag{3}$$

In this talk, we first prove several new interpretations of a kind of q-Narayana polynomials along with their corresponding γ-expansions using pattern avoiding permutations, and consider such (-1)-phenomenon for these subsets as well. Moreover, we enumerate the alternating permutations avoiding simultaneously two patterns, namely $(2413, 3142)$ and $(1342, 2431)$, of length four.

Part I: Catalan families

We define the q-Narayana polynomials $N_n(t,q)$ as

$$N_n(t,q) = \sum_{\pi \in \mathfrak{S}_n(321)} t^{\mathsf{exc}\,\pi} q^{\mathsf{inv}-\mathsf{exc}\,\pi}.$$

We can state our main contribution:

Theorem 1. *The q-Narayana polynomials $N_n(t,q)$ also have the following ten interpretations*

$$N_n(t,q) = \sum_{\pi \in \mathfrak{S}_n(\tau)} t^{\mathsf{des}\,\pi} q^{\mathsf{stat}\,\pi},$$

with τ being a pattern of length 3, and stat being a permutation statistic. Ten choices for the pair (τ, stat) are listed in Table 1.

Table 1: Ten choices for (τ, stat)

#	τ	stat	#	τ	stat
1	231	13-2	6	132	2-31
2	231	adi*	7	231	31-2
3	312	2-13	8	312	2-31
4	312	adi	9	213	13-2
5	213	31-2	10	132	2-13

Theorem 2. *For $n \geq 1$, the following γ-expansions formula holds true*

$$N_n(t,q) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}(q) t^k (1+t)^{n-1-2k}, \tag{4}$$

where

$$\gamma_{n,k}(q) = \sum_{\pi \in \widetilde{\mathfrak{S}}_{n,k}(321)} q^{\mathsf{inv}\,\pi - \mathsf{exc}\,\pi} \tag{5}$$

$$= \sum_{\pi \in \widetilde{\mathfrak{S}}_{n,k}(213)} q^{(31\text{-}2)\,\pi} = \sum_{\pi \in \widetilde{\mathfrak{S}}_{n,k}(312)} q^{(2\text{-}13)\,\pi} \tag{6}$$

$$= \sum_{\pi \in \widetilde{\mathfrak{S}}_{n,k}(132)} q^{(2\text{-}31)\,\pi} = \sum_{\pi \in \widetilde{\mathfrak{S}}_{n,k}(231)} q^{(13\text{-}2)\,\pi}. \tag{7}$$

Remark 3.

1. When $q = 1$, equation (4) reduce to a classical result of Narayana polynomials [2, Section 4.3].
2. The main tool of the proof of Theorem 1 is the continued fraction method. The proof of Theorem 2 is based on the so-called *Modified Foata-Strehl action*. The definitions of permutation statistics.

Part II: alternating permutations avoiding two patterns of length four

A permutation is said to be alternating (or up-down) if it starts with an ascent and then descents and ascents come in turn. We denote by $\mathfrak{A}_n$ the set of alternating permutations of length n, and by $\mathfrak{A}_n(p_1, p_2, \cdots, p_m)$ the set of alternating permutations of length n that avoid patterns $p_1, p_2, \ldots, p_m$.

Theorem 4. *Let $r_n := |\mathfrak{A}_{2n+1}(2413, 3142)|$, $n \geq 0$, $R(x) := \sum_{n=1}^{\infty} r_n x^n$, then*

$$R(x) = x(R(x)+1)^2 + x(R(x)+1)^3. \tag{8}$$

Consequently, $r_0 = 1$ and for $n \geq 1$,

$$r_n = \frac{2}{n} \sum_{i=0}^{n-1} 2^i \binom{2n}{i} \binom{n}{i+1}. \tag{9}$$

Theorem 5. *Let $t_n := |\mathfrak{A}_{2n}(2413, 3142)|$, $n \geq 1$, $T(x) := \sum_{n=1}^{\infty} t_n x^n$, then*

$$\frac{1}{2} R(x) = \frac{1}{2} R(x) \cdot T(x) + T(x). \tag{10}$$

Consequently, $t_1 = 1$ and for $n \geq 2$,

$$t_n = \frac{4}{n-1} \sum_{i=0}^{n-2} 2^i \binom{2n-1}{i} \binom{n-1}{i+1}. \tag{11}$$

Theorem 6. *Let $u_n := |\mathfrak{A}_{2n+1}(1342, 2431)|$ and $U(x) := \sum_{n=0}^{\infty} u_n x^n$, then*

$$U(x) = \frac{\sqrt{1-4x}}{\sqrt{1-4x} - 2x} = \cfrac{1}{1 - \cfrac{2x}{1 - \cfrac{2x}{1 - \cfrac{x}{1 - \cfrac{x}{\ddots}}}}}. \tag{12}$$

Remark 7. The proof of all the theorems can be found in the article [1].

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Hung P. Hoang ETH Zurich

This talk is based on joint work with Elizabeth Hartung, Torsten Mütze, and Aaron Williams

In this work we propose an algorithmic framework for exhaustively generating different classes of pattern-avoiding permutations, generalizing the well-known Steinhaus-Johnson-Trotter algorithm. We present a simple greedy algorithm that works for a broad class of classical patterns, vincular patterns, multiset patterns, as well as for conjunctions and disjunctions of those, and for the more general setting of bounding the number of occurences of a pattern. We thus obtain Gray codes for various pattern-avoiding permutations, including vexillary, skew-merged, X-shaped, separable, Baxter and twisted Baxter permutations, etc. Using standard bijections, these Gray codes also translate into Gray codes for other combinatorial objects, in particular for rectangulations, which are divisions of a square into n rectangles; see Figure 1.

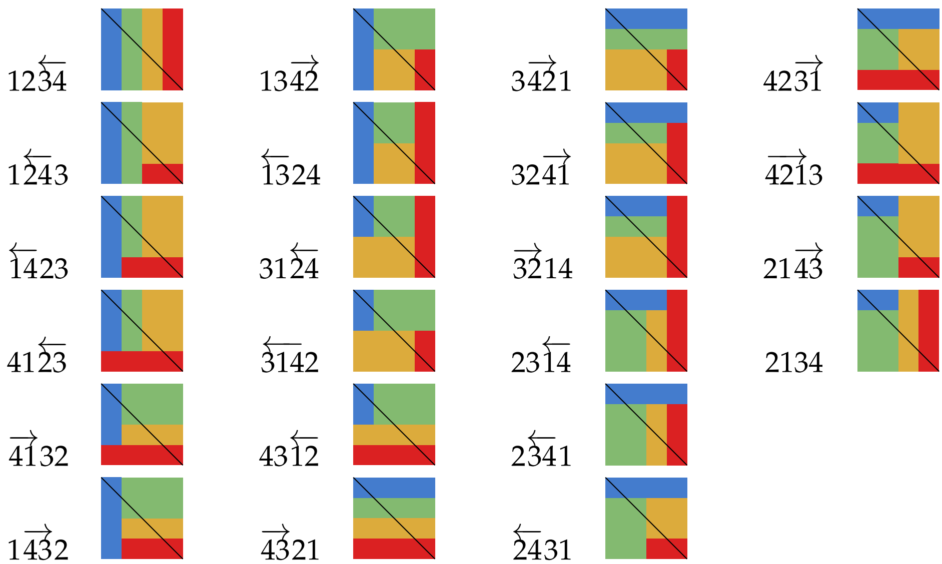

Figure 1: Twisted Baxter permutations ($2\underline{41}3 \wedge 3\underline{41}2$-avoiding) of length $n = 4$ generated by our algorithm and resulting Gray code for diagonal rectangulations. The arrows indicate the substring rotations that create the next permutation.

Introduction The Steinhaus-Johnson-Trotter order, also known as plain change order, is a well-known Gray code that orders all $n!$ permutations of $\{1, \ldots, n\}$, so that any two consecutive permutations differ in an adjacent transposition [6, 12]. In this work we propose an algorithmic framework for exhaustively generating different classes of pattern-avoiding permutations, generalizing the Steinhaus-Johnson-Trotter algorithm. Specifically, in our Gray codes successive permutations differ by a *jump*, which means that one value of the permutation is moved to the left or right over some number of smaller values, which corresponds to a cyclic substring rotation. Furthermore, the Gray codes only use *minimal* jumps, meaning that a shorter jump of the same value

Table 1: Permutation patterns and corresponding combinatorial objects and orderings generated by our algorithm.

Patterns	Combinatorial objects and ordering	References/OEIS
none	permutations by adjacent transpositions $\rightarrow$ plain change order	A000142 [6, 12]
$231 = 2\underline{31}$	*Catalan families* • binary trees by rotations • triangulations by edge flips • Dyck paths by hill flips	A000108 [9]
$\underline{23}1$	set partitions by exchanges $\rightarrow$ Kaye's Gray code order	A000110 [7]
$132 \wedge 231 = \underline{13}2 \wedge 2\underline{31}$: permutations without peaks	binary strings by bitflips $\rightarrow$ reflected Gray code order (BRGC)	A000079 [5]
2143: vexillary permutations		[8], A005802
conjunction of v_k tame patterns with $v_2 = 35, v_3 = 91, v_4 = 2346$: k-vexillary permutations $(k \geq 1)$		[3], A224318, A223034, A223905
$2143 \wedge 3412$: skew-merged permutations		[11], A029759
$2143 \wedge 2413 \wedge 3142 \wedge 3412$: X-shaped permutations		[4], A006012
$2413 \wedge 3142$ separable permutations	slicing floorplans (=guillotine partitions) by flips	[1], A006318
$2\underline{41}3 \wedge 3\underline{14}2$: Baxter $2\underline{41}3 \wedge 3\underline{41}2$: twisted Baxter $2\underline{14}3 \wedge 3\underline{14}2$	mosaic floorplans (=diagonal rectangulations=R-equivalent rectangulations) by flips	[1], A001181
$2\underline{14}3 \wedge 3\underline{41}2$	S-equivalent rectangulations by flips	[2], A214358
$2\underline{14}3 \wedge 3\underline{41}2 \wedge 2413 \wedge 3142$	S-equivalent guillotine rect. by flips	[2], A078482
$3\underline{51}24 \wedge 3\underline{51}42 \wedge 24\underline{51}3 \wedge 42\underline{51}3$: 2-clumped permutations	generic rectangulations (=rectangular drawings) by flips and wall slides	[10] (not in OEIS)
conjunction of c_k tame patterns with $c_k = 2(k/2)!(k/2+1)!$ for k even and $c_k = 2((k+1)/2)!^2$ for k odd: k-clumped permutations		[10] (not in OEIS)

would create the forbidden pattern. For example, when considering 231-avoiding permutations, the permutation 42135 could be followed by 21435. This is because the value 4 jumps to the right over two smaller values and a shorter jump would produce the forbidden permutation 24135.

Overview of our results We propose the following simple greedy algorithm to generate each permutation from the previous one: Perform a minimal jump with the largest possible value that yields a new permutation in the list. We show that this algorithm generates the Johnson-Trotter-Steinhaus order as well several classes of pattern-avoiding permutations. These patterns include any classical pattern in which the largest value is not in the first or last position, and any vincular pattern in which the only pair of adjacent entries includes the largest value. For example, our algorithm generates all 231-avoiding permutations of any fixed length n, as well as all $2\underline{31}$-avoiding permutations, where the underlining denotes a vincular pair. Table 1 lists several more examples of patterns for which our algorithm solves the exhaustive generation problem successfully. The algorithm also works for additional types of patterns that are given by conjunctions such as $132 \wedge 231$, meaning that both patterns must be avoided, and disjunctions such as $132 \vee 231$, meaning that at least one of the patterns must be avoided. We can also form more complicated propositional formulas of patterns connected by $\wedge$ and $\vee$ in this way. In the most general setting, we can prescribe any upper bound for the number of occurences of a pattern in the formula, where the case of zero occurences is pattern-avoidance. The algorithm also works for multiset patterns such as $121 = 132 \wedge 231$, which in this particular case generates all permutations without peaks. A a result, we obtain minimal jump Gray codes for various pattern-avoiding permutations, including vexillary, skew-merged, X-shaped, separable, Baxter and twisted Baxter permutations.

Moreover, the jump orderings of permutations generated by our algorithm translate into Gray codes for other combinatorial objects that are in ono-to-one correspondence to permutations. These objects and Gray codes are shown in Table 1, and the bijections are explained in the listed papers. In this way, our algorithm provides a unified description of four known classical Gray codes, namely the aforementioned Steinhaus-Johnson-Trotter algorithm, the binary reflected Gray code [5], and two Gray codes for binary trees [9] and set partitions [7]. Moreover, we obtain new Gray codes for five different classes of rectangulations, also known as floorplans, which are divisions of a square into n rectangles, subject to different conditions; see Figure 1.

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Catalan words avoiding pairs of length three patterns

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This talk is based on joint work with Jean-Luc Baril and Vincent Vajnovszki

Catalans words are particular growth-restricted words over the set of non-negative integers: the word $\boldsymbol{w} = w_1 w_2 \ldots w_n$ is called Catalan word if $w_1 = 0$ and $0 \leq w_i \leq w_{i-1} + 1$ for $i = 2, 3, \ldots, n$. We denote by $\mathcal{C}_n$ the set of length n Catalan words, and $|\mathcal{C}_n|$ is the nth Catalan number, see for instance [4, exercise 6.19.u, p. 222].

A pattern $\pi = p_1 p_2 \ldots p_k$ is said to be contained in the word $\boldsymbol{w}$ if there is a subsequence of $\boldsymbol{w}$, $w_{i_1} w_{i_2} \ldots w_{i_k}$, order-isomorphic with $p_1 p_2 \ldots p_k$. If $\boldsymbol{w}$ does not contain π, we say that $\boldsymbol{w}$ avoids π. If π is a pattern (a set of patterns), $\mathcal{C}_n(\pi)$ denotes the words in $\mathcal{C}_n$ avoiding π (each pattern in π), and $c_n(\pi) = |\mathcal{C}_n(\pi)|$.

Sequel of the work initiated in [1] where, among other things, Catalan words avoiding a length three pattern are enumerated, in this article we almost complete the enumeration of Catalan words avoiding *a pair* of length three patterns (the remaining difficult case is left as open problem) and obtain the Wilf classification of these pairs of patterns. Some of the resulting enumerating sequences are not yet recorded in [3] .

Our methods include structural characterization, recurrence relations, constructive bijections and (bivariate) generating functions. For some pairs of patterns we give the descent distribution and popularity on the set of Catalan words avoiding these patterns.

At the end of this introductory part, we recall results from [1] summarized in Table 1.

Pattern π	Sequence $c_n(\pi)$	Generating function	OEIS
012, 001, 010	2^{n-1}	$\frac{1-x}{1-2x}$	A011782
021	$(n-1)\cdot 2^{n-2}+1$	$\frac{1-4x+5x^2-x^3}{(1-x)(1-2x)^2}$	A005183
102, 201	$\frac{3^{n-1}+1}{2}$	$\frac{1-3x+x^2}{(1-x)(1-3x)}$	A007051
120, 101	F_{2n-1}	$\frac{1-2x}{1-3x+x^2}$	A001519
011	$\frac{n(n-1)}{2}+1$	$\frac{1-2x+2x^2}{(1-x)^3}$	A000124
000	–	$\frac{1-2x^2}{1-x-3x^2+x^3}$	–
100	$\left\lceil\frac{(1+\sqrt{3})^{n+1}}{12}\right\rceil$	$\frac{1-2x-x^2+x^3}{1-3x+2x^3}$	A057960
110	$\frac{1}{2}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n+1}{2k+1}2^k-\frac{n-1}{2}$	$\frac{1-3x+2x^2+x^3}{(1-x)^2(1-2x-x^2)}$	–
210	–	$\frac{1-5x+7x^2-x^3-x^4}{(1-2x)(1-4x+3x^2+x^3)}$	–

Table 1: The cardinality of $\mathcal{C}_n(\pi)$ for each pattern π of length three.

Avoiding a length two and a length three pattern

Proposition For $n \geq 3$ we have:

- $c_n(00,\sigma) = \begin{cases} 0 & \text{if } \sigma = 012, \\ 1 & \text{otherwise.} \end{cases}$
- $c_n(01,\sigma) = \begin{cases} 0 & \text{if } \sigma = 000, \\ 1 & \text{otherwise.} \end{cases}$
- $c_n(10,\sigma) = \begin{cases} F_{n+1} & \text{if } \sigma = 000, \\ n & \text{if } \sigma \in \{001, 011, 012\}, \\ 2^{n-1} & \text{otherwise.} \end{cases}$

Avoiding two length three patterns

Our enumerating results for Catalan words avoiding a pairs of patterns of length three are encompassed in Table 2. The enumeration is given either by a closed expression or by a (bivariate) generating function according to the length (and number of descents). As a byproduct, we give the descent distribution and the descent popularity on the set of Catalan words avoiding some of these pairs of patterns.

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$\pi \setminus \sigma$	000	001	010	100	011	110	101	012	021	102	201	120	210
000	Table 1	F_{n+1}	F_{n+1}	$c_n(000)$	1,2,3,3,3	NEW	2^{n-1}	1,2,3,3,0,0	NEW	NEW	NEW	NEW	NEW
001	-	2^{n-1}	n	$F_n - 1$	n	$\frac{n(n-1)}{2}+1$	2^{n-1}	n	$\frac{n(n-1)}{2}+1$	2^{n-1}	2^{n-1}	$\frac{n(n-1)}{2}+1$	$\binom{n+1}{3}+n+1$
010	-	-	2^{n-1}	2^{n-1}	n	2^{n-1}	2^{n-1}	n	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}
100	-	-	-	A057960	$2(n+1)$	NEW	Pell number	$\frac{n(n-1)}{2}+1$	$2^n - n$	?	F_{2n-1}	A034943	A267905
011	-	-	-	-	$\frac{n(n-1)}{2}+1$	$\frac{n(n-1)}{2}+1$	$\frac{n(n-1)}{2}+1$	n	$\frac{n(n-1)}{2}+1$	$\frac{n(n-1)}{2}+1$	$\frac{n(n-1)}{2}+1$	$2(n-1)$	$\frac{n(n-1)}{2}+1$
110	-	-	-	-	-	Table 1	$2^n - n$	$\frac{n(n-1)}{2}+1$	NEW	A116702	NEW	A034943	NEW
101	-	-	-	-	-	-	F_{2n-1}	$\frac{n(n-1)}{2}+1$	$2^n - n$	F_{2n-1}	F_{2n-1}	$2^n - n$	$(n-1)2^{n-2}+1$
012	-	-	-	-	-	-	-	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}
021	-	-	-	-	-	-	-	-	$(n-1)2^{n-2}+1$	A116702	$(n-1)2^{n-2}+1$	A045623	$(n-1)2^{n-2}+1$
102	-	-	-	-	-	-	-	-	-	$\frac{3^{n-1}+1}{2}$	NEW	$(n-1)2^{n-2}+1$	NEW
201	-	-	-	-	-	-	-	-	-	-	$\frac{3^{n-1}+1}{2}$	F_{2n-1}	NEW
120	-	-	-	-	-	-	-	-	-	-	-	F_{2n-1}	F_{2n-1}
210	-	-	-	-	-	-	-	-	-	-	-	-	Table 1

Table 2: Number of Catalan words avoiding two pattern of length three.

