Séminaire Lotharingien de Combinatoire 85B (2021)
Article #47, 12 pp.

Proceedings of the 33rd Conference on Formal Power Series and Algebraic Combinatorics (Ramat Gan)

Young tableaux with periodic walls: counting with the density method

Cyril Banderier*[1] and Michael Wallner†[2]

[1]*LIPN, University Sorbonne Paris Nord, Villetaneuse, France*
[2]*Institute of Discrete Mathematics and Geometry, TU Wien, Vienna, Austria*

Abstract. We consider a generalization of Young tableaux in which we allow some consecutive pairs of cells with decreasing labels, conveniently visualized by a "wall" between the corresponding cells. Some shapes can be enumerated by variants of hook-length type formulas. We focus on families of tableaux (like the so-called "Jenga tableaux") having some periodic shapes, for which the generating functions are harder to obtain. We get some interesting new classes of recurrences, and a surprisingly rich zoo of generating functions (algebraic, hypergeometric, D-finite, differentially-algebraic). Some patterns lead to nice bijections with trees, lattice paths, or permutations. Our approach relies on the density method, a powerful way to perform both random generation and enumeration of linear extensions of posets.

Keywords: Young tableaux, analytic combinatorics, generating functions, D-finite functions, hypergeometric functions, differentially-algebraic functions, random generation, density method, linear extensions of posets

1 Introduction

Counting the number of linear extensions of a poset is known to be a hard problem; it was even proven to be #P-complete by Brightwell and Winkler [9]. The enumeration is even still #P-complete when restricted to posets of height 2; see Dittmer and Pak [11]. This enumeration challenge is also strongly connected to the question of uniform random generation. While there exist thousands of *ad hoc approaches* to generate combinatorial structures (see, e.g., [18, 22]), there are few *generic methods* for their uniform random generation: one could name rejection algorithms and Markov chain sampling [17], the recursive method [15, 22], generating trees [2], and Boltzmann sampling [12]. Another important method that we want to promote and to add to this list is the *density method*. We will illustrate its power and its flexibility in this article by applying it to many different posets.

*https://lipn.fr/~banderier https://orcid.org/0000-0003-0755-3022
†https://dmg.tuwien.ac.at/mwallner/ https://orcid.org/0000-0001-8581-449X
Michael Wallner was supported by the Austrian Science Fund (FWF): J 4162 and P 34142.

What we call the density method is an appropriate combination of recurrences and integral representations of order polytope volumes in order to enumerate poset structures. For this reason, as suggested by one referee, it could also be called the *polytope volume method*. Some ancestors of this natural idea can be found in [5,7,13,23]. It should also be mentioned that several works by Stanley (see for example his nice survey [26]) contributed to propagating interest in this idea, e.g., in connection with variants of the enumeration of zig-zag permutations (permutations which have a periodic succession of rises and falls [1,8]); this led to the articles [6,19,21]. Together with Philippe Marchal, we further developed this density method in [3,4], as a way to analyse structures like permutations, trees, Young tableaux, all with additional order constraints on their labels.

In this article, we consider a generalization of Young tableaux by allowing two consecutive cells to have decreasing values. We put a bold red edge between the cells which are allowed (but not imposed) to be decreasing (we call these edges "walls"), and consider structures where the location of the walls obey some periodicity rules. More precisely, let a tableau $\mathcal{Y}$ with periodic walls be the concatenation (as shown in Figure 1) of n copies of a building block $\mathcal{B}$ of cells (i.e., $\mathcal{Y} = \mathcal{B}^n$) and then filled with all integers from $\{1, \ldots, |\mathcal{B}|n\}$ respecting the induced order constraints.

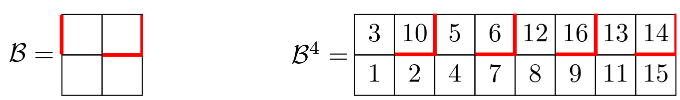
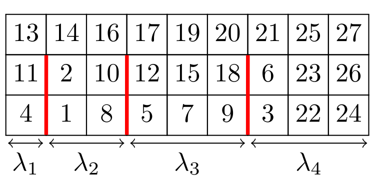


Figure 1: Left: example of a block $\mathcal{B}$ of shape 2×2. Right: a Young tableau with periodic walls at positions imposed by concatenations of $\mathcal{B}$.



Some of these tableaux are in bijection with other combinatorial structures. Just to give a small example, if all the λ_i's $= 1$, the tableaux on the left are in bijection with partitions of $\{1, \ldots, 3n\}$ into n sets of size 3.

In [3], we introduced rectangular Young tableaux with walls and explored their links with hook-length-like formulas, the Chung–Feller theorem, and studied their uniform random generation. In this article, we introduce other several families of tableaux with periodic walls to illustrate the rich diversity of the corresponding generating functions, and some of their unexpected closure properties.