Pattern-Avoiding Fillings of Skew Shapes

Vít Jelínek Charles University, Prague

This talk is based on joint work with Mark Karpilovskij

A *polyomino* is a finite collection of unit squares (called *cells*) in the plane whose vertices have integer coordinates. A *filling* of a polyomino is a mapping assigning a value 0 or 1 to each cell of the polyomino. A *transversal filling* (or just a *transversal*) is a filling with exactly one 1-cell in each row and column. With this terminology, a permutation diagram can be seen as a transversal of a square-shaped polyomino. For consistency with permutation diagrams, we shall number the rows of polyominoes from bottom to top, starting with the first (i.e., lowest) non-empty row, and columns are numbered from left to right. We will identify a permutation with its diagram, and view diagrams as special cases of fillings.

To *delete a row* from a polyomino (or a filling of a polyomino) means to remove all the cells from that row and then move all the cells above the deleted row downwards by one step to close the gap. Column deletion is defined analogously. We say that a polyomino P contains a polyomino Q if Q can be obtained from P by a sequence of row deletions and column deletions. An *erasure* in a filling is an operation that changes a 1-cell into a 0-cell. We say that a filling F contains a filling G, if G can be obtained from F by a sequence of row deletions, column deletions and erasures. Note that if F and G are permutation diagrams, then F contains G if and only if the permutation represented by F contains the permutation represented by G in the sense of the classical Wilf containment.

For our purposes we will only be interested in a special type of polyominoes known as skew shapes. A *skew shape* is a polyomino whose boundary is a union of two internally disjoint lattice paths consisting of up-steps and right-steps; see Figure 1. Clearly, a square is also a skew shape, so permutation diagrams are special cases of transversals of skew shapes.

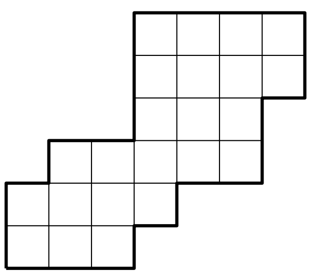


Figure 1: A skew shape

We say that a permutation π is *more restrictive for permutations* than a permutation σ, written $\pi \leqslant_{\mathrm{p}} \sigma$, if the number of π-avoiding permutations of size n is no greater than the number of σ avoiding permutations of size n, i.e., if $|\mathsf{Av}_n(\pi)| \leq |\mathsf{Av}_n(\sigma)|$. Similarly, we say that a permutation π is *more restrictive for skew transversals* than a permutation

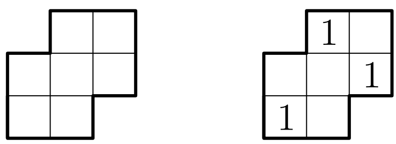
σ, written $\pi \leqslant_{\mathrm{s}} \sigma$, if for every skew shape S, the number of π-avoiding transversals of S is no greater than the number of those that avoid σ. We will extend $\leqslant_{\mathrm{p}}$ and $\leqslant_{\mathrm{s}}$ to sets of patterns in a straightforward way.

The partial order $\leqslant_{\mathrm{p}}$ is known as the *Wilf order* of permutation patterns. Since a permutation diagram is a special case of a skew transversal, we observe that $\pi \leqslant_{\mathrm{s}} \sigma$ implies $\pi \leqslant_{\mathrm{p}} \sigma$. However, our main motivation for the study of pattern avoidance in skew shapes stems from the following easy fact, which is based on standard arguments on patterns in polyominoes (see, e.g., Backelin et al. [1]).

Fact 1. *If $\pi \leqslant_s \sigma$ for some permutations π and σ, then for any two permutations α and β, we also have $\alpha \ominus \pi \ominus \beta \leqslant_s \alpha \ominus \sigma \ominus \beta$, and consequently $\alpha \ominus \pi \ominus \beta \leqslant_p \alpha \ominus \sigma \ominus \beta$.*

To make Fact 1 useful, we need nontrivial examples of patterns comparable in the $\leqslant_{\mathrm{s}}$ order. Unfortunately, the only known such examples are based on the following fact, which can be derived from the results of Burstein and Pantone [2, Lemmas 1.4 and 1.5] or Jelínek [3, Lemmas 29 and 30].

Fact 2. *Let F_0 be the skew filling from Figure 2. For each skew shape S, its number of 21-avoiding transversals is equal to its number of $\{12, F_0\}$-avoiding transversals. It follows that $21 \leqslant_s 12$, and consequently, for any α and β, $\alpha \ominus 21 \ominus \beta \leqslant_p \alpha \ominus 12 \ominus \beta$.*



S_0 F_0

Figure 2: The skew shape S_0 (left) and its filling F_0 (right). The empty cells of the filling correspond to zeros.

Computer-generated data suggest that the inequality $21 \leqslant_{\mathrm{s}} 12$ might be a special case of a more general result, which we state here as a conjecture (see [3, Conjecture 26]).

Conjecture 3. *Let $\iota_k = 12 \cdots k$ be the identity permutation of size k, and let $\delta_k = k \cdots 21$ be the decreasing permutation of size k. Then $\delta_k \leqslant_s \iota_k$.*

Our results

We obtained several results that either strengthen Fact 2 or establish special cases of Conjecture 3.

Our first result shows that the Wilf order relationship $\alpha \ominus 21 \ominus \beta \leqslant_{\mathrm{p}} \alpha \ominus 12 \ominus \beta$ can be strengthened to a Wilf equivalence result.

Theorem 4. *For any pair of nonempty permutations α and β, there is a finite set X of permutations, such that the permutation $\alpha \ominus 21 \ominus \beta$ is Wilf-equivalent to the set of permutations $X \cup \{\alpha \ominus 12 \ominus \beta\}$.*

We remark that a special case of this result has been proven by Burstein and Pantone [2], who showed that 4321 is Wilf-equivalent to $\{4231, 5276143\}$.

Another of our results extends the equivalence between 21 an $\{12, F_0\}$ from Fact 2 to non-transversal fillings.

Theorem 5. *Let S be a skew shape. Let $S(21)$ be the set of (not necessarily transversal) 21-avoiding fillings of S, and similarly let $S(12, F_0)$ be the set of $\{12, F_0\}$-avoiding fillings of S. Then $|S(21)| = |S(12, F_0)|$.*

We note that our proof of Theorem 5 is different from the known proofs of Fact 2, which do not seem to generalize to non-transversal fillings.

Our last theorem offers a partial support for Conjecture 3, as well as a generalization of known results of Backelin, West and Xin [1] for Ferrers diagrams.

Theorem 6. *Let S be a skew shape that does not contain the shape S_0 from Figure 2, and let $k \geq 1$ be an integer. With ι_k and δ_k as in Conjecture 3, the number of ι_k-avoiding transversals of S is equal to the number of δ_k-avoiding transversals of S.*

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On a New Parameter of Permutations Arising in a Context of Testing for Forbidden Patterns

Gil Laufer Department of Computer Science, University of Haifa, Israel

This talk is based on joint work with Yuri Rabinovich

The algorithmic problem of checking whether a given real-valued sequence $f = (f(1), f(2), \ldots, f(n))$ contains a fixed order-pattern π of length k (usually thought of as a permutation) has a long history, and is of considerable interest. A break-through sophisticated algorithm of Guillemot and Marx [2] provides a solution for this problem in a time linear in n, namely $2^{O(k^2 \log k)} \cdot n$.

More recently, this problem was studied in the framework of Property Testing, a novel and rapidly developing direction in both theoretical and practical CS. Newman et al. [3] posed, among other things, the followed problem (we discuss here only what is professionally called the one-sided error nonadaptive version, with a fixed ϵ, say $\epsilon = 0.1$, and with a particular distance function defined below).

Call a sequence f *π-free* if f does not contain π as a subpattern. That is, f does contain a π-copy, i.e., a subset of indices $(i_1, i_2, \cdots, i_k)$ ordered in increasing order, such that $f(i_x) > f(i_y)$ whenever $\pi(x) > \pi(y)$. Call a sequence f *π-abundant*, or *ϵ-far from being π-free*, if no matter how one alters ϵn arbitrarily chosen values of f, the resulting f' will still contain a π-copy.

Testing for π-freeness means separating between the π-free and the π-abundant sequences f. More concretely, it is a randomized procedure that produces a (small) sample $S \subset \{1, 2, \ldots, n\}$, such that for any π-abundant f, the probability that $f|_S$, the restriction of f on S, is *not* π-free is > 0.999. The *query complexity* of this procedure is the size of S as a function of n. And the question is: *What is the query complexity of testing for π-freeness, i.e., the query complexity of the best possible procedure for doing it?*

One of the main results of Newman et al. [3] was to show that, on one hand, the query complexity of testing for a (any) *monotone* π is $\tilde{\Theta}_\epsilon(1)$, while on the other hand, for any non-monotone permutation it is $\tilde{\Omega}_\epsilon(n^{0.5})$. (The notation hides the multiplicative terms polynomial in $\log n$ and ϵ^{-1}). The latter bound is tight, e.g., for $\pi = (132)$. A *trivial* upper bound for any π of length k is $O_\epsilon(n^{1-1/k})$. Newman et al. [3] have also constructed infinite families of permutations with query complexity $\Omega_\epsilon(n^{1-2/k+1})$.

The study of Newman et al. [3] was continued and greatly extended by Ben-Eliezer and Canonne [1]. In particular, they have established an upper bound of $O_\epsilon(n^{1-1/k-1})$ for any π of length k, and constructed infinite families of permutations with query complexity $O_\epsilon(n^{1-1/\ell})$ for any fixed ℓ. Their main result, at least as far as the present paper is concerned, is as follows.

After having defined a new numeric parameter $u(\pi)$ of a permutation π, ranging in $\{1, 2, \cdots, k-1\}$ where k is the size of π, Ben-Eliezer and Canonne [1] established a lower bound of $\tilde{\Omega}_\epsilon(n^{1-1/u(\pi)})$ for testing π-freeness. This lower bound is tight in all known cases; they conjecture that it is *always* tight.

Ben-Eliezer and Canonne [1] have also established the following properties of the parameter $u(\pi)$. It is 1 iff π is monotone. It is $(k-1)$ iff the maximum of π is adjacent to its minimum. And, for k large, π is almost surely either $(k-2)$ or $(k-3)$.

The main problem with this interesting new parameter is its involved and rather unnatural definition, which makes the computation, or even the estimation, of $u(\pi)$ extremely difficult. To upper bound $u(\pi)$, Ben-Eliezer and Canonne [1] have introduced another parameter $m(\pi)$, and used in a number of cases to compute $u(\pi)$.

The current paper is a natural addition to the study of Ben-Eliezer and Canonne [1]. Our main results are as follows.

As mentioned above, $u(\pi) \leq m(\pi)$. We demonstrate that $m(\pi)$ is easily computable (in time linear in k), and that it provides a 3-approximation of $u(\pi)$:

Theorem 1. $u(\pi) \ \leq \ m(\pi) \ \leq 3u(\pi)$

This result clarifies the meaning of $u(\pi)$. Since $m(\pi)$, as opposed to $u(\pi)$, is quite well understood, it seems that a natural first step towards verifying the conjecture of Ben-Eliezer and Canonne would be to show that the query complexity of testing for π-freeness is $\tilde{O}_\epsilon(n^{1-1/m(\pi)})$.

We also close a small gap left in [1] about the asymptotic behavior of $u(\pi)$.

Theorem 2. *Let π be a random uniform permutation from S_k. Then, as k grows, the probability that $u(\pi) = k-2$ converges to $5/6$, and the probability that $u(\pi) = k-3$ converges to $1/6$.*

Key Definitions

Definition 3. A *partition* $\Lambda = (\sigma_1, ..., \sigma_l)$ of the permutation π is a sequence of disjoint sub-permutations of π, that is, $\sigma_1 = \pi|_{[j_0+1,j_1]}$, $\sigma_2 = \pi|_{[j_1+1,j_2]}$, $\ldots$, $\sigma_l = \pi|_{[j_{l-1}+1,j_l]}$, where $j_0 = 0$, $j_l = k$, and the $j's$ are strictly increasing.

A *signed partition* Λ^* is a partition $\Lambda = (\sigma_1, ..., \sigma_l)$, where each σ_i is assigned a sign $s(\sigma_i) \in \{+, -\}$.

Definition 4. Given a signed partition Λ^*, where $\Lambda = (\sigma_1, ..., \sigma_l)$, the sequence f_{Λ^*} of length k^2 is defined as follows. It will be convenient to view f_{Λ^*} as a function with domain $[1, k^2]$, and range $\bigcup_{r=0}^{k-1} \{i + 1/2k, i + 2/2k, ..., i + k/2k\} \ \subset \ (0, k)$.

Let the support of σ_i, $i = 1, \ldots, l$ be $[j_{i-1}+1, j_i]$ as before. The *length* of σ_i, $|\sigma_i| = j_i - j_{i-1}$, will be denoted by ℓ_i. Define the interval $I_i = [kj_{i-1}+1, kj_i]$ of length $k\ell_i$. Clearly, the intervals I_i, $i = 1, \ldots, l$ constitute a partition of $[1, k^2]$.

The following function $f_{\sigma_i^*}$ supported on I_k is defined as follows. For $r = 1, \ldots, k-1$ and $q = 1, \ldots, \ell_i$, if $s(\sigma_i) = +$, set $f_{\sigma_i^*}(kj_{i-1} + r\ell_i + q) = r + \sigma_i(q)/2k$; else, if $s(\sigma_i) = -$, set $f_{\sigma_i^*}(kj_{i-1} + r\ell_i + q) = (k-1-r) + \sigma_i(q)/2k$.

Finally, given an integer $x \in [1, k^2]$ such that $x \in I_i$, set $f_{\Lambda^*}(x) = f_{\sigma_i^*}(x)$.
See Figure 1 for a visualization of f_{Λ^*}.

Observe the following property of f_{Λ^*}: the set of all $x \in [1, k^2]$ satisfying $r < f_{\Lambda^*}(x) < r+1$ for some integer $r \in [0, k-1]$ forms a π-copy in f_{Λ^*}. Such a π-copy will be called *trivial*. Thus, f_{Λ^*} has k trivial copies.

Definition 5. A signed partition Λ^* of π will be called *unique*, if all the π-copies contained in f_{Λ^*} are trivial.

The parameter $u(\pi)$ is the maximal possible size of a unique signed partition Λ^* of the permutation π.

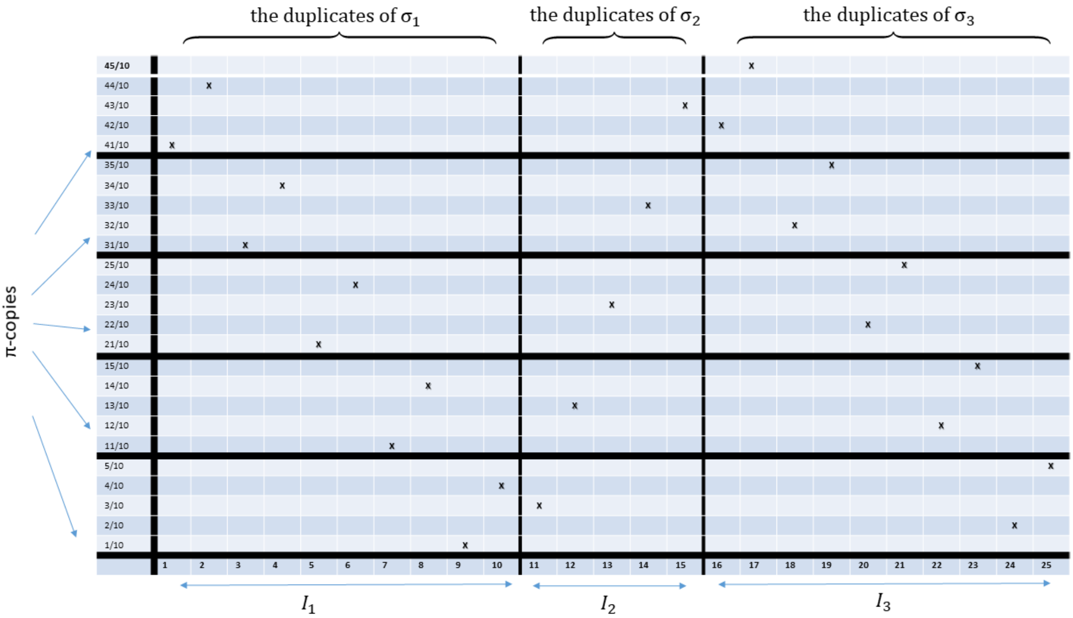

Figure 1: Permutation $\pi = (1,4,3,2,5)$ and its sign-partition $\Lambda^* = (\sigma_1, \sigma_2, \sigma_3)$, where $\sigma_1 = (1,4)$, $\sigma_2 = (3)$, $\sigma_3 = (2,5)$, and $s(\sigma_1) = -$, $s(\sigma_2) = +$, $s(\sigma_3) = -$. This is a unique sign-partition, with one trivial π-copy for each range $(r, r+1)$, $r = 0,1,2,3,4$.

The other important parameter $m(\pi)$ is defined as follows.

Definition 6. A partition $\Lambda = (\sigma_1, ..., \sigma_l)$ of π is called *covering* if $\cup_{i=1}^{l} Range_\pi(\sigma_i) = [1,k]$, where each $Range_\pi(\sigma_i) = [\min(\sigma_i), \max(\sigma_i)]$ is a continuous real interval.

The parameter $m(\pi)$ is the maximal possible size of a covering partition Λ of the permutation π.

It was shown in [1] that a unique (signed) partition is necessarily covering, hence $m(\pi) \geq u(\pi)$.

(For lack of space, the proofs are delayed to the full version.)

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A Generalization of Dyck Paths and Catalan numbers

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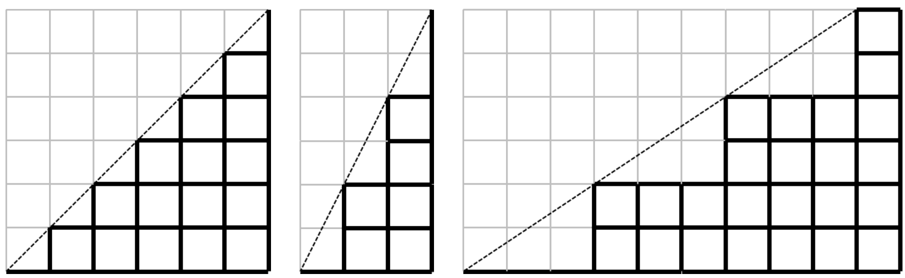
A generalization of Dyck paths

In this talk, motivated by tennis ball problem [1] and regular pruning problem [2], we will present generating functions of the generalized Catalan numbers.

Definition 1. For $r < m$, let $\mathcal{D}_n^{(m,r)}$ denote a (m,r)-Dyck path of order n, which is a lattice path from $(0,0)$ to $(n,(m-r)\lfloor \frac{n}{r} \rfloor)$ with steps $E = (1,0)$ and $N = (0,1)$ that never go above the path $(E^r N^{m-r})^{\lfloor \frac{n}{r} \rfloor} E^{n-r\lfloor \frac{n}{r} \rfloor}$ as illustrated in Figure 1.



(a) $\mathcal{D}_6^{(2,1)}$ (b) $\mathcal{D}_3^{(3,1)}$ (c) $\mathcal{D}_{10}^{(5,3)}$

Figure 1: (m,r)-Dyck paths

Figure 1 shows some examples of (m,r)-Dyck paths.

Definition 2. Let $c_n^{(m,r)}$ denote (m,r)-Catalan number, the number of the (m,r)-Dyck paths of order n. The generating function is denoted by

$$C^{(m,r)}(z) = \sum_{n \geq 0} c_n^{(m,r)} z^n.$$

Remark 3. It is well known that the number of $(m,1)$-Dyck paths of order n is given by m-Catalan numbers [3], defined by

$$c_n^{(m,1)} = \frac{1}{mn+1}\binom{mn+1}{n} = \frac{1}{(m-1)n+1}\binom{mn}{n} = \frac{1}{n}\binom{mn}{n-1}.$$

Let us recall that the generating function $\omega = C^{(m,1)}(z)$ satisfies the following equation:

$$\omega = 1 + z\omega^m.$$

Remark 4. The number of $(m,m-1)$-Dyck paths of order n is enumerated by [3]

$$c_n^{(m,m-1)} = \frac{1+d}{m\nu+1+d}\binom{m\nu+1+d}{\nu} = \frac{1+d}{(m-1)\nu+1+d}\binom{m\nu+d}{\nu} = \frac{1+d}{\nu}\binom{m\nu+d}{\nu-1},$$

where $\nu = \lfloor \frac{n}{m-1} \rfloor$ and $d = n - (m-1)\nu$.

A generalization of Catalan numbers

For any $r < m$, we wish to show the generating functions of (m,r)-Catalan numbers in the following theorems.

Lemma 5. *If m and r are coprime, the algebraic equation in ω*

$$(\omega - 1)^r = z^r \omega^m \tag{1}$$

has a solution called $\omega_0(z)$ in the polynomial ring $\mathbb{Q}[z]$. In the polynomial ring $\mathbb{C}[z]$, it has r solutions called $\omega_k(z)$, $k = 0, \cdots, r-1$, where $\omega_k(z) = \omega_0(\phi^k z)$ and $\phi = e^{i\frac{2\pi}{r}}$.

Theorem 6. *If m and r are coprime, the generating function $C^{(m,r)}$ is given by*

$$C^{(m,r)}(z) = \sum_{p=1}^{r} (z-1)^{p-1} e_p(\omega_0, \cdots, \omega_{r-1}) = \frac{-1 + \prod_{k=0}^{r-1}(1 + (z-1)\omega_k(z))}{z-1}, \tag{2}$$

where $\omega_k = \omega_k(z)$, $k = 0, \cdots, r-1$, are r solutions of (1), and $e_p(\omega_0, \cdots, \omega_{r-1})$, $p = 0, \cdots, r$, are the elementary symmetric polynomials in r solutions, which is given by

$$e_p(\omega_0, \cdots, \omega_{r-1}) = \sum_{0 \le k_1 < \cdots < k_p < r} \prod_{j=1}^{p} \omega_{k_j}(z).$$

Suppose $\phi = e^{i\frac{2\pi}{r}}$ denotes the r-th primitive root of unity. As an application of the discrete Fourier transform, let $C_{(k)}(z)$, $k = 0, \cdots r-1$, denote the generating functions of the subsequences $\{c_{rn+k}\}_{n \ge 0}$:

$$C_{(k)}(z) = \sum_{n \ge 0} c_{rn+k} z^{rn+k} = \frac{1}{r} \sum_{j=0}^{r-1} \phi^{-jk} C(\phi^j z).$$

Theorem 7. *If m and r are coprime, the generating function $C^{(gm,gr)}$ is given by*

$$\begin{aligned} C^{(gm,gr)}(z) &= \frac{-1 + \prod_{l=0}^{g-1} \left(1 + (z-1) \sum_{k=0}^{r-1} \psi^{-kl} C_{(k)}^{(m,r)}(\psi^l z)\right)}{z-1} \qquad (3) \\ &= \frac{-1 + \prod_{l=0}^{g-1} \left(1 + (z-1) \sum_{n \ge 0} \phi^{ln} \sum_{k=0}^{r-1} c_{rn+k}^{(m,r)} z^{rn+k}\right)}{z-1}, \qquad (4) \end{aligned}$$

where $\psi = e^{i\frac{2\pi}{gr}}$ and $\phi = \psi^r = e^{i\frac{2\pi}{g}}$.

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The growth of the Möbius function on the permutation poset

David Marchant The Open University

In this talk, we will show that the growth of the principal Möbius function, $\mu[1, \pi]$, on the permutation poset is at least exponential in the length of the permutation.

Background

The Möbius function $\mu[\sigma, \pi]$ is defined on an interval of a poset as follows: for $\sigma \not\leq \pi$, $\mu[\sigma, \pi] = 0$; for all π, $\mu[\pi, \pi] = 1$; and for $\sigma < \pi$,

$$\mu[\sigma, \pi] = - \sum_{\lambda \in [\sigma, \pi)} \mu[\sigma, \lambda]$$

The problem of the Möbius function on the permutation pattern poset was first raised by Wilf [6]. Earlier work by Smith [4] showed that the growth was at least $O(n^2)$, and recently Jelínek, Kantor, Kynčl and Tancer [3] improved this lower bound to $O(n^7)$. In the other direction, Brignall, Jelínek, Kynčl and Marchant [1] show that the proportion of permutations of length n with principal Möbius function equal to zero is asymptotically bounded below by $(1 - 1/e)^2 \geq 0.3995$.

2413-balloon permutations

We show that, given some permutation β, we can construct a permutation that we call the "2413-balloon" of β. This permutation will have four more points than β, and a generic example is shown in Figure 1. We then show that if π is a 2413-balloon of β, and β is itself a 2413-balloon, then $\mu[1, \pi] = 2\mu[1, \beta]$. From this we deduce that the growth of the principal Möbius function is at least exponential.

If $\beta = 25314$ (which is a 2413-balloon), then we can construct a hereditary class that contains only the simple permutations $\{1, 12, 21, 2413, 25314\}$, where the growth of the principal Möbius function is exponential, answering questions in Burstein et al [2] and Jelínek et al [3].

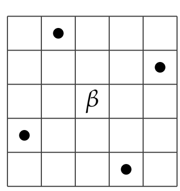

Figure 1: The 2413-balloon of the permutation β.

Outline of the proof

We start by assuming that β is a 2413-balloon, and that π is the 2413-ballon of β. We partition the chains in the poset interval $[1, \pi]$ into into three sets, $\mathcal{R}$, $\mathcal{G}$, and $\mathcal{B}$. We then show that there are parity-reversing involutions on the sets $\mathcal{G}$ and $\mathcal{B}$, and therefore, by a Corollary to Hall's Theorem [5, Proposition 3.8.5], the contribution to $\mu[1, \pi]$ of the chains in these sets is zero. It follows that $\mu[1, \pi]$ is determined by the chains in $\mathcal{R}$. We then show that the Hall sum of $\mathcal{R}$ can be written in terms of $\mu[1, \beta]$, which leads to:

Theorem 1. *Let π be a 2413-balloon of β, where β is itself a 2413-balloon. Then $\mu[1, \pi] = 2\mu[1, \beta]$.*

We define $\max_\mu(n) = \max\{|\mu[1, \pi]| : |\pi| = n\}$, and we define a method of constructing a permutation $\pi^{(n)}$ of length n,

$$\pi^{(n)} = \begin{cases} 1 & \text{If } n = 1 \\ 12 & \text{If } n = 2 \\ 132 & \text{If } n = 3 \\ 2413 & \text{If } n = 4 \\ 2413 \odot \pi^{(n-4)} & \text{Otherwise} \end{cases}$$

Note that for $n > 8$, $\pi^{(n)}$ is a double 2413-balloon.

It is now simple to show that:

Theorem 2. *For all n, $\max_\mu(n) \geq 2^{\lfloor n/4 \rfloor - 1}$.*

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Enumeration of Permutation Classes by Inflation of Independent Sets of Graphs

Émile Nadeau Reykjavik University

This talk is based on joint work with Christian Bean and Henning Ulfarsson

We present a way to obtain permutation classes by inflation of independent sets of certain graphs. We cover classes of the form $\mathrm{Av}(2314, 3124, P)$ and $\mathrm{Av}(2413, 3142, P)$. These results allow us to enumerate a total of 48 classes, with bases containing only length 4 patterns. Using a modified approach, we also demonstrate a result for classes of the form $\mathrm{Av}(2134, 2413, P)$ that allows us to enumerate eight more classes described by bases containing only length 4 patterns. We finally use our results to prove an unbalanced Wilf-equivalence between $\mathrm{Av}(2134, 2413)$ and $\mathrm{Av}(2314, 3124, 12435, 13524)$.

Inflating the up-core

Bean, Tannock, and Ulfarsson [1] show a link between the permutations in $\mathrm{Av}(123)$ and independent sets of certain graphs U_n whose vertices are the cells of the staircase grid of size n. These graphs are called up-cores. We extend their results to enumerate $\mathrm{Av}(2314, 3124)$, and moreover certain sub-classes avoiding patterns of the form $1 \oplus \pi$ where π is skew-indecomposable.

More precisely, we choose an independent set of size k in the graph U_n together with a list of k non-empty permutations in $\mathrm{Av}(2314, 3124, P)$ where P is a set of skew-indecomposable permutations. We establish a bijection between these objects and permutations in $\mathrm{Av}(2314, 3124, 1 \oplus P)$. From [1], we get that the number of independent set of size k in an up-core of a staircase grid of size n is given by the coefficient of $x^n y^k$ in the generating function $F(x, y)$ satisfying

$$F = 1 + xF + \frac{xyF^2}{1 - y(F - 1)}.$$

Using it, we get the enumeration of these classes:

Theorem 1. *Let P be a set of skew-indecomposable permutations and $A(x)$ be the generating function of* $\mathrm{Av}(2314, 3124, P)$. *Then* $\mathrm{Av}(2314, 3124, 1 \oplus P)$ *is enumerated by $F(x, A - 1)$.*

This can be used to enumerate eight classes avoiding length 4 patterns, and many more avoiding longer patterns.

Moreover, using the down-core, also introduced in [1], we state a similar theorem for the class $\mathrm{Av}(2413, 3142, 1 \oplus P)$ where P is a set of sum-indecomposable permutations. This can be used to enumerate eight more classes avoiding length 4 patterns.

Theorem 2. *Let P be a set of sum-indecomposable permutations and $A(x)$ be the generating function of* $\mathrm{Av}(2413, 3142, P)$. *Then* $\mathrm{Av}(2413, 3142, 1 \oplus P)$ *is enumerated by $F(x, A - 1)$.*

New cores

We also describe new graphs on the staircase grid of size n. We prove that the number of independent sets of size k for such a graph is given by the coefficient of $x^n y^k$ in the generating function $G(x, y)$ that satisfies

$$G = 1 + xG + \frac{xyG}{1 - x(y+1)}.$$

Using a similar bijection, as we did for Theorem 1, we prove two theorems that enumerate 32 classes avoiding patterns of length 4.

Theorem 3. *Let P be a set of skew-indecomposable permutations and $A(x)$ be the generating function of* $\mathrm{Av}(2314, 3124, 3142, P)$ *(resp.* $\mathrm{Av}(2314, 3124, 2413, P)$*). Then the generating function of* $\mathrm{Av}(2314, 3124, 3142, 1 \oplus P)$ *(resp.* $\mathrm{Av}(2314, 3124, 2413, 1 \oplus P)$*) is $G(x, A-1)$.*

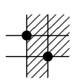
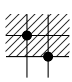
An small modification of G to track the number of rows of the independent set with a third variable also allows to handle classes of the type $\mathrm{Av}(2413, 3142, 3124, 1 \oplus P)$ and $\mathrm{Av}(2413, 3142, 2413, 1 \oplus P)$ for P a set of sum-indecomposable permutations.

Avoiding 2134 and 2413

We use independent sets in a core graph with marked cells in the staircase grid to enumerate of $\mathrm{Av}(2134, 2413)$ and certain sub-classes. The class is symmetric to $\mathrm{Av}(3142, 4312)$ enumerated by Albert, Atkinson, and Vatter [2]. We define

$$_\times\pi = \begin{cases} \alpha & \text{if } \pi = 1 \oplus \alpha \\ \pi & \text{otherwise} \end{cases} \quad \text{and} \quad \pi^\times = \begin{cases} \alpha & \text{if } \pi = \alpha \oplus 1 \\ \pi & \text{otherwise} \end{cases}.$$

We show that for a set of patterns P satisfying that for all $\pi \in P$

- π is skew-indecomposable,
- π avoids  and
- π contains  or $\pi = \alpha \oplus 1$ with α skew-indecomposable.

Theorem 4. *The generating function of* $\mathrm{Av}(2134, 2413, P)$ *is*

$$H\left(xB, \frac{x}{1-x}, B-1, xC\right)$$

where

- *$B(x)$ is the generating function of* $\mathrm{Av}(2134, 2413, {}_\times P)$,
- *$C(x)$ is the generating function of* $\mathrm{Av}(213, {}_\times P^\times)$,

- $H(x,y,z,s) = \frac{s(y+1)-1}{syz+(1-s)x+(1-x)sy+s-1}$

The proofs of all theorems nicely highlight the structure of all the permutation classes. For example, we can extract the structure of the skew-indecomposable permutations in Av$(2134, 2413)$ as seen in Figure 1.

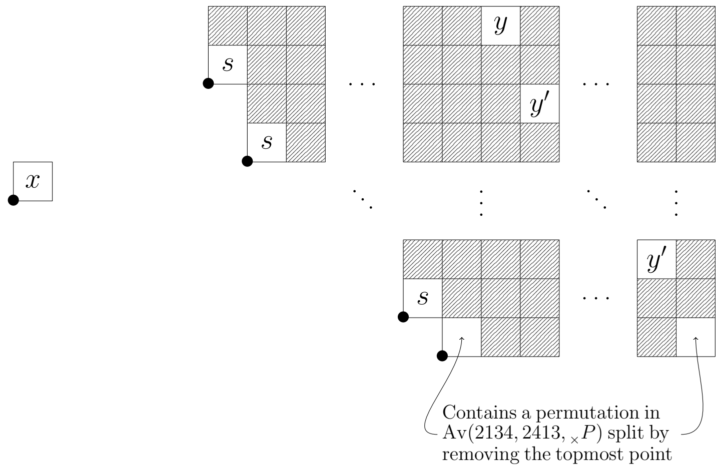


Figure 1: Structure of the two types of skew-indecomposable permutations in Av$(2134, 2413)$. The cells marked s contain a permutation in Av(213), the cell marked x contains a permutation in Av$(2134, 2413)$, the cell marked y contains a non-empty decreasing sequence and the cells marked y' contain a decreasing sequence.

Unbalanced Wilf-equivalence

We demonstrate the Wilf-equivalence of the classes Av$(2314, 3124, 13524, 12435)$ and Av$(2134, 2413)$. We first compute $A(x)$, the generating function for the first class using Theorem 1. We get that $A(x) = F(x, B-1)$ where B is the generating function of Av$(2314, 3124, 2413, 1324)$. Then, with Theorem 3, we get that $B(x) = G(x, C-1)$ where $C(x)$ is the generating function for Av(213). Since $C(x)$ is known, $A(x)$ can be computed explicitly. Moreover, by Theorem 4, the generating function $D(x)$ of Av$(2134, 2413)$ satisfies $D(x) = H\left(xD, \frac{x}{1-x}, D-1, xC\right)$. Solving and comparing with $A(x)$ shows the Wilf-equivalence. This fact leads us to believe that a direct proof using the core structure might be possible.

Several other unbalanced Wilf-equivalences can be derived using our theorems.

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Solving hard problems effectively on permutations of small grid-width

Michal Opler Charles University

This talk is based on joint work with Vít Jelínek

In this talk, we will introduce a grid-like decomposition of permutations and a corresponding parameter called grid-width, which are well-suited for designing dynamic programming algorithms. We will provide some examples of hard problems that can be effectively solved on permutations of small grid-width.

Grid-width

Definition 1. An *interval family* I is a set of pairwise disjoint integer intervals with the natural ordering $I_1, \ldots, I_n$ such that for $j < k$, $I_j < I_k$. For two interval families $\mathcal{I}$ and $\mathcal{J}$, let $\mathcal{I} \times \mathcal{J}$ denote the naturally defined set of boxes in the plane. For a point set A in the plane, let $x(A)$ denote its projection on the x-axis and equivalently $y(A)$ its projection on the y-axis. The *intervalicity* of a set $A \subseteq [n]$, denoted by $I(A)$, is the size of the smallest interval family whose union is equal to A.

Definition 2. A *grid tree* of a permutation $\pi \in \mathcal{S}_n$ is a rooted binary tree T with n leaves, each leaf being labeled by a distinct point of the permutation diagram $\{(i, \pi_i);\ i \in [n]\}$. Let S_v^T denote the point set of the labels on the leaves in the subtree of T rooted in v. The *grid-width* of a vertex v in T is the maximum of the intervalicities $I(x(S_v^T))$ and $I(y(S_v^T))$, and the *grid-with* of T, denoted by $\mathrm{gw}^T(\pi)$, is the maximum grid-width of a vertex of T. Finally, the *grid-width* of a permutation π, denoted by $\mathrm{gw}(\pi)$, is the minimum of $\mathrm{gw}^T(\pi)$ over all grid trees T of π. See Figure 1.

The following observations show that the notion of small grid-width lines up with our current view of "well-behaved" permutation classes.

Proposition 3. *A permutation π has grid-width equal to 1 if and only if it is separable.*

Proposition 4. *Any permutation class that contains only finitely many simple permutations has bounded grid-width.*

Relation to other decompositions

The notion of grid-width is very closely related to the notion of tree-complexity defined by Ahal and Rabinovich [1]. Furthermore, Ahal and Rabinovich [1, Proposition 3.6] showed that the tree-complexity of a permutation π is, up to a constant multiplicative factor, equivalent to the tree-width of the adjacency graph G_π, which we define below. We will omit the definition of tree-complexity itself, and focus on the relation between grid-width of π and tree-width of G_π.