Plan of the article: In Section 2, we consider the class of Young tableaux of shape $n \times 2$, where adding walls enrich known bijections with trees and lattice paths. In Section 3, we use the density method to enumerate certain poset structures (the new Jenga tableaux), which lead to unexpected closed forms, and sometimes to D-finite generating functions. In Section 4, we show that some simple classes of Young tableaux with periodic walls lead to complicated asymptotic formulas. In Section 5, we characterize (except for two cases) Young tableaux with periodic walls built of 2×2 blocks with respect to the nature of their counting sequences: simple product, algebraic, hypergeometric, or D-algebraic.

2 Young tableaux of shape $n \times 2$ and binary trees

In this section, just to illustrate a little bit more the diversity of combinatorial objects which can be related to tableaux with walls, we consider Young tableaux of shape $n \times 2$, allowing some walls between their two columns and map them bijectively to leaf-marked binary trees; see Figure 2. The following proposition gives a new combinatorial meaning to several OEIS[1] entries, such as A000108, A000984, A002457, A002802, and A020918.

Proposition 2.1. *The generating function of $n \times 2$ Young tableaux with k walls is equal to*

$$V_k(z) := \sum_{n\geq 0} v_{n,k} z^n = \frac{Cat(k-1)z^{k-1}}{(1-4z)^{(2k-1)/2}} \qquad \text{where} \qquad Cat(n) = \frac{1}{n+1}\binom{2n}{n},$$

and the corresponding generating polynomial with respect to the number of walls is

$$v_n(u) := \sum_{k=0}^{n} v_{n,k} u^k = Cat(n)((1+u)^{n+1} - u^{n+1}). \tag{2.1}$$

Proof. In [3, Theorem 2.1], using a link with Dyck paths and the Chung–Feller theorem, we proved that the number $v_{n,k}$ of $n \times 2$ Young tableaux with k vertical walls is equal to

$$v_{n,k} = \frac{1}{n+1-k}\binom{n}{k}\binom{2n}{n}.$$

The formula for $v_n(u)$ follows by summing $v_{n,k}$ with respect to n. What is more, a simple rewriting shows that $v_{n,k} = \frac{(n)_{k-1}}{k!}\binom{2n}{n}$ for $k \geq 1$. This shows

$$\sum_{n\geq 0} v_{n,k} z^n = \sum_{n\geq k-1} \frac{(n)_{k-1}}{k!}\binom{2n}{n} z^n = \frac{z^{k-1}}{k!}\frac{d^{k-1}}{dz^{k-1}} \sum_{n\geq 0}\binom{2n}{n} z^n = \frac{z^{k-1}}{k!}\frac{d^{k-1}}{dz^{k-1}}\frac{1}{\sqrt{1-4z}}. \qquad \square$$

It is noteworthy that $v_{n,k}$ is at the same time divisible by $\mathrm{Cat}(n)$ and $\mathrm{Cat}(k-1)$, and, obviously, (2.1) demands a simple combinatorial explanation. The following classical lemma will allow us to give a bijective explanation of all these facts.

Lemma 2.2. *Young tableaux of shape $n \times 2$ are in bijection with binary trees that have n internal nodes.*

Proof. The key observation is that every element in the first column corresponds to an internal node and every element in the second column to a leaf. For the bijection we iterate through the cells in increasing order. We start with an internal node for the entry 1. Depending on the column of m, we add an internal node or a leaf to the next available empty position in depth-first order. At the end, we add a leaf to the left-most branch of the root. $\square$

[1] OEIS stands for the On-Line Encyclopedia of Integer Sequences, accessible via https://oeis.org.

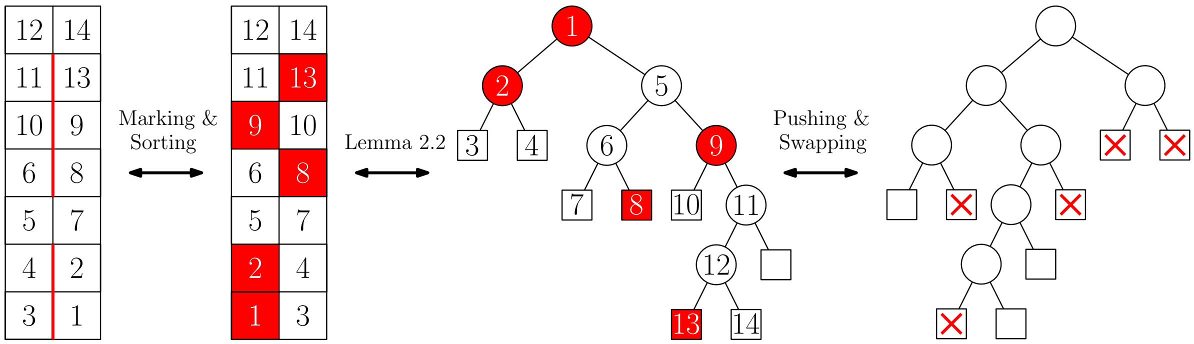


Figure 2: The bijection between $n \times 2$ Young tableaux with k walls and binary trees with k marked leaves from Theorem 2.3. Here $n = 14$ and $k = 5$.