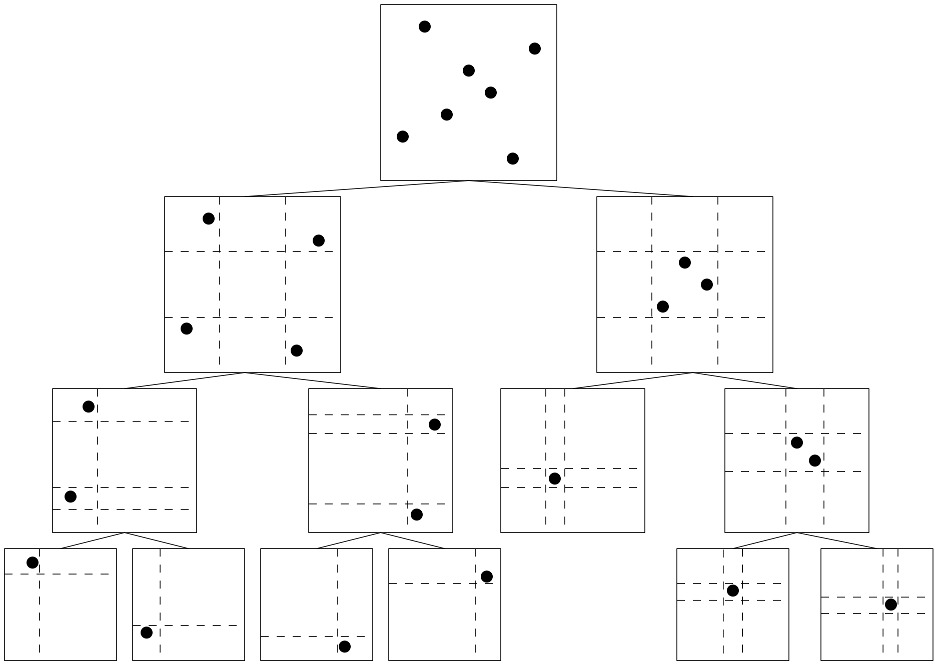

Figure 1: Possible grid tree decomposition of permutation 2735416 with grid-width 2. The dashed lines highlight consecutive intervals in each projection.

Definition 5. For a permutation $\pi \in \mathcal{S}_n$, its *adjacency graph* G_π is defined as follows. The set of vertices of G_π is the set $\{(i, \pi_i);\ i \in [n]\}$. Two vertices $(x_1, y_1), (x_2, y_2)$ are adjacent if $|x_1 - x_2| = 1$ or $|y_1 - y_2| = 1$.

Proposition 6. *For any permutation $\pi \in \mathcal{S}_n$, it holds that $\frac{1}{8}\mathrm{tw}(G_\pi) \leq \mathrm{gw}(\pi) \leq \mathrm{tw}(G_\pi) + 2$. Moreover, given a tree decomposition of width k for G_π, we can compute a grid tree of π of grid-width at most $k + 2$ in linear time.*

The previous proposition allows us to use general tree-width approximating algorithms to obtain good approximation of grid-width. In particular, applying the algorithm due to Bodlaender et al. [2] we obtain the following result.

Corollary 7. *We can compute a grid tree of permutation π with grid-width at most $5\mathrm{gw}(\pi) + O(1)$ in time $2^{O(\mathrm{gw}(\pi))}n$.*

Algorithmic applications

We illustrate how one can use grid trees of small grid-width to obtain efficient algorithms on the following NP-complete problems.

Permutation Pattern Matching (PPM)	
Input:	Permutations π of size k and τ of size n.
Question:	Is π contained in τ?

Longest Common Subpattern (LCP)	
Input:	Permutations π of size k and τ of size n.
Output:	A pattern of maximal size contained in both π and τ.

Theorem 8. *We can solve both* LCP *and* PPM *in time* $O(n^{O(\mathrm{gw}(\pi))})$.

Longest $\mathcal{C}$ Subpattern (L$\mathcal{C}$S)	
Input:	Permutation π of size n.
Output:	Longest permutation $\sigma \in \mathcal{C}$ contained in π.

Previously, L$\mathcal{C}$S was known to be polynomially solvable for separable permutations [3]. We introduce more general framework that allows us to infer polynomial tractability of L$\mathcal{C}$S for many different classes.

Definition 9. We say that a permutation class $\mathcal{C}$ is *grid tree* recognizable, or GT-recognizable for short, if there is an algorithm A that receives the grid-width g and outputs a tree automaton that recognizes $\mathcal{C}$ over grid trees of grid-width at most g.

Theorem 10. *For a GT-recognizable class $\mathcal{C}$ such that every $\pi \in \mathcal{C}$ has grid-width bounded by some constant,* L$\mathcal{C}$S *can be solved in polynomial time.*

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Consecutive permutation patterns in trees and mappings

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In this presentation, we will give results concerning avoidance and occurrence of consecutive permutation patterns in labelled trees and mappings.

Introduction

In the combinatorial and probabilistic literature various studies of quantities related to the labelling of trees, where the vertices of objects of size n are labelled with distinct integers of $[n] := \{1, 2, \ldots, n\}$, can be found. However, only very recently studies concerning avoidance [2] or occurrence [1] of classical permutation patterns in families of labelled trees and forests have been initiated. Analyses of certain consecutive permutation patterns also appear in literature, in particular, occurrences of the pattern 12, i.e., ascents, in rooted labelled trees have been treated in [3], and there are a huge number of results for trees avoiding the pattern 21, so-called increasing trees. Furthermore, alternating (or intransitive) trees, which avoid the set of patterns $\{123, 321\}$, have been treated for several tree families (see, e.g., [5] and references therein).

It seems that apart from the before mentioned work almost no further results on the occurrence or avoidance of consecutive permutation patterns of length 3 or higher are available for trees. Here we initiate such a study by treating the enumeration problem when avoiding a single pattern of length 3, and analysing the number of occurrences of a single pattern of length 3, for rooted labelled trees, also called Cayley-trees (the enumeration formula n^{n-1} for the number of rooted labelled trees of size n is attributed to A. Cayley). We assume here edges in the tree as oriented towards the root node and the occurrence of a consecutive permutation pattern $s = s(1) \ldots s(k) \in S_k$ of length k corresponds to a directed path $p = (v_1, v_2, \ldots, v_k)$ of k vertices, whose sequence of labels is order-isomorphic to s.

Moreover, we consider n-mappings, i.e., functions $f : [n] \to [n]$ and the corresponding functional digraphs $G_f = (V, E)$, i.e., the directed graph with vertex-set $V = [n]$ and edge-set $E = \{(i, f(i)) : i \in [n]\}$, and extend the notion of consecutive permutation pattern occurrence/avoidance to them. Although structural properties of the functional digraphs of random mappings, where one of the n^n n-mappings is chosen with equal probability, have widely been studied, there are only few results concerning label patterns in mappings. As one example we mention the study [6] of alternating mappings, i.e., functions f, for which the iteration sequences $i = f^0(i), f^1(i), f^2(i), \ldots$ are always forming an alternating sequence, i.e., where the functional digraph G_f avoids the set of patterns $\{123, 321\}$.

The structure of the functional digraph of a mapping is rather simple and is well described in [4]: the weakly connected components of such graphs are just cycles of Cayley trees. This connection, although slightly more involved when taking into account the labels of the nodes, also allows to gain results concerning consecutive permutation patterns in mappings from corresponding results in trees.

Results for avoiding a pattern of length 3

Due to obvious symmetry arguments, the permutation patterns $s = s(1)\dots s(k)$ and $\tilde{s} = \tilde{s}(1)\dots\tilde{s}(k)$, with $\tilde{s}(j) = k+1-s(j)$, for $1 \le j \le k$, are strongly consecutive-Wilf equivalent. For patterns s of length 3 this yields the three equivalence classes $123 \cong 321$, $132 \cong 312$, and $231 \cong 213$.

Theorem 1. *The exponential generating functions $T^{[s]}(z)$ of the number $T_n^{[s]}$ of rooted labelled trees of size n that avoid a given consecutive pattern s of length 3 are all characterized as solutions of certain functional equations given below. Moreover, the exponential generating functions $M^{[s]}(z)$ of the number $M_n^{[s]}$ of n-mappings that avoid the corresponding pattern s can be expressed via the function $T^{[s]}(z)$ as stated below.*

Pattern class $Av(s)$	$T := T^{[s]}(z)$	$M := M^{[s]}(z)$
$Av(123)$	$z = e^{-T}\int_0^T \frac{e^t}{1+t}dt$	$M = \frac{1}{1-z(1+T)}$
$Av(321)$		
$Av(132)$	$z = \int_0^T e^{-t-(T-t)e^{-t}}dt$	$M = \frac{e^{T-1+e^{-T}}}{1-e^T\int_0^T e^{-2t-(T-t)e^{-t}}dt}$
$Av(312)$		
$Av(231)$	$z = e^{-T}\int_0^T e^{t-1+e^{-t}}dt$	$M = \frac{1}{1-ze^{1-e^{-T}}}$
$Av(213)$		

Theorem 2. *The numbers $T_n^{[s]}$ and $M_n^{[s]}$ of rooted labelled trees of size n and n-mappings, respectively, that avoid a given consecutive pattern s of length 3 are asymptotically, for $n \to \infty$, given as follows:*

$$T_n^{[s]} \sim c_T \cdot \gamma^n \cdot n^{n-1}, \quad M_n^{[s]} \sim c_M \cdot \gamma^n \cdot n^n,$$

with $\gamma = \frac{1}{e\rho}$, where ρ is the radius of convergence of the corresponding generating function $T^{[s]}(z)$ characterised via solutions of certain functional equations, and where c_T, c_M are some computable constants. Numerical values of the occurring constants are given below.

Pattern s	ρ	γ	c_T	c_M
123	0.42718536...	0.86117050...	1.53000135...	1.53000135...
132	0.44084481...	0.83448739...	1.74299311...	1.83550666...
231	0.44922576...	0.81891883...	2.23735314...	2.23735314...

Remark 3.

- The numbers $T_n^{[s]}$ are for $n \in [8]$ given as follows.

Pattern s	$T_n^{[s]}$, $n \in [8]$							
	1	2	3	4	5	6	7	8
123	1	2	8	50	426	4606	60418	932282
132	1	2	8	49	407	4280	54537	816905
231	1	2	8	49	406	4248	53740	797786

- Only the enumeration sequence of $T_n^{[123]}$ occurs in OEIS as sequence $A225052$, but without giving a combinatorial meaning. Now we can provide such one as rooted labelled trees without double-ascents.
- According to Theorem 2 one obtains the following asymptotic relation for the enumeration sequences $T_n^{[s]}$:

$$T_n^{[231]} \ll T_n^{[132]} \ll T_n^{[123]}.$$

- For $s = 123$ and $s = 231$ one has the relation $M_n^{[s]} = nT_n^{[s]}$, which can also be shown by a combinatorial argument. This relation does not hold, not even asymptotically, for the pattern $s = 132$, for which we get $M_n^{[s]} \sim 1.0530\ldots \cdot nT_n^{[s]}$.

Results for occurrences of patterns of length 3

Theorem 4. *The exponential generating functions $F^{[s]}(z,v)$ of the number $F_{n,m}^{[s]}$ of rooted labelled trees of size n with m occurrences of a given consecutive pattern s of length 3 are characterized as solutions of certain functional equations given below. Moreover, the exponential generating functions $G^{[s]}(z,v)$ of the number $G_{n,m}^{[s]}$ of n-mappings with m occurrences of the corresponding pattern s can be expressed via the functions $F^{[s]}(z,v)$ as stated below.*

Pattern s	$F := F^{[s]}(z,v)$	$G := G^{[s]}(z,v)$
123	$z = e^{-F}\int_0^F e^t(1-(v-1)t)^{\frac{1}{v-1}}dt$	$G = \frac{1}{1-z(1-(v-1)F)^{-\frac{1}{v-1}}}$
132	$z = \int_0^F e^{-t-(F-t)e^{(v-1)t}}dt$	$G = \frac{e^{\frac{(v-1)F+1-e^{(v-1)F}}{v-1}}}{1-e^F\int_0^F e^{(v-2)t-(F-t)e^{(v-1)t}}dt}$
231	$z = e^{-F}\int_0^F e^{\frac{(1-v)t-1+e^{(v-1)t}}{1-v}}dt$	$G = \frac{1}{1-ze^{\frac{1-e^{(v-1)F}}{1-v}}}$

Theorem 5. *Let $X_n^{[s]}$ and $Y_n^{[s]}$ be the random variables counting the number of occurrences of the pattern s of length 3 in a randomly chosen size-n tree or n-mapping, respectively. Then mean and variance of these r.v. are given as follows:*

<table>
<tr><th></th><th>123</th><th>132</th><th>231</th></tr>
<tr><td>$\mathbb{E}(X_n^{[s]})$</td><td colspan="3" rowspan="2">$\frac{n}{6} - \frac{1}{2} + \frac{1}{3n} \sim \frac{1}{6}n$</td></tr>
<tr><td>$\mathbb{E}(Y_n^{[s]})$</td></tr>
<tr><td>$\mathbb{V}(X_n^{[s]})$</td><td rowspan="2">$\frac{n}{5} - \frac{2}{3} + \frac{1}{3n} + \frac{2}{3n^2} - \frac{8}{15n^3}$</td><td>$\frac{2n}{15} - \frac{1}{3} + \frac{1}{3n^2} - \frac{2}{15n^3}$</td><td rowspan="2">$\frac{7n}{60} - \frac{1}{6} - \frac{7}{12n} + \frac{7}{6n^2} - \frac{8}{15n^3}$</td></tr>
<tr><td>$\mathbb{V}(Y_n^{[s]})$</td><td>$\frac{2n}{15} - \frac{1}{4} - \frac{1}{2n} + \frac{5}{4n^2} - \frac{19}{30n^3}$</td></tr>
</table>

Furthermore, after suitable normalization, the r.v. $X_n^{[s]}$ and $Y_n^{[s]}$ converge in distribution to a standard normal distribution $\mathcal{N}(0,1)$, i.e., $\frac{X_n^{[s]}-\mathbb{E}(X_n^{[s]})}{\sqrt{\mathbb{V}(X_n^{[s]})}} \xrightarrow{(d)} \mathcal{N}(0,1)$, analogous for $Y_n^{[s]}$.

Remark 6.

- As expected, the r.v. $X_n^{[s]}$ and $Y_n^{[s]}$ satisfy a central limit theorem with linear mean and variance. However, interestingly the variance, and thus the normalization constants, are different for the three pattern classes of length 3.
- For the patterns 123 and 231 one gets the relation $G_{n,m} = nF_{n,m}$, for $n \geq 1$, which can also be shown via a pattern-preserving bijection from marked labelled trees to mappings. For the pattern 132 this relation does not hold, but the r.v. $X_n^{[s]}$ and $Y_n^{[s]}$ have the same limiting behaviour.

Outlook

The study of consecutive patterns in labelled trees could be extended in various ways. We mention a few such directions for which we obtained some preliminary results via the method presented.

- Sets of patterns. There seem to be several interesting classes of sets of patterns of length 3; some (but as it seems, not all) of them could be treated by using a decomposition w.r.t. the largest or smallest labelled vertex.
- Patterns of length 4 or higher. Although computations quickly get quite involved, there is some hope to obtain at least partial results.
- Other tree families. There are other combinatorial tree families, most notably labelled ordered trees and labelled binary trees, where the approach presented could be applied. Again, computations are more involved, since one has to take into account the number of possible "attachment points" to reconstruct a tree after the decomposition.

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How many chord diagrams have no short chords?

Jay Pantone Marquette University

This talk is based on joint work with Peter Doyle and Everett Sullivan

A *chord diagram* consists of $2n$ points labeled $1, 2, \ldots 2n$ arranged in a circle in increasing order and connected in pairs by *chords*. A *linear chord diagram* is a chord diagram in which the points have been arranged linearly, with labels increasing from left to right. Figure 1 shows a chord diagram on the left and a linear chord diagram on the right.

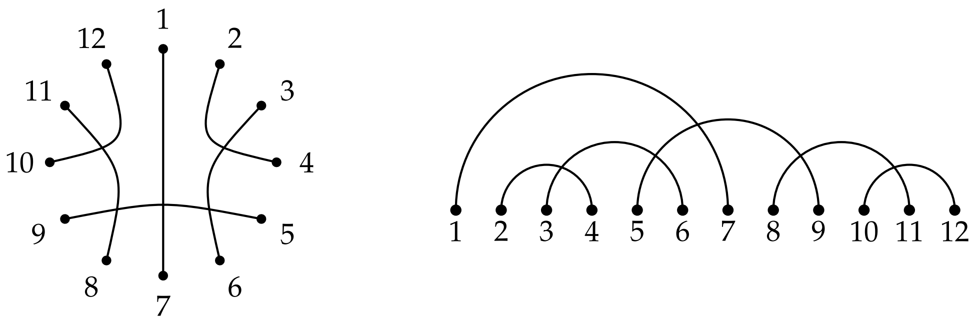

Figure 1: On the left, a chord diagram. On the right, the linear chord diagram that results from breaking apart the chord diagram on the left between points 1 and $2n$ and "straightening".

Chord diagrams and their linear counterparts (known often in the literature as *matchings*) are foundational combinatorial structures, appearing for example in the analysis of RNA folding [5], interconnection networks [4, Section V.4], and the representation theory of Lie algebras [2]. The introductory section of [7] catalogs a number of other applications and interesting references.

Among enumerative combinatorialists, the study of patterns in (linear) chord diagrams has proved to be a particularly fruitful endeavor that closely parallels the study of patterns in permutations. In her 2015 thesis, Jefferson [5] ported the substitution decomposition and several related techniques from the realm of permutation patterns to that of linear chord diagrams. Several other authors have investigated classes of linear chord diagrams that avoid certain types of patterns, e.g., Bloom and Elizalde [1], Chen, Mansour, and Yan [3], and Jelínek [6].

In this talk, we consider a type of pattern avoidance distinct from the references above; our notion is more akin to *consecutive* permutation patterns, while those above are analogues of *classical* permutation patterns—our patterns must occur locally.

For the rest of this abstract we refer only to *linear* chord diagrams, referring to them simply as chord diagrams. The analysis here can be readily adapted to the circular variety with some care.

Definition 1. A *partial chord diagram* is an arrangement of n points in a line such that each point is either connected to no other point (*isolated*) or connected to exactly one other point by a chord. Figure 2 depicts a partial chord diagram with two chords and three isolated points.

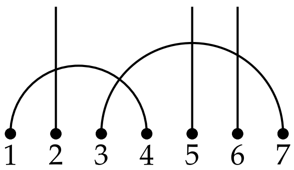

Figure 2: A partial chord diagram with two chords and three isolated points. For visual emphasis, we draw isolated points with a straight vertical line connected to no other points.

A partial chord diagram is the result of restricting a chord diagram to an interval of points.

Definition 2. Given a chord diagram C with n chords, the *restriction of C to the interval of integers* $[\alpha, \beta] \in \{1, 2, \ldots, 2n\}$, denoted $C^{[\alpha,\beta]}$, is the partial chord diagram formed from C by deleting all endpoints with labels not in $[\alpha, \beta]$, deleting all chords that have an endpoint not in $[\alpha, \beta]$, and relabeling the remaining points $1, 2, \ldots, \beta - \alpha + 1$. Note that this may leave isolated points. Figure 3 shows $C^{[3,9]}$, with C the chord diagram on the right side of Figure 1.

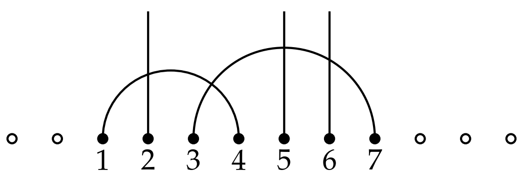

Figure 3: The partial chord diagram $C^{[3,9]}$ for the chord diagram C shown on the right side of Figure 1.

We now define analogues of pattern avoidance and permutation classes for linear chord diagrams. These definitions are adaptations of those found in the thesis of Sullivan [8], and his later work [9].

Definition 3. The chord diagram C with n chords *contains* the partial chord diagram P if there is some integer interval $[\alpha, \beta] \subseteq \{1, \ldots, 2n\}$ such that $C^{[\alpha,\beta]} = P$. If C does not contain P, then C *avoids* P.

Definition 4. Given partial chord diagrams $P_1, \ldots, P_k$, we define $\mathsf{Av}(P_1, \ldots, P_k)$ to be the set of chord diagrams that avoid each of $P_1, \ldots, P_k$.

We call $\mathcal{C} = \mathsf{Av}(P_1, \ldots, P_k)$ a *chord diagram class*, and like permutation classes we are interested in properties of the counting sequence of $\mathcal{C}$. Letting $\mathcal{C}_n$ denote the chord diagrams in $\mathcal{C}$ with n chords, we may define the generating function

$$f_{\mathcal{C}}(z) = \sum_{n \geq 0} \mathcal{C}_n z^n.$$

Based on extensive empirical evidence we present the following conjecture, which contrasts sharply with the situation in both classical and consecutive permutation patterns.

Conjecture 5. *For any finite set of partial chord diagrams $P_1, \ldots, P_k$, the chord diagram class $\mathcal{C}$ has a D-finite generating function. That is, $f_{\mathcal{C}}(z)$ satisfies a linear differential equation with coefficients in $\mathbb{Q}[z]$.*

In this talk, we prove a specialization of this conjecture.

Definition 6. The *length* of a chord with endpoints $\alpha < \beta$ is $\beta - \alpha$. A chord diagram is said to have *minimum-length-k* if all of its chords have length at least k.

Let $\mathcal{M}_k$ be the set of all chord diagrams with minimum-length-k. It is easily verified that each $\mathcal{M}_k$ is a chord diagram class avoiding a finite set of partial chord diagrams.

By combining several symbolic and analytic tools, including finite state automata, the sieve method, creative telescoping, and a trick we call the "gap method", we prove our main theorem below.

Theorem 7. *For all $k \geq 1$, the generating function $f_{\mathcal{M}_K}(z)$ is D-finite.*

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Pattern Hopf algebras on marked permutations and enriched set species

Raul Penaguiao University of Zürich

In this talk we introduce pattern Hopf algebras in combinatorial structures. We start by considering the functions that count patterns. These pattern functions satisfy a product relation, and we are able to endow the linear span of pattern functions with a compatible coproduct. In this way, several combinatorial objects generate a Hopf algebra. For example, the Hopf algebra on permutations studied by Vargas in [6] is a particular case of this construction.

Questions of algebraic nature arise when dealing with these Hopf algebras: freeness, the character group, and so on. We discuss here the freeness of these Hopf algebras.

A particular case of such a Hopf algebra structure, defined on marked permutations, is of interest to us. These objects have an inherent multiplication structure that stems from the inflation operation on permutations, and this product operation is central in establishing that this Hopf algebra is a free algebra.

Introduction

The notion of substructures is important in mathematics, and particularly in combinatorics. In graph theory, minors and induced subgraphs are the main examples of studied substructures. Other objects also get attention in this topic: set partitions, trees, paths and, to a bigger extent, permutations, where the study of substructures leads us to the concept of a pattern in permutations.

A priori unrelated, Hopf algebras are a natural tool in algebraic combinatorics to study graphs, set compositions and permutations. For instance, the celebrated Hopf algebra on permutations named after Malvenuto and Reutenauer sheds some light on the structure of shuffles in permutations. Other examples of Hopf algebras in combinatorics are the word quasisymmetric functions with a basis indexed by set compositions, and the permutation pattern Hopf algebra introduced by Vargas in [6]. A seminal work on the interactions between combinatorics and Hopf algebras is [4].

With that in mind, we build upon the notion of species, as presented in [1] by Aguiar and Mahajan, in order to connect these two areas of algebraic combinatorics, introducing in [5] the notion of presheaf.

In the 1950's, Chen, Fox and Lyndon introduced to us, in [3], the concept of a Lyndon word, and used it to establish that the shuffle algebra on words is free. This gave our comunity a readily availabe tool to establish the freeness of a Hopf algebra, with a surprising amount of flexibility. Indeed, Vargas used this same method in [6] to establish the freeness of the pattern algebra on permutations.

The particular case of a pattern Hopf algebra that interests us is the one on marked permutations. We exploit the remarkable product structure that arises from the inflation of two marked permutations. The study of the freeness of the pattern Hopf algebra on marked permutations reduces to a unique factorisation theorem, presented below, and a careful tunning of the application of the theory of Lyndon words to establish the freeness of an algebra.

Pattern functions, cover numbers and the inflation of marked permutations

For us, a permutation in a set S is a pair of orders in S, as described in [2]. A marked permutation π^* on a set I is a pair of orders in $I \sqcup \{*\}$, that is a permutation on $I \sqcup \{*\}$, and its size is $|\pi^*| = \#I$. If $J \subseteq I$, then $\pi^*|_J$ is the restriction of the underlying permutation to $J \sqcup \{*\}$, and so is a marked permutation on the set J.

An occurence of a marked permutation π^* in another marked permutation τ^* is an occurence of the underlying permutation where the marked element on both permutations match. Equivalently, is a set $J \subseteq I$ such that $\pi^*|_J$ and τ^* are isomorphic marked permutations.

Consider the following marked permutations: $\pi_1^* = 3\bar{2}1, \pi_2^* = \bar{2}1, \pi_3^* = 14\bar{2}3$. Then π_2^* is a pattern in π_1^*, but not in π_3^*.

Let us write $\mathbf{p}_{\pi^*}(\tau^*)$ for the number of occurences of π as a pattern in τ. Then, in the previous examples, we have that $\mathbf{p}_{\pi_2^*}(\pi_1^*) = 1$ and $\mathbf{p}_{\pi_2^*}(\pi_3^*) = 0$.

Our first observation, is that these pattern functions have a product formula.

Proposition 1. *Let π_1, π_2^*, τ^* be marked permutations, then*

$$\mathbf{p}_{\pi_1^*}(\tau^*)\,\mathbf{p}_{\pi_2^*}(\tau^*) = \sum_{\sigma^*} \binom{\sigma^*}{\pi_1^*, \pi_2^*} \mathbf{p}_{\sigma^*}(\tau^*)\,, \tag{1}$$

where the coefficients $\binom{\sigma^*}{\pi_1^*, \pi_2^*}$ have an explicit combinatorial interpretation, and where the sum runs over marked permutations σ^* up to isomorphism, with size at most $|\pi_1^*| + |\pi_2^*|$. In particular, span$\{\mathbf{p}_{\pi^*} \,|\pi^*$ marked permutation$\} = \mathcal{A}(\mathtt{MPer})$ is a graded algebra.

If π_1^*, π_2^* are two marked permutations, we can define the inflation product $\pi_1^* * \pi_2^*$ of these marked permutations by inflating the marked element of π_1^* with the marked permutation π_2^*. In Fig.1, we have a graphical expression of $1\bar{3}2 * \bar{2}1$.

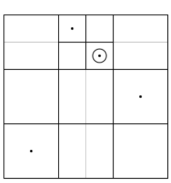

Figure 1: The inflation product of the marked permutations $1\bar{3}2$ and $\bar{2}1$ is $14\bar{3}2$.

A marked permutation π^* is called simple when any factorisation $\pi^* = \tau_1^* \cdot \tau_2^*$ has $\tau_1^* = \bar{1}$ or $\tau_2^* = \bar{1}$. For example $\bar{1}423$ is a simple marked permutation, although it has a decomposition as an $\oplus$ product.

We remark that the inflation product in marked permutations is not commutative. However, factorisations into simples are not unique. For instance, both $\bar{1}32$ and $21\bar{3}$ are simple, but $\bar{1}32 * 21\bar{3} = 21\bar{3} * \bar{1}32$. This is an example of a *two factor transposition*.

Define the coproduct $\Delta\, \mathbf{p}_{\pi^*} = \sum_{\pi^* = \tau_1^* \cdot \tau_2^*} \mathbf{p}_{\tau_1^*} \otimes \mathbf{p}_{\tau_2^*}$ in $\mathcal{A}(\texttt{MPer})$. This coproduct is compatibe with the product of pattern functions. With this, $\mathcal{A}(\texttt{MPer})$ is a Hopf algebra with a basis indexed by marked permutations.

Proposition 2. *Let α^* be a marked permutation with two factorisations $a_1^* * \cdots * a_k^*$ and $b_1^* * \cdots * b_j^*$ into simple marked permutations. Then these factorisations can be obtained from one to the other by explicit two factor transpositions. In particular, $k = j$ and the simple factors are the same up to order.*

The freeness follows via a careful application of the theory of Lyndon words.

Theorem 3. *The pattern algebra on marked permutations is free commutative.*

Other pattern algebras

The process of building a Hopf algebra out of combinatorial structures with a restriction and a product, that we went through above, is quite general.

For instance, the family of graphs with the disjoint union is a monoidal presheaf, and the family of permutations with the $\oplus$ product is one as well. It turns out that having a commutative product is enough to guarantee a unique factorisation theorem, hence:

Theorem 4. *Any pattern algebra with a commutative monoidal structure is a free commutative algebra.*

The surprising feature of this result is that the coalgebra structure, which is the one that pertains the monoidal strucutre, dictates whether the algebra structure is free. As a consequence, pattern algebras in graphs, posets and matroids are free.

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Scaling limits of permutation classes with a finite specification: a dichotomy

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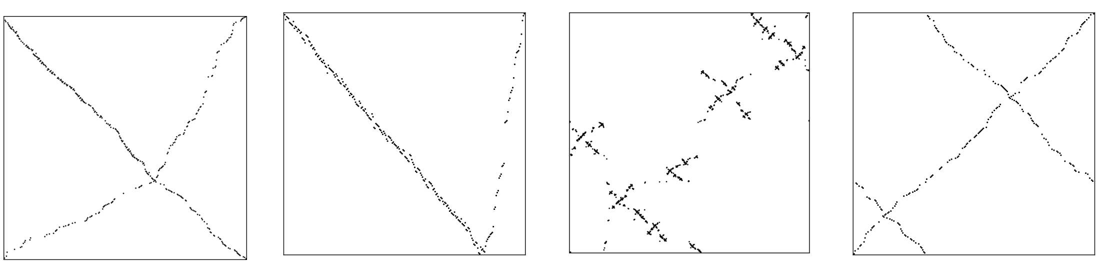
This talk is based on joint work with Frédérique Bassino, Mathilde Bouvel, Valentin Féray, Lucas Gerin and Mickaël Maazoun. The full paper is available at https://arxiv.org/abs/1903.07522

We are interested in the description of the asymptotic properties of a uniform random permutation of large size in a permutation class. We consider uniform random permutations in classes having a finite combinatorial specification for the substitution decomposition. These classes include all permutation classes having finitely many simple permutations, and also some classes having infinitely many simple permutations. Our goal is to study their limiting behavior in the sense of permutons.

The limit depends on the structure of the specification restricted to families with the largest growth rate. When it is strongly connected, two cases occur. If the associated system of equations is linear, the limiting permuton is a deterministic X-shape. Otherwise, the limiting permuton is the Brownian separable permuton, a random object that already appeared as the limit of most substitution-closed permutation classes, among which the separable permutations. Moreover these results can be combined to study some non strongly connected cases.

To prove our result, we use a characterization of the convergence of random permutons by the convergence of random subpermutations. Key steps are the combinatorial study of families of permutations with marked elements inducing a given pattern, and the singularity analysis of the corresponding generating functions.



(a) (b) (c) (d)

Figure 1: Large uniform random permutations in four different finitely specified classes. These four cases are covered by the present paper.

Context and background

We see a permutation σ as its *diagram*, *i.e.* a square grid with dots at coordinates $(i, \sigma(i))$. For θ a permutation of size d, the substitution $\theta[\pi^{(1)}, \dots, \pi^{(d)}]$ is obtained by inflating each point $\theta(i)$ of θ by a square containing the diagram of $\pi^{(i)}$. Each permutation can be decomposed in a canonical way as successive substitutions, starting from the indecomposable elements, which are called *simple permutations*.

We are interested in classes $\mathcal{C}$ with a nice recursive description, namely a *finite* system of combinatorial equations for $\mathcal{C}$, called *specification*. Example: the class $\mathcal{C}_{\text{sep}}$ of *separable permutations* has the following specification, where $\mathcal{C}_{\text{sep}}^{\text{not}\oplus}$ (resp. $\mathcal{C}_{\text{sep}}^{\text{not}\ominus}$) is the set of separable permutations that cannot be written as $12[\pi^{(1)},\pi^{(2)}]$ (resp. $21[\pi^{(1)},\pi^{(2)}]$):

$$\begin{cases} \mathcal{C}_{\text{sep}} &= \{\bullet\} \uplus 12[\mathcal{C}_{\text{sep}}^{\text{not}\oplus},\mathcal{C}_{\text{sep}}] \uplus 21[\mathcal{C}_{\text{sep}}^{\text{not}\ominus},\mathcal{C}_{\text{sep}}]; \\ \mathcal{C}_{\text{sep}}^{\text{not}\oplus} &= \{\bullet\} \uplus 21[\mathcal{C}_{\text{sep}}^{\text{not}\ominus},\mathcal{C}_{\text{sep}}]; \\ \mathcal{C}_{\text{sep}}^{\text{not}\ominus} &= \{\bullet\} \uplus 12[\mathcal{C}_{\text{sep}}^{\text{not}\oplus},\mathcal{C}_{\text{sep}}]. \end{cases} \tag{1}$$

The example of $\mathcal{C}_{\text{sep}}$ is a particular case of a more general family of permutation classes, that of *substitution-closed classes*. All these classes have specifications with three equations. In [BBF+19], we obtained all the possible limiting shapes for such classes with a unified combinatorial approach and a careful analysis.

Another sufficient condition for having a specification is that the class contains finitely many simple permutations. [BBP+17] provides an algorithmic way to compute a specification for such a class, implemented in [Maa19]. Unlike for substitution-closed classes, the number of equations is not fixed, making a uniform analysis much harder. We also note that a class may have a specification, while containing infinitely many simple permutations (example: the class of *pin-permutations* [BHV08b, BBR11]).

A specification provides in an automatic way a random sampler for permutations in the class. We show in Figure 1 large permutations in several classes obtained in this way (using Boltzmann generators). As we can see on these examples, various qualitative asymptotic behaviors may occur. The results of the present paper apply in particular to each of these four cases, giving an explicit limit shape result.

Our limiting results are phrased in the framework of *permutons*, which can be thought of as infinite rescaled permutations. A permuton is a measure on $[0,1]^2$, whose projections on the horizontal and vertical axes are the uniform measure on $[0,1]$. Every permutation defines a permuton, by considering its rescaled diagram. The set of permutons is endowed with the weak convergence topology of measures, providing a natural notion of convergence for permutations.

Presentation of the results

We consider a permutation class $\mathcal{C}$ with a specification. This specification involves several families of permutations $\mathcal{C}_0 = \mathcal{C}$, $\mathcal{C}_1$, ..., $\mathcal{C}_d$. Among these families, the ones with the smallest radius of convergence play a prominent role in the asymptotics; we call such families *critical*. In our case, the class $\mathcal{C}$ is always critical.

An important information to study $\mathcal{C}$ through its specification is to know which families appear in the equation defining each $\mathcal{C}_i$ in the specification. This is traditionally encoded in a directed graph with vertex set $\{\mathcal{C}_0,\dots,\mathcal{C}_d\}$, called *dependency graph* of the specification. A standard assumption to study specifications is that this graph is strongly connected, implying in particular that all families are critical. This assumption is too strong in our context. We shall instead assume that the dependency graph *restricted to the critical families* is strongly connected.

Moreover, we restrict ourselves to specifications satisfying an analytic condition, which is a weak assumption informally saying that the equations appearing in this system are all analytic at the radius of convergence. Then, under the strong connectivity assumption above, there are two possible asymptotic behaviors for a uniform random permutation $\boldsymbol{\sigma}_n$ in $\mathcal{C}$.

- Either the combinatorial equation defining each *critical* family $\mathcal{C}_i$ is linear in every *critical* family (it may depend nonlinearly on non-critical families). This is referred to as the *essentially linear case*. In this case, we prove the convergence of $\boldsymbol{\sigma}_n$ in distribution towards a deterministic permuton, that has a shape of an X, *i.e.* is supported by four line segments from the corners of $[0,1]^2$ to a common central point. This permuton depends on the class $\mathcal{C}$ only through a quadruple $\boldsymbol{p}$ whose components are in $[0,1]$, sum up to 1 and indicate the mass of the four line segments (thus determining the coordinates of the central point). The simulations (a) and (b) of Figure 1 fit in this framework (in the second case, the limiting X-permuton is in some sense degenerate: only two components of its quadruple $\boldsymbol{p}$ are nonzero, explaining the V-shape).

- The other possibility (called *essentially branching case*) is that the equation defining some *critical* family $\mathcal{C}_i$ involves a product of at least two *critical* families (which may be the same). In this case, we prove that $\boldsymbol{\sigma}_n$ converges in distribution towards a biased Brownian separable permuton, as introduced in [BBF+19, Maa17]. In this case, the limit depends only on $\mathcal{C}$ through a single real parameter $p \in [0,1]$. The simulation (c) of Figure 1 illustrates this behavior.

Unlike the X-permuton, the Brownian separable permuton already appeared in our previous works [BBF+19, BBF+18] as a universal limit of substitution-closed permutation classes. The second item above shows that the universality class of the Brownian separable permuton extends further than the substitution-closed classes. The first item reveals another (new) universality class, with a simple limiting object: the X-permuton.

Our main results stated above do not apply to the not strongly connected case. However, we describe a strategy to reduce the study of such cases to the strongly connected one. This strategy applies in particular to the class in the simulation (d) of Figure 1: the limit in this case is a juxtaposition of two X-permutons of random relative sizes.

Proof tools: analytic combinatorics of algebraic systems

Our main results are convergence results of random permutations in some class $\mathcal{C}$ in the topology of permutons. A general result relates such convergence to the convergence, for each $k \geq 1$, of the substructure, *i.e.* the *pattern*, induced by k random elements of the permutation. This can be done by enumerating, for each π, the family $\mathcal{C}_\pi$ of permutations in $\mathcal{C}$ with k marked elements inducing the pattern π. It turns out that the specification for $\mathcal{C}$ can be refined to a specification for $\mathcal{C}_\pi$.

We analyze the resulting specifications with analytic combinatorial tools. Namely, it is standard to translate specifications into systems of equations for the associated generating series. When the equations are analytic on a sufficiently large domain and when the dependency graph of the system is strongly connected, two different kinds of behavior might happen:

- either the system is linear, and the series have all polar singularities at their radius of convergence [BD15];
- or the system is called *branching*, and the series have all square-root singularities (this is known as Drmota-Lalley-Woods theorem in the literature [FS09, Drm09]).

We need however to adapt the hypotheses of these theorems to our setting, and more importantly, to make explicit the coefficients in the first-order asymptotic expansion of the series.

We will apply these theorems to the critical series in our (refined) specifications, considering the non-critical series as parameters. Once we know the kind of the singularities of the series, the transfer theorem of analytic combinatorics [FS09] gives us the asymptotic number of elements in $\mathcal{C}$ and $\mathcal{C}_\pi$ for all π. We deduce from this the probability that k marked elements in a uniform permutation in $\mathcal{C}$ induce a given pattern π. Comparing these probabilities to those in the candidate limiting permutons, this proves the desired convergence.

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Given a pattern ρ and a finite set S, let $f_S(\rho)$ be the maximum number of copies of ρ in any member of S, and call any member of S that achieves this maximum *ρ-optimal*. For example, $f_{\mathcal{S}_n}(12) = \binom{n}{2}$ since the increasing permutation $I_n = 1 \cdots n$ of length n is 12-optimal with $\binom{n}{2}$ copies of the pattern 12. Further, the packing density of ρ is defined as

$$d(\rho) = \lim_{n\to\infty} \frac{f_{\mathcal{S}_n}(\rho)}{\binom{n}{|\rho|}}.$$

It is easy to see that $d(I_k) = 1$, since every subsequence of length k in I_n is a copy of I_k. It takes more work to show $d(132) = 2\sqrt{3} - 3$. Exact values of $d(\rho)$ are known for some $\rho \in \mathcal{S}_4$, while for others there are bounds on $d(\rho)$ with a conjecture of the precise packing density. (See, for example, [1, 2, 3, 4].)

In this talk, our primary object of interest is $f_{\mathcal{S}_n(\sigma)}(\rho)$. From this computation, we also determine the *σ-restricted packing density* of ρ, namely

$$d_\sigma(\rho) = \lim_{n\to\infty} \frac{f_{\mathcal{S}_n(\sigma)}(\rho)}{\binom{n}{|\rho|}},$$

for any choice of $\sigma, \rho \in \mathcal{S}_3$. We are also interested in packing patterns in the set of alternating permutations A_n, which is the set of permutations that avoid both 123 and 321 consecutively. While the primary goal of this talk is to determine $f_S(\rho)$ and the corresponding restricted packing densities, we will highlight bijections with other mathematical objects as appropriate.

Packing in classical pattern classes

First, we consider $f_{\mathcal{S}_n(\sigma)}(\rho)$ and $d_\sigma(\rho)$ for $\sigma, \rho \in \mathcal{S}_3$. By symmetry, we may restrict our attention to $\rho = 123$ and $\rho = 132$. Of course, $d_\rho(\rho) = 0$, so we assume $\sigma \neq \rho$.

In the case of $\rho = 123$, it is already known that $d(123) = 1$ because every subsequence of length 3 in the permutation I_n is a 123 pattern. Similarly, for $\sigma \in \mathcal{S}_3 \setminus \{123\}$, $d_\sigma(123) = 1$ because $I_n \in \mathcal{S}_n(\sigma)$.

Values of $d_\sigma(132)$ are more interesting. Because the optimal 132-packing permutation in S_n is layered, it avoids 231 and 312. Therefore,

$$d_{231}(132) = d_{312}(132) = d(132) = 2\sqrt{3} - 3.$$

On the other hand, both $\left\{f_{\mathcal{S}_n(123)}(132)\right\}_{n\geq 1}$ and $\left\{f_{\mathcal{S}_n(213)}(132)\right\}_{n\geq 1}$ are given by OEIS entry A200067, which is the "maximum sum of all products of absolute differences and distances between element pairs among the integer partitions of n." This enumeration implies

$$d_{123}(132) = d_{213}(132) = \frac{4}{9}.$$

While $\left\{f_{\mathcal{S}_n(321)}(132)\right\}_{n\geq 1}$ is new to the OEIS, it can be computed recursively, and the restricted packing density is

$$d_{321}(132) = \frac{3}{13}.$$

Packing in alternating permutations

While the set A_n of alternating permutations is not a classical pattern class, packing in alternating permutations still produces interesting results, especially for the case when $\rho = I_k$. In this case, the ρ-optimal permutation in A_n is the layered permutation

$$\pi^* = \begin{cases} 1 \oplus 21 \oplus 21 \oplus \cdots \oplus 21 \oplus 1 & n \text{ even} \\ 1 \oplus 21 \oplus 21 \oplus \cdots \oplus 21 & n \text{ odd} \end{cases}.$$

While the structure of π^* is unsurprising, the enumerations $f_{A_n}(I_k)$ themselves are interesting. In particular,

- $f_{A_n}(12) = \left\lceil \frac{(n-1)^2}{2} \right\rceil$ (A000982), which has a variety of geometric interpretations.
- $f_{A_n}(123)$ is given by A168380 (the atomic numbers of the augmented alkaline earth metals group in the periodic table of chemical elements). It is striking to see a sequence only listed for chemistry reasons in the OEIS appear in a permutation enumeration context. Although $\{f_{A_n}(123)\}_{n\geq 1}$ has many terms beyond the number of elements in the alkaline earth group, we give a bijection between copies of 123 in π^* and legal tuples of integers used to described electron orbitals in physical chemistry.
- $f_{A_n}(1234)$ is given by A072819 when n is odd and 4 times A006325 when n is even. The sequence for the odd case has an interpretation in terms of random walks, while the sequence for the even case is a 4-dimensional analog of centered polygonal numbers.

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Avoiding Baxter-like patterns

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This talk is based on joint work with Mathilde Bouvel and Veronica Guerrini

In this talk, we collect classical results, recent results, preliminary investigations and open problems on the enumeration of permutations avoiding one to four *Baxter-like patterns*, which we define as the vincular patterns in $\{2\text{-}14\text{-}3, 2\text{-}41\text{-}3, 3\text{-}14\text{-}2, 3\text{-}41\text{-}2\}$. Our approach is to use generating trees and succession rules, in a unified manner for all considered families.

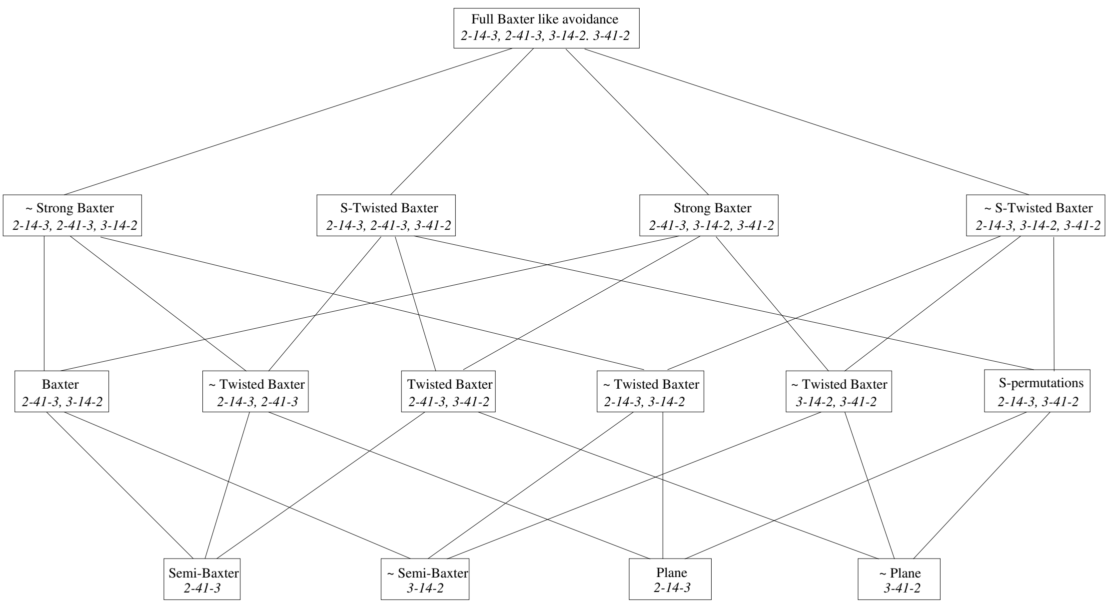

Figure 1: The families of permutations avoiding one to four Baxter-like pattern(s).

Known cases

Avoiding one pattern: Up to symmetry, there are just two families avoiding a single Baxter-like pattern: Av(2-41-3) (called *semi-Baxter permutations* – see [5]) and Av(2-14-3) (called *plane permutations* – see [4]). Using generating trees, we proved in [5] that both families are enumerated by the OEIS sequence A117106. The associated succession rule is later denoted Ω_{semi}.

Avoiding two patterns: The famous *Baxter permutations* are defined as Av(2-41-3, 3-14-2). Generating trees and successions rules, combined with the "obstinate" kernel method, can be used to prove that they are enumerated by the Baxter numbers – see [3]. The corresponding succession rule is denoted Ω_{Bax} in the sequel.

Four other families, all symmetric of one another, are also enumerated by the Baxter numbers. One representative is the family of *twisted Baxter permutations*: Av(2-41-3, 3-41-2). Although enumerated by the same sequence, the succession rule associated with twisted Baxter permutations in [6] is different from Ω_{Bax}, and denoted Ω_{TBax}. The final family avoiding two Baxter-like patterns, called S-permutations, is Av(2-14-3, 2-14-3). It is not enumerated by the Baxter numbers, but by the OEIS sequence A214358 – see [1]. The proof of this result does not use generating trees.

Avoiding three patterns: So far, only the family Av(2-41-3, 3-14-2, 3-41-2) (called *strong Baxter permutations*) has been enumerated – see [5]. (Note that the family Av(2-41-3, 3-14-2, 2-14-3) is a symmetry of these strong Baxter permutations.) The proof uses again generating trees, and the corresponding succession rule is denoted Ω_{strong}. The OEIS sequence enumerating strong Baxter permutations is A281784.

Comparison of succession rules: For all families above (except the S-permutations), the generating trees are obtained by letting permutations grow "on the right", *i.e.* by insertion of a final element. In addition, the encoding of the generating trees into succession rules is done in the same way for all considered cases (namely, recording the numbers of active sites above and below the final element). These two facts, together with the inclusion relations among the considered families, imply that the succession rule for a smaller family is a specialization of the one for a larger family, in the sense that we discuss in [2, Section 4.1]. This appears clearly if one lists the succession rules (all with root $(1,1)$) as follows:

$$
\begin{array}{llllllllll}
\Omega_{semi}: & (h,k) & \rightsquigarrow & (1,k+1) & \dots & (h-1,k+1) & (h,k+1) & (h+k,1) & \dots & (h+1,k) \\
\Omega_{Bax}: & (h,k) & \rightsquigarrow & (1,k+1) & \dots & (h-1,k+1) & (h,k+1) & (h+1,1) & \dots & (h+1,k) \\
\Omega_{TBax}: & (h,k) & \rightsquigarrow & (1,k) & \dots & (h-1,k) & (h,k+1) & (h+k,1) & \dots & (h+1,k) \\
\Omega_{strong}: & (h,k) & \rightsquigarrow & (1,k) & \dots & (h-1,k) & (h,k+1) & (h+1,1) & \dots & (h+1,k).
\end{array}
$$

Open cases

Avoiding two patterns: Even though the enumeration of the S-permutations has been solved in [1], the proof (which does not use generating trees) does not allow to include the S-permutations in the comparison of succession rules shown above. We show that it is possible to write a succession rule for S-permutations, obtained as previously by letting these permutations grow "on the right" and recording the numbers of active sites above and below the final element. The resulting succession rule, specializing Ω_{semi}, is however a *colored succession rule* with three colors, namely:

$$
\Omega_S : \begin{cases}
(1,1) \\
(h,k) \rightsquigarrow (h+1,1)^\sharp \dots (h+1,k-1)^\sharp \; (h+1,k) \; (h,k+1) \; (h-1,k+1)^\flat \dots (1,k+1)^\flat \\
(h,k)^\sharp \rightsquigarrow (h+1,1)^\sharp \dots (h+1,k-1)^\sharp \; (h+1,k) \; (h,k)^\flat \; (h-1,k)^\flat \dots (1,k)^\flat \\
(h,k)^\flat \rightsquigarrow (h,1)^\sharp \dots (h,k-1)^\sharp \; (h,k)^\sharp \; (h,k+1) \; (h-1,k+1)^\flat \dots (1,k+1)^\flat
\end{cases}
$$

(The three colors red, blue and black are also indicated by the superscripts $\sharp$, $\flat$, or no superscript.)

We are working on re-deriving the enumeration of S-permutations from this succession rule, using a variant of the kernel method.

Avoiding three patterns: Up to symmetry, only the family Av(2-14-3, 3-14-2, 3-41-2) (of twisted Baxter S-permutations) remains to be enumerated. Specializing the above succession rule, we have also provided a colored succession rule for this family, with three colors. The enumeration of the family from this succession rule is work in progress. The first terms of the enumerating sequence (easily obtained from the succession rule) are $1, 2, 6, 21, 80, 322, 1354, 5901, 26494, 121960, 573458, 2745991$.

Avoiding all four Baxter-like patterns: To our knowledge, the family Av(2-14-3, 2-41-3, 3-14-2, 3-41-2) has not yet been considered in the literature. Again, a succession rule with three colors specializing the ones above can be obtained, and the derivation of the enumeration from this succession rule remains to be done. The first terms of the sequence enumerating Av(2-14-3, 2-41-3, 3-14-2, 3-41-2) are $1, 2, 6, 20, 72, 274, 1088, 4470, 18884, 81652, 360054, 1614618$.

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Cyclic Schur-positive permutation sets

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This talk is based on joint work with Jonathan Bloom, Sergi Elizalde

We introduce a notion of *cyclic Schur-positivity* for sets of permutations, which is a natural extension of the notion of Schur-positivity. Cyclic Schur-positive sets are always Schur-positive, but the converse does not hold. Most Schur-positive sets of permutations, including inverse descent classes, Knuth classes and conjugacy classes are not cyclic Schur-positive.

In this paper we show that certain pattern-avoiding classes of permutations which are invariant under either horizontal or vertical rotation, i.e., under rotation of the positions or of the values, are cyclic Schur-positive. In the process, we also prove a conjecture from [7] regarding the equidistribution of descent sets on vertical and horizontal rotations of inverse descent classes.

Definitions

Let $[n] = \{1, 2, \ldots, n\}$ and let $\mathfrak{S}_n$ denote the symmetric group on $[n]$. Recall that the *descent set* of a permutation $\pi \in \mathfrak{S}_n$ is

$$\mathrm{Des}(\pi) := \{i \in [n-1] : \ \pi(i) > \pi(i+1)\}.$$

Given any subset $A \subseteq \mathfrak{S}_n$, we define the quasi-symmetric function

$$\mathcal{Q}(A) := \sum_{\pi \in A} \mathscr{F}_{n,\mathrm{Des}(\pi)},$$

where $\mathscr{F}_{n,D}$ is Gessel's *fundamental quasi-symmetric functions*, first defined in [8]. A symmetric function is called *Schur-positive* if all the coefficients in its expansion in the basis of Schur functions are nonnegative. A subset $A \subseteq \mathfrak{S}_n$ is called *Schur-positive* if $\mathcal{Q}(A)$ is symmetric and Schur-positive.

It is possible to characterize Schur-positive permutation sets using standard Young tableaux (SYT). This characterization is useful because it does not require computing quasisymmetric functions, but rather finding a bijection from the set of permutations to a multiset of SYT that preserves the descent set. As with permutations, there is a well-studied notion of the descent set of a SYT. Let λ/μ denote a skew shape, where λ and μ are partitions such that where the Young diagram of μ is contained in that of λ, and let $\mathrm{SYT}(\lambda/\mu)$ denote the set of standard Young tableaux of shape λ/μ. The *descent set* of $T \in \mathrm{SYT}(\lambda/\mu)$ is

$$\mathrm{Des}(T) := \{i \in [n-1] : \ i+1 \text{ is in a lower row than } i \text{ in } T\}.$$

For $J \subseteq [n-1]$, let $\mathbf{x}^J := \prod_{i \in J} x_i$.

Theorem 1 ([2, Prop. 9.1]). *A subset $A \subseteq \mathfrak{S}_n$ is Schur-positive if and only if there exist nonnegative integers $(m_\lambda)_{\lambda \vdash n}$ such that*

$$\sum_{\pi \in A} \mathbf{x}^{\mathrm{Des}(\pi)} = \sum_{\lambda \vdash n} m_\lambda \sum_{T \in \mathrm{SYT}(\lambda)} \mathbf{x}^{\mathrm{Des}(T)}. \tag{1}$$

The following problem was first posed by Sagan and Woo.

Problem 2. Find symmetric/Schur-positive pattern-avoiding classes in $\mathfrak{S}_n$.

Cyclic Schur-positive permutation sets

The standard *cyclic descent set* for permutations was defined by Cellini [4] and later studied by by Dilks, Petersen, Stembridge [5], and others. For $\pi \in \mathfrak{S}_n$ let

$$\mathrm{cDes}(\pi) := \{i \in [n] : \ \pi(i) > \pi(i+1)\}, \tag{2}$$

with the convention $\pi(n+1) := \pi(1)$.

The cyclic descent set for rectangular SYT was introduced by Rhoades [10] and generalized to all skew shapes in [1]. For an explicit combinatorial approach, see [9].

Theorem 3 ([1, Theorem 1.1]). *Let λ/μ be a skew shape. There exists a cyclic descent extension for* $\mathrm{SYT}(\lambda/\mu)$ *if and only if λ/μ is not a connected ribbon. Furthermore, for all $J \subseteq [n]$, all such cyclic extensions* (cDes, ψ) *share the same cardinalities* $|\mathrm{cDes}^{-1}(J)|$.

In this paper we study the following cyclic analogue of Schur-positive permutation sets. A subset $A \subseteq \mathfrak{S}_n$ is *cyclic Schur-positive* (*cSp*) if there exists a collection of nonnegative integers $(m_{\lambda/\mu})_{\lambda/\mu \vdash n}$ such that

$$\sum_{\pi \in A} \mathbf{x}^{\mathrm{cDes}(\pi)} = \sum_{\lambda/\mu \vdash n} m_{\lambda/\mu} \sum_{T \in \mathrm{SYT}(\lambda/\mu)} \mathbf{x}^{\mathrm{cDes}(T)}, \tag{3}$$

where cDes on permutations in the LHS is defined by Eq. (2), and cDes on SYT in the RHS is the cyclic descent extension defined in [1, 9]. Note that the sum in the RHS is over skew shapes of size n, for which there exists a cyclic descent extension (i.e., are not connected ribbons).

It can be shown that cSp sets of permutations are always Schur-positive, but the converse does not hold. Most known Schur-positive sets of permutations are not cSp.

Problem 4. Find cSp subsets in $\mathfrak{S}_n$.

Main results

Let c_n denote the n-cycle $(1,2,\ldots,n) = 23\ldots n1 \in \mathfrak{S}_n$, and let $C_n := \langle c_n \rangle$ be the cyclic subgroup generated by c_n. Any set $A \subseteq \mathfrak{S}_{n-1}$ can be interpreted as a subset of $\mathfrak{S}_n$ by identifying $\mathfrak{S}_{n-1}$ with the set of permutations in $\mathfrak{S}_n$ that fix n. With this interpretation, we define the *horizontal* (respectively, *vertical*) *rotation closure* of $A \subseteq \mathfrak{S}_{n-1}$ as the set $AC_n \subseteq \mathfrak{S}_n$ (respectively, $C_nA \subseteq \mathfrak{S}_n$). Our first main result states that horizontal rotations of Schur-positive sets are always cSp.

Theorem 5. *For every Schur-positive set $A \subseteq \mathfrak{S}_{n-1}$, the set $AC_n \subseteq \mathfrak{S}_n$ is cSp.*

For every $J \subseteq [n-2]$, define the *descent class* $D_{n-1,J} := \{\pi \in \mathfrak{S}_{n-1} : \operatorname{Des}(\pi) = J\}$. The proof of the next theorem involves cDes-preserving operations on grid classes, as defined in [3].

Theorem 6. *For every positive integer $n > 1$ and every subset $J \subseteq [n-2]$ the distribution of* cDes *on $C_n D^{-1}_{n-1,J}$ is the same as on $D^{-1}_{n-1,J} C_n$.*

It follows from Theorems 5 and 6 that the set $C_n D^{-1}_{n-1,J}$ is cSp. In particular, it is Schur-positive, providing an affirmative solution to [7, Conjecture 10.2].

Example: arc permutations

As shown in [6], the set of arc permutations can be characterized in terms of pattern avoidance as

$$\mathfrak{A}_n := \mathfrak{S}_n(1324, 1342, 2413, 2431, 3124, 3142, 4213, 4231).$$

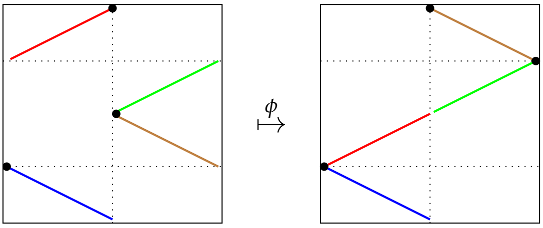
Arc permutations satisfy $\mathfrak{A}_n = C_n \mathrm{E}_{n-1}$, where $\mathrm{E}_{n-1} = \mathfrak{S}_{n-1}(132, 312)$ is the set of left-unimodal permutations. Using Theorems 5 and 6, we deduce the following.

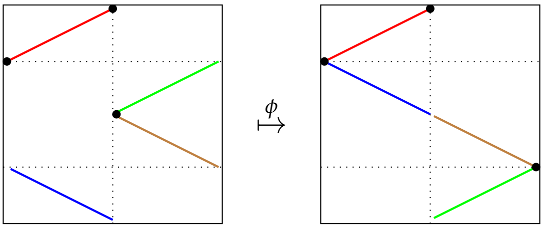
Corollary 7. *The set $\mathfrak{A}_n$ is cSp.*

The diagrams below give a pictorial description of a cDes-preserving bijection

$$\phi : D^{-1}_{n-1,[i]} C_n \to C_n D^{-1}_{n-1,[i]},$$

which proves Theorem 6 in the special case $J = [i]$. Having i range between 0 and n, we obtain a bijection between $\mathrm{E}_{n-1} C_n$ and $\mathfrak{A}_n = C_n \mathrm{E}_{n-1}$.

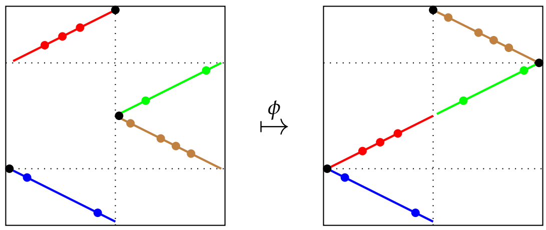




For $\pi = \sigma c_n^{n-j} \in \mathrm{E}_{n-1}C_n$ with $\pi(j) = n$, the two above cases correspond to $\pi(1) < \pi(j+1)$ and $\pi(1) > \pi(j+1)$, respectively (note that when $j = n$, $\phi(\pi) = \pi$).

Here is an example of the bijection:

$$\phi(3\,2\,11\,12\,13\,1\,14\,8\,7\,9\,6\,5\,4\,10) = 3\,2\,4\,5\,6\,1\,14\,13\,7\,12\,11\,10\,8\,9.$$



An explicit set of SYT on which cDes has the same distribution as it has on arc permutations is described in the full version of this paper.

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This talk is based on joint work with Miklós Bóna

We will present results and open questions on permutations p such that both p and p^2 avoid a given pattern q.

Introduction

The standard definition of pattern avoidance does not consider the other perspective from which permutations can be studied, namely that of the *symmetric group*, where the product of two permutations is defined, and the notion of a permutation's inverse is defined. Therefore, it is not surprising that pattern avoidance questions become much more difficult if the symmetric group concept is present in them. (See [1], [2] or [5] for a few results in this direction.) One exception to this is the straightforward observation [3] that if p avoids q, then its inverse permutation p^{-1} avoids q^{-1}.

We consider the following family of questions. Let us call a permutation p *strongly q-avoiding* if both p and p^2 avoid q. Let $\mathrm{Sav}_n(q)$ denote the number of strongly q-avoiding permutations of length n. What can be said about the numbers $\mathrm{Sav}_n(q)$?

The pattern $12 \cdots k$

For all k, if p is long enough, then either p or p^2 must contain an increasing subsequence of length k. That is, we have the following theorem:

Theorem 1. *Let k be a positive integer, and let $n \geq (k-1)^3 + 1$. Then* $\mathrm{Sav}_n(12 \cdots k) = 0$.

The bound on the length of n relative to k for a strongly $12 \cdots k$-avoiding permutation given above is tight for at least small values of k. This is trivially true for $k = 1, 2$. Moreover, one example of a strongly $12 \cdots k$-avoiding permutation of length $(k-1)^3$ for $k = 3$ is $p = 75863142 = (1746)(2538)$ whose square is $p^2 = 43218765$. Similar examples can be given for $k = 4, 5, 6$.

The pattern 312

In a 312-avoiding permutation, all entries on the left of the entry 1 must be smaller than all entries on the right of 1, or a 312-pattern would be formed with the entry 1 in the middle. Therefore, if $p = p_1 p_2 \cdots p_n$ is a 312-avoiding permutation, and $p_i = 1$, then p maps the interval $[1, i]$ into itself, and the interval $[i+1, n]$ into itself. That means that p will be strongly 312-avoiding if and only if its restrictions to those two intervals are strongly 312-avoiding. In other words, each non-empty strongly 312-avoiding permutation p uniquely decomposes as $p = LR$, where L is a strongly 312-avoiding permutation ending in the entry 1, and R is a (possibly empty) strongly 312-avoiding permutation.

Therefore, if $\mathrm{Sav}_{312}(z) = \sum_{n\geq 0} \mathrm{Sav}_n(312)z^n$, and $B(z)$ is the ordinary generating function for the number of strongly 312-avoiding permutations ending in 1, then the equality

$$\mathrm{Sav}_{312}(z) = 1 + B(z)\mathrm{Sav}_{312}(z), \tag{1}$$

holds. This motivates our analysis of strongly 312-avoiding permutations that end in the entry 1, specifically:

Theorem 2. *For any permutation p ending in 1, the following two statements are equivalent.*

(A) The permutation p is strongly 312-avoiding.

(B) The permutation p has form $p = (k+1)(k+2)\cdots n\ k(k-1)(k-2)\cdots 1$ where $k \geq \frac{n}{2}$. That is, p is unimodal beginning with its $n-k \leq \frac{n}{2}$ largest entries in increasing order followed by the remaining k smallest entries in decreasing order.

Rearranging (1), we get the equality

$$\mathrm{Sav}_{312}(z) = \frac{1}{1-B(z)}. \tag{2}$$

It follows from Theorem 2 that there are $\lfloor \frac{n}{2} \rfloor$ strongly 312-avoiding permutations of length n and ending in 1 if $n \geq 2$, and there is one such permutation if $n = 1$. Therefore,

$$B(z) = z + \frac{(z+1)z^2}{(1-z^2)^2} = \frac{z^4 - z^3 + z}{(z-1)^2(z+1)}.$$

So (2) yields

$$\mathrm{Sav}_{312}(z) = \frac{-z^3 + z^2 + z - 1}{z^4 - 2z^3 + z^2 + 2z - 1}.$$

So in particular, $\mathrm{Sav}_{312}(z)$ is rational. Its root of smallest modulus is about 0.4689899435, so the exponential growth rate of the sequence of the numbers $\mathrm{Sav}_{312}(n)$ is the reciprocal of that root, or about 2.132241882. The first few elements of the sequence, starting with $n = 1$, are 1, 2, 4, 9, 19, 41, 87, 186, 396, 845.

Interestingly, the sequence is in the Encyclopedia of Integer Sequences [6] as Sequence A122584, where it is mentioned in connection to work in Quantum mechanics [7].

The pattern 321

We give a lower bound for the numbers $\mathrm{Sav}_{321}(n)$ that shows that for large n, the inequality $\mathrm{Sav}_{321}(n) > \mathrm{Sav}_{312}(n)$ holds.

Indeed, let us call a permutation $p = p_1p_2\cdots p_n$ *block-cyclic* if it has the following properties.

1. It is possible to cut p into blocks $B_1, B_2, \ldots, B_t$ of entries in consecutive positions so that for all $i < j$, the block B_i is on the left of the block B_j, and each entry in B_i is smaller than each entry in B_j, and
2. Each block is either a singleton, or its entries can be written in one-line notation as $(a+i)\,(a+i+1)\cdots(a+k)\,(a+1)\cdots(a+i-1)$, for some integers $1 < i \leq k$. That is, each block is a singleton or a power of the cycle $(a+1\ a+2\cdots a+k)$ that is not the identity.

Note that each block-cyclic permutation is 321-avoiding. Furthermore, *any* power of a block-cyclic permutation is block-cyclic, and so it is also 321-avoiding. Therefore, block-cyclic permutations are all 321-avoiding. Let h_n be the number of block-cyclic permutations of length n, and let $H(z) = \sum_{n\geq 0} h_n z^n$. The number of allowed blocks of size k is 1 if $k = 1$, and $k-1$ if $k > 1$ (since longer blocks cannot be monotone increasing), leading to the formula

$$H(z) = \frac{1}{1 - z - \sum_{k\geq 2}(k-1)z^k} = \frac{1}{1 - z - \frac{z^2}{(1-z)^2}} = \frac{(1-z)^2}{1 - 3z + 2z^2 - z^3}.$$

The singularity of smallest modulus of the denominator is about 0.430159709, so the exponential growth rate of the sequence h_n is the reciprocal of that number, or about 2.324717957. As $h_n \leq \mathrm{Sav}_{321}(n)$ for all n, we have the following.

Corollary 3. *The inequality*

$$2.3247 \leq \limsup_{n\to\infty} \sqrt[n]{\mathrm{Sav}_{321}(n)}$$

holds.

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Hopping from Chebyshev polynomials to permutation statistics

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This talk is based on joint work with Yan Zhuang

We express exponential generating functions counting permutations by the peak number, valley number, double ascent number, and double descent number statistics in terms of the exponential generating function for Chebyshev polynomials. We give a combinatorial proof of this result using monomino-domino tilings, inclusion-exclusion, and the modified Foata–Strehl action ("valley hopping"). We also give a cyclic analogue.

Eulerian and peak polynomials evaluated at -1

Our work beings with a few observations about permutation statistic polynomials evalutated at -1. For a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$, we say that $i \in \{1,\ldots,n-1\}$ is a *descent* if $\pi_i > \pi_{i+1}$, and $i \in \{2,\ldots,n-1\}$ is a *peak* if $\pi_{i-1} < \pi_i > \pi_{i+1}$. Define $\mathsf{des}(\pi)$ to be the number of descents of π and $\mathsf{pk}(\pi)$ to be the number of peaks of π. We can use these statistics to define the *Eulerian polynomials*

$$A_n(t) := \sum_{\pi\in\mathfrak{S}_n} t^{\mathsf{des}(\pi)}$$

and the *peak polynomials*

$$P_n^{\mathsf{pk}}(t) := \sum_{\pi\in\mathfrak{S}_n} t^{\mathsf{pk}(\pi)},$$

whose exponential generating functions have the following expressions.

$$A(t;x) := \sum_{n=1}^{\infty} A_n(t)\frac{x^n}{n!} = \frac{e^{(1-t)x}-1}{1-te^{(1-t)x}};$$

$$P^{\mathsf{pk}}(t;x) := \sum_{n=1}^{\infty} P_n^{\mathsf{pk}}(t)\frac{x^n}{n!} = \frac{1}{\sqrt{1-t}\coth(x\sqrt{1-t})-1}.$$

It is known [1, Théorème 5.6] that evaluating the Eulerian polynomials at -1 gives a signed version of the tangent numbers. A combinatorial argument shows these count alternating permutations of odd length. Evaluating the peak polynomials at -1 gives the sequence $0,\ 1,\ 2,\ 2,\ -8,\ -56,\ -112,\ 848,\ldots$, which has no known direct combinatorial interpretation. This sequence appears on the OEIS [4, A006673], and its exponential generating function is the logarithmic derivative of that of the Pell numbers $1,\ 0,\ 1,\ 2,\ 5,\ 12,\ldots$. Interestingly, the exponential generating function for the Eulerian polynomials evaluated at -1 is the logarithmic derivative of the exponential generating function of an even simpler sequence: $1,\ 0,\ 1,\ 0,\ldots$. While it is not hard to verify these facts by manipulating generating functions, we provide combinatorial proofs using the *modified Foata-Strehl action* ("valley hopping").

Chebyshev polynomials

To state our main result, we first generalize from the Pell numbers and the sequence 1, 0, 1, 0, ... to polynomials $U_n(s,t)$ defined by the recurrence

$$U_n(s,t) = 2tU_n(s,t) - sU_n(s,t) \tag{1}$$

for $n \geq 2$ with initial values $U_0(s,t) = 1$ and $U_1(s,t) = 2t$. These are a two-parameter variant of the Chebyshev polynomials of the second kind. They are related to the usual Chebyshev polynomials of the second kind $U_n(t)$ by the formulas $U_n(t) = U_n(1,t)$ and $U_n(s,t) = U_n(s^{-1/2}t)s^{n/2}$. From the recurrence (1), it is not hard to see that $U_n(s,t)$ counts tilings of a $1 \times n$ rectangle with two types of monominoes, each weighted t, and one type of domino, weighted $-s$.

We will work with the shifted exponential generating function

$$V(s,t;x) := \sum_{n=0}^{\infty} U_n(s,t)\frac{x^{n+2}}{(n+2)!} = \frac{x^2}{2!} + 2t\frac{x^3}{3!} + (4t^2 - s)\frac{x^4}{4!} + \cdots .$$

Note that $1 + V(-1,0)$ and $1 + V(-1,1)$ are the exponential generating functions for the sequences we saw earlier, 1, 0, 1, 2, 5, 12, ... (the Pell numbers) and 1, 0, 1, 0, ..., respectively. An expression for the exponential generating function of the polynomials $U_n(t)$ is known (see [3, p. 301]) and we can use it to obtain the closed-form expression

$$V(s,t;x) = \frac{1}{s}\left(1 - \cosh(x\sqrt{t^2 - s})e^{tx} + \frac{te^{tx}\sinh(x\sqrt{t^2 - s})}{\sqrt{t^2 - s}}\right). \tag{2}$$

Hopping to permutation statistics

Definition 1. For a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$, we adopt the convention that $\pi_0 = \pi_{n+1} = \infty$ and characterize each $i \in \{1, \ldots, n\}$ as either a peak if $\pi_{i-1} < \pi_i > \pi_{i+1}$, a *valley* if $\pi_{i-1} > \pi_i < \pi_{i+1}$, a *double ascent* if $\pi_{i-1} < \pi_i < \pi_{i+1}$, or a *double descent* if $\pi_{i-1} > \pi_i > \pi_{i+1}$. We denote the number of valleys, double ascents, and double descents by val, dasc, and ddes, respectively. We also write $\mathsf{dbl} = \mathsf{dasc} + \mathsf{ddes}$.

For any list of permutation statistics $\mathrm{st}_1, \mathrm{st}_2, \ldots, \mathrm{st}_m$ and variables $t_1, t_2, \ldots, t_m$, we define the exponential generating function

$$P^{(\mathrm{st}_1,\mathrm{st}_2,\ldots\mathrm{st}_m)}(t_1, t_2, \ldots, t_m; x) := \sum_{n=1}^{\infty} \sum_{\pi \in \mathfrak{S}_n} t_1^{\mathrm{st}_1(\pi)} t_2^{\mathrm{st}_2(\pi)} \cdots t_m^{\mathrm{st}_m(\pi)} \frac{x^n}{n!}.$$

We can now state our main theorem, which expresses the exponential generating function for the $(\mathsf{pk}, \mathsf{dbl})$ polynomials as the logarithmic derivative of $1 - sV(s,t;x)$ (divided by $-s$).

Theorem 2. $P^{(\mathsf{pk},\mathsf{dbl})}(s,t;x) = \dfrac{\frac{\partial}{\partial x}V(s,t;x)}{1 - sV(s,t;x)}$

While this can be verified with the known exponential generating function formulas, we present a combinatorial proof using the monomino-domino tiling interpretation of $U_n(s,t)$, inclusion-exclusion, and the valley hopping group action based on a classical group action of Foata and Strehl [2].

The exponential generating function for the quadruple distribution $(\mathsf{pk}, \mathsf{val}, \mathsf{dasc}, \mathsf{ddes})$ then follows from

$$P^{(\mathsf{pk},\mathsf{val},\mathsf{dasc},\mathsf{ddes})}(s,t,u,v;x) = tP^{(\mathsf{pk},\mathsf{dbl})}\left(st, \frac{1}{2}(u+v); x\right).$$

A cyclic analogue

Definition 3. Given $\pi = \pi_1\pi_2\cdots\pi_n$, we say that π_i is either: a *cyclic peak* if $i < \pi_i > \pi_{\pi_i}$, a *cyclic valley* if $i > \pi_i < \pi_{\pi_i}$, a *cyclic double ascent* if $i < \pi_i < \pi_{\pi_i}$, a *cyclic double descent* if $i > \pi_i > \pi_{\pi_i}$, or a *fixed point* if $i = \pi_i$. We denote the number of each of these by cpk, cval, cdasc, cddes, and fix respectively. We also write $\mathsf{cdbl} = \mathsf{cdasc} + \mathsf{cddes}$.

We use a similar method with "cyclic valley hopping", due to Sun and Wang [5], to give a combinatorial proof of the following.

Theorem 4. $1 + P^{(\mathsf{cpk},\mathsf{cdbl},\mathsf{fix})}(s,t,u;x) = \dfrac{e^u}{1 - sV(s,t;x)}$

From there, we can generalize to the quintuple distribution $(\mathsf{cpk}, \mathsf{cval}, \mathsf{cdasc}, \mathsf{cddes}, \mathsf{fix})$ over all permutations. See [6] for details.

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k-PARTIAL PERMUTATIONS AND THE CENTER OF THE WREATH PRODUCT $\mathcal{S}_k \wr \mathcal{S}_n$ ALGEBRA

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We will introduce k-partial permutations and we will use them to show a polynomiality property for the structure coefficients of the center of the wreath product $\mathcal{S}_k \wr \mathcal{S}_n$ algebra. More details can be found in [2].

The algebra of k-partial permutations

If i and k are two positive integers, the k-tuple $p_k(i)$ is the following set of size k :

$$p_k(i) := \{(i-1)k+1, (i-1)k+2, \cdots, ik\}.$$

Suppose we have a set d that is a disjoint union of some k-tuples

$$d = \bigsqcup_{i=1}^{r} p_k(a_i),$$

where a_i is a positive integer for any $1 \leq i \leq r$. We define the group $\mathcal{B}_d^k$ to be the following group of permutations:

$$\mathcal{B}_d^k := \{\omega \in \mathcal{S}_d \mid \forall 1 \leq i \leq r, \exists 1 \leq j \leq r \text{ with } \omega(p_k(a_i)) = p_k(a_j)\},$$

where $\mathcal{S}_d$ is the group of permutations of the set d. In other words, the group $\mathcal{B}_d^k$ consists of permutations that permute the blocks of the set d. It was shown in [1] that $\mathcal{B}_{[kn]}^k$ is isomorphic to the wreath product $\mathcal{S}_k \wr \mathcal{S}_n$, where $[kn] := \{1, 2, \ldots, kn\}$.

Definition 1. Let n be a non-negative integer. A k-partial permutation of n is a pair (d, ω) where $d \subset [kn]$ is a disjoint union of some k-tuples and $\omega \in \mathcal{B}_d^k$. The set of all k-partial permutation of n will be denoted $\mathcal{P}_{kn}^k$.

For a permutation $\omega \in \mathcal{B}_d^k$ and a partition $\rho = (\rho_1, \ldots, \rho_l)$ of k we will construct the partition $\omega(\rho)$ as follows. First decompose ω as a product of disjoint cycles. Consider the collection of cycles $C_1, \ldots, C_l$ such that C_1 contains ρ_1 elements of a certain k-tuple $p_k(i)$, C_2 contains ρ_2 elements of the same k-tuple $p_k(i)$, etc. Now add the part m to $\omega(\rho)$ if m is the number of k-tuples that form the cycles $C_1, \ldots, C_l$. We define type(ω) to be the following family of partitions

$$\text{type}(\omega) := (\omega(\rho))_{\rho \vdash k}.$$

The type of a k-partial permutation (d, ω) of n is the type of its permutation ω.

Example 2. Consider the 3-partial permutation (d, ω), where $d = p_3(1) \cup p_3(2) \cup p_3(4) \cup p_3(6)$ and

$$\omega = \left(\begin{array}{ccc|ccc|ccc|ccc} 1 & 2 & 3 & 4 & 5 & 6 & 10 & 11 & 12 & 16 & 17 & 18 \\ 12 & 10 & 11 & 4 & 5 & 6 & 16 & 18 & 17 & 1 & 2 & 3 \end{array}\right).$$

Its cycle decomposition is $(1, 12, 17, 2, 10, 16)(3, 11, 18)(4)(5)(6)$ and its type is formed by $\omega(2, 1) = (3)$ and $\omega(1^3) = (1)$.

If (d_1, ω_1) and (d_2, ω_2) are two k-partial permutations of n, we define their product as follows:

$$(d_1, \omega_1)(d_2, \omega_2) = (d_1 \cup d_2, \omega_1 \omega_2),$$

where the composition $\omega_1 \omega_2$ is made after extending both ω_1 and ω_2 by identity to $d_1 \cup d_2$. Let $\mathcal{I}^k_{kn}$ be the algebra over $\mathbb{C}$ generated by the following formal sums indexed by families of partitions $\Lambda = (\lambda(\rho))_{\rho \vdash k}$ with $|\Lambda| := \sum_{\rho \vdash k} |\Lambda(\rho)| < n$

$$\mathbf{C}_{\Lambda;n} := \sum_{(d,\omega)} (d, \omega),$$

where the sum is taken over all $(d, \omega) \in \mathcal{P}^k_{kn}$ such that $|d| = k|\Lambda|$ and $\operatorname{type}(\omega) = \Lambda$. There is a surjective homomorphism between the algebras $\mathcal{I}^k_{kn}$ and $Z(\mathbb{C}[\mathcal{B}^k_{kn}])$, the center of the group algebra $\mathbb{C}[\mathcal{B}^k_{kn}]$, defined by

$$\psi(\mathbf{C}_{\Lambda;n}) = \binom{n - |\Lambda| + m_1(\Lambda(1^k))}{m_1(\Lambda(1^k))} \mathbf{C}_{\underline{\Lambda}_n},$$

where $\underline{\Lambda}_n$ is the family of partitions Λ except that $\Lambda(1^k)$ is replaced by $\Lambda(1^k) \cup (1^{n-|\Lambda|})$.

Let $\mathcal{P}^k_\infty$ be the group of all the k-partial permutations with a finite support and consider the algebra $\mathcal{I}^k_\infty$ generated by the elements $\mathbf{C}_\Lambda$, indexed by families of partitions, and defined by

$$\mathbf{C}_\Lambda = \sum_{(d,\omega)} (d, \omega),$$

where the sum runs over all k-partial permutations $(d, \omega) \in \mathcal{P}^k_\infty$ such that d is a union of $|\Lambda|$ k-tuples and ω has type Λ. The projection homomorphism from $\mathcal{I}^k_\infty$ to $\mathcal{I}^k_{kn}$ is defined by $\operatorname{Proj}_n(\mathbf{C}_\lambda) = 0$ if $|\Lambda| > n$ and if $|\Lambda| \leq n$, $\operatorname{Proj}_n(\mathbf{C}_\Lambda) = \mathbf{C}_{\Lambda;n}$.

Remark 3. In [3], the algebra $\mathcal{I}^1_\infty$ appeared for the first time to prove a polynomiality property for the structure coefficients of the center of the symmetric group algebra.

Structure coefficients of the center of $\mathcal{B}^k_{kn}$ algebra

A family of partitions Λ is called proper if the partition $\Lambda(1^k)$ does not have any part equal to one. Let Λ and Δ be two proper families of partitions with $|\Lambda|, |\Delta| \leq n$. In the algebra $\mathcal{I}^k_\infty$, we can write the product $\mathbf{C}_\Lambda \mathbf{C}_\Delta$ as a linear combination of the basis elements, that is

$$\mathbf{C}_\Lambda \mathbf{C}_\Delta = \sum_\Gamma c^\Gamma_{\Lambda\Delta} \mathbf{C}_\Gamma,$$

where the sum runs over the families of partitions Γ satisfying $\max(|\Lambda|, |\Delta|) \leq |\Gamma| \leq |\Lambda| + |\Delta|$ and $c^\Gamma_{\Lambda\Delta}$ are non-negative integers independent of n. If we apply $\psi \circ \operatorname{Proj}_n$ to this equality we get the following identity in the center of the group $\mathcal{B}^k_{kn}$ algebra:

$$\mathbf{C}_{\underline{\Lambda}_n} \mathbf{C}_{\underline{\Delta}_n} = \sum_\Gamma c^\Gamma_{\Lambda\Delta} \binom{n - |\Gamma| + m_1(\Gamma(1^k))}{m_1(\Gamma(1^k))} \mathbf{C}_{\underline{\Gamma}_n}.$$

If we sum up all the partitions that give $\mathbf{C}_{\underline{\Gamma}_n}$, the above sum can be turned into a sum over proper families of partitions.

Theorem 4. *Let Λ, Δ and Γ be three proper families of partitions satisfying* $\max(|\Lambda|, |\Delta|) \leq |\Gamma| \leq |\Lambda| + |\Delta|$. *For any integer $n \geq |\Gamma|$ we have*

$$c_{\underline{\Delta}_n \underline{\Delta}_n}^{\underline{\Gamma}_n} = \sum_{r=1}^{n-|\Gamma|} c_{\Lambda\Delta}^{\Gamma_{|\Gamma|+r}} \binom{n-|\Gamma|}{r},$$

where $c_{\Lambda\Delta}^{\Gamma_{|\Gamma|+r}}$ are non-negative integers independent of n.

Corollary 5. *Let Λ, Δ and Γ be three proper families of partitions satisfying* $\max(|\Lambda|, |\Delta|) \leq |\Gamma| \leq |\Lambda| + |\Delta|$. *The structure coefficient $c_{\underline{\Delta}_n \underline{\Delta}_n}^{\underline{\Gamma}_n}$ is a polynomial in n with non-negative integer coefficients and of degree at most $|\Lambda| + |\Delta| - |\Gamma|$.*

Example 6. There are only two partitions of 2, namely (1^2) and (2). Thus the elements generating $\mathcal{I}_\infty^2$ are indexed by families of partitions $\Lambda = (\Lambda(1^2), \Lambda(2))$. Take for instance $\Lambda = ((1),(2))$, then $C_{((1),(2))}$ is the set of all 2-partial permutations with type $((1),(2))$. For example, $(\{3,4,7,8,9,10\}, (3,7,4,8)(9)(10)) \in C_{((1),(2))}$. We have the following two complete expressions in $\mathcal{I}_\infty^2$:

$$\mathbf{C}_{((1),\emptyset)} \mathbf{C}_{((1),(1))} = 2\mathbf{C}_{((1),(1))} + 2\mathbf{C}_{((1^2),(1))} \tag{1}$$

and

$$\mathbf{C}_{(\emptyset,(2))} \mathbf{C}_{(\emptyset,(2))} = 2\mathbf{C}_{((1^2),\emptyset)} + 2\mathbf{C}_{(\emptyset,(1^2))} + 2\mathbf{C}_{(\emptyset,(2^2))} + 3\mathbf{C}_{((3),\emptyset)}. \tag{2}$$

For example the first coefficient 2 in the above first equation is due to the fact that there are only two pairs $(x,y) \in C_{((1),\emptyset)} \times C_{((1),(1))}$ that satisfy $xy = (\{1,2,3,4\}; (1)(2)(3,4))$. Namely (x,y) can be one and only one of the following pairs:

$$\Big((\{1,2\}; (1)(2)), (\{1,2,3,4\}; (1)(2)(3,4)) \Big) \text{ or } \Big((\{3,4\}; (3)(4)), (\{1,2,3,4\}; (1)(2)(3,4)) \Big).$$

Apply now $\psi \circ \mathrm{Proj}_n$ for the above two expressions to get the following results in the center of the hyperoctahedral group algebra $Z(\mathbf{C}[\mathcal{B}_{2n}^2])$:

$$\mathbf{C}_{((1^n),\emptyset)} \mathbf{C}_{((1^{n-1}),(1))} = \mathbf{C}_{((1^{n-1}),(1))} \text{ for any } n \geq 3$$

and for any $n \geq 5$

$$\mathbf{C}_{((1^{n-2}),(2))} \mathbf{C}_{((1^{n-2}),(2))} = n(n-1)\mathbf{C}_{((1^n),\emptyset)} + 2\mathbf{C}_{((1^{n-2}),(1^2))} + 2\mathbf{C}_{((1^{n-4}),(2^2))} + 3\mathbf{C}_{((1^{n-3},3),\emptyset)}.$$

The first equation comes with no surprise since $C_{((1^n),\emptyset)}$ is the identity class.

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Classes of Sum-Decomposable Affine Permutations

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This talk is based on joint work with Neal Madras

In this talk, we characterize the affine permutation classes whose elements are all sum-decomposable. This characterization involves the infinite increasing oscillation. We also enumerate the sum-decomposable affine permutations in a class in terms of the corresponding ordinary permutation class, and we discuss the asymptotic number of sum-decomposable affine permutations in a class according to whether the class is supercritical or subcritical.

By the time you hear my talk, you hopefully will have heard about affine permutations from invited speaker Neal Madras. To refresh your memory: an *affine permutation* of size n is a bijection $\omega\colon \mathbb{Z} \to \mathbb{Z}$ such that

$$\omega(i+n) = \omega(i) + n \quad \text{for all } i \in \mathbb{Z}$$

and

$$\sum_{i=1}^{n} \omega(i) = \sum_{i=1}^{n} i \quad \text{(a “centering” condition).}$$

Here is a new definition: we say an affine permutation is *sum-decomposable*, or just *decomposable*, if it is a diagonal shift of an infinite direct sum $(\cdots \oplus \pi \oplus \pi \oplus \cdots)$ for some (ordinary) permutation π (see Figure 1 for an example). Being sum-decomposable

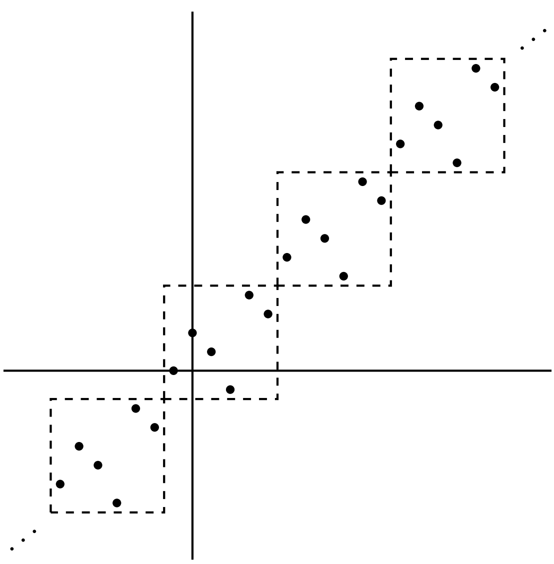

Figure 1: An affine permutation of size 6, whose values on $1, \ldots, 6$ are $[1, -1, 4, 3, 6, 8]$. It is sum-decomposable, because it is the infinite direct sum of the permutation 234165, shifted diagonally downwards by 2 units.

is a stronger condition than the boundedness condition from Neal's talk (an affine permutation ω of size n is *bounded* if $|\omega(i) - i| < n$ for all $i \in \mathbb{Z}$).

We can talk about pattern containment and pattern avoidance for affine permutations in much the same way as for ordinary permutations (after deciding a few minor details), and then we can define an *affine permutation class* as a set of affine permutations that is downwards closed in the containment order. If R is a set of affine or ordinary permutations, we let $\mathrm{AvA}(R)$ denote the class of affine permutations that avoid all the elements of R. Containing an ordinary permutation is a special case of containing an affine permutation: indeed, for an affine permutation ω and an (ordinary) permutation σ, ω contains σ if and only if ω contains the affine permutation $(\cdots \oplus \sigma \oplus \sigma \oplus \cdots)$.

The infinite oscillation and recognizing decomposability

A class of affine permutations may have the property that every element of the class is decomposable. We will say that a class with this property is *decomposable*. There is an easy way to tell whether a class is decomposable, involving an affine permutation called the *infinite (increasing) oscillation*. This is the permutation $\mathcal{O} = [3,0] = (\ldots, 1, -2, 3, 0, 5, 2, \ldots)$ shown in Figure 2. This permutation has made an

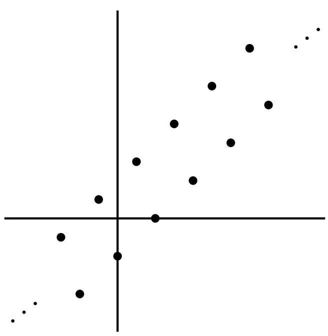

Figure 2: The infinite oscillation, $\mathcal{O} = [3,0] = (\ldots, 1, -2, 3, 0, 5, 2, \ldots)$.

appearance in research on antichains in the permutation containment order [1] and on growth rates of permutation classes [3]. Here it arises as the main obstruction to decomposability.

Theorem 1. *(a) An affine permutation is decomposable if and only if it avoids $\mathcal{O}$.*

(b) An affine permutation class is decomposable if and only if it does not have $\mathcal{O}$ as an element if and only if the class is a subset of $AvA(\mathcal{O})$.

If σ and τ are (ordinary) permutations and ω is an affine permutation, then ω contains $\sigma \oplus \tau$ if and only if ω contains both σ and τ. Thus, when considering affine permutations that avoid an ordinary permutation, we need only consider sum-indecomposable patterns. The sum-indecomposable permutations contained in $\mathcal{O}$, which we may call finite oscillations, are:

$$1, 21, 312, 231, 3142, 2413, 31524, 24153, 315264, 241635, \ldots$$

(there are two of each size after the first two). Thus, Theorem 1 tells us that, for a set R of ordinary permutations, $\mathrm{AvA}(R)$ is decomposable if and only if one of the elements of R has a block that is a finite oscillation.

Exact and asymptotic enumeration

Now that we can easily recognize when an affine permutation class is decomposable, we turn to the problem of enumerating the elements of a decomposable class. For the remainder of this abstract, let $\mathcal{C}$ be a *sum-closed* ordinary permutation class (meaning the direct sum of permutations in $\mathcal{C}$ is in $\mathcal{C}$). Define $\oplus\mathcal{C}$ to be the affine permutation class consisting of diagonal shifts of infinite direct sums $(\cdots \oplus \pi \oplus \pi \oplus \cdots)$ for $\pi \in \mathcal{C}$. We remark that, by Theorem 1, $\oplus\mathrm{Av}(R) = \mathrm{AvA}(R \cup \{\mathcal{O}\})$.

Let a_n, $\widetilde{a}_n$, and c_n be respectively the number of size-n permutations in $\mathcal{C}$, the number of size-n affine permutations in $\oplus\mathcal{C}$, and the number of size-n indecomposable permutations in $\mathcal{C}$. Define the generating functions

$$A(x) = \sum_{n\geq 0} a_n x^n \quad \text{and} \quad \widetilde{A}(x) = \sum_{n\geq 1} \widetilde{a}_n x^n \quad \text{and} \quad C(x) = \sum_{n\geq 1} c_n x^n.$$

Theorem 2. $\widetilde{A}(x) = x\, C'(x)\, A(x) = \dfrac{xA'(x)}{A(x)} = x\dfrac{d}{dx}\log(A(x)).$

Theorem 2 allows us to easily obtain the number of affine permutations in $\oplus\mathcal{C}$ from the number of ordinary permutations in $\mathcal{C}$. It also leads to:

Corollary 3. $\max\{a_n, nc_n\} \leq \widetilde{a}_n \leq na_n$.

In contrast, for $\mathcal{C} = \mathrm{Av}(321)$, the number of *bounded affine* permutations avoiding 321 is asymptotically $(n^2/2)a_n$, as seen in Neal's talk. Corollary 3 also implies that the exponential growth rate of $\oplus\mathcal{C}$ is the same as that of $\mathcal{C}$.

Time permitting, the talk will include asymptotic results for $\widetilde{a}_n$ that depend on whether $\mathcal{C}$ is subcritical, critical, or supercritical (see [2]). In the subcritical case we get $\widetilde{a}_n \sim \lambda n a_n$ for a constant λ, and in the supercritical case we get $\widetilde{a}_n \sim \lambda a_n$ for a constant λ.

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