Theorem 2.3. *Young tableaux of shape $n \times 2$ with k walls are in bijection with binary trees with n internal nodes and k marked leaves.*

Proof. The following bijection consists of (possibly) 3 steps and is shown on an example in Figure 2. First, we mark every entry in the second column that is in a row with a wall and remove the wall. Then, we sort each row to get a standard $n \times 2$ Young tableau (yet, with k markers).

Second, we transform this tableau together with its markers into a binary tree using Lemma 2.2. If no internal nodes are marked, then we are finished; yet if some internal nodes are marked, then we perform the following step.

Third, we inductively transform the binary tree with markers into a binary tree with marked leaves. Observe that if an internal node on the right-most branch of the root is marked, all internal nodes in the left subtree are marked as well, but no leaf. Vice versa, if in such a subtree at least one leaf is marked, no internal node is marked. This follows from the depth-first traversal of Lemma 2.2 as we only append a new internal node to the right-most branch of the root if the subtableau corresponding to the previous nodes is a valid Young tableau. Now, start from the right-most leaf in the right branch of the root and move upwards to the root. If an internal node is marked, push all markers to the leaves of the left subtree and thereafter swap the left and right subtree. Continue until you reach the root.

For the reverse bijection, we distinguish two cases: Either the right-most leaf is marked or not. If it is not marked we reverse only steps 1 and 2, while if it is marked, we reverse all three steps. $\square$

In the next section, we introduce the main tool of this article: the density method. We apply it on different variants of tableaux with walls, leading here to unexpectedly well-structured generating functions (e.g., hypergeometric or D-finite).

3 Jenga tableaux and the density method

The towers of the game Jenga[2] inspired the following fruitful generalization of Young tableaux. Consider a column of n cells to which one attaches at row i, ℓ_i cells to the left and r_i cells to the right. The $N := n + \sum_i^n \ell_i + r_i$ cells of this structure are then filled with the integers 1 to N under the constraint that each row and the middle column have increasing labels, and each label appears only once; see Figure 3.

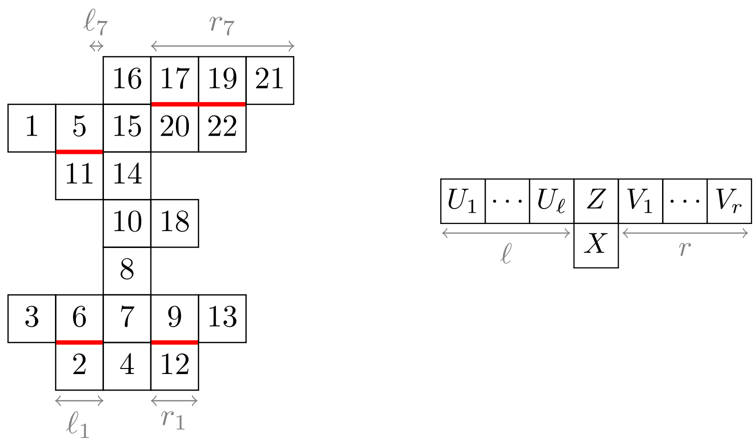


Figure 3: Left: a Jenga tableau with $n = 7$ rows and the left and right subsequences $(l_i)_{i=1}^7 = (1, 2, 0, 0, 1, 2, 0)$ and $(r_i)_{i=1}^7 = (1, 2, 0, 1, 0, 2, 3)$, respectively. Right: the building block used here in the density method to generate each row iteratively.

The density method is the key to enumerate such objects. We used it in [4, 20, 21] for other classes of tableaux. Let us sketch its principle on the example of Jenga tableaux.

The density method builds on a geometric interpretation of the problem. Consider an N-tuple α (with non-equal coordinates) that is an element of the hypercube $[0, 1]^N$. Then, we associate to each of these N coordinates one of the N cells of $\mathcal{Y}$: if the jth coordinate of α is the ith biggest element, then we assign the value i to the cell j. This filling is not (yet) respecting all increasing constraints, but this operation is readily reversed by associating to every legitimate filling of $\mathcal{Y}$ a region of $[0, 1]^N$ which corresponds to a polytope. The key observation now is that the volume of this polytope is equal to $1/N!$. Let $\mathcal{P}$ be the set of all polytopes corresponding to correct fillings of $\mathcal{Y}$. Then, a uniformly random element $\mathcal{P}$ corresponds to a uniformly random filling of $\mathcal{Y}$. Note that $\mathcal{P}$ is also known as the "order polytope" in poset theory.

We build now on this geometric viewpoint and describe how the density method works. Consider the generic building block of a row shown in Figure 3. It consists of the ℓ cells $U_1, \ldots, U_\ell$, the r cells $V_1, \ldots, V_r$, one cell Z, and one cell X. To each of these cells we assign a random number from $[0, 1]$. Then, we define a sequence of polynomials $f_n(z)$ which encode the order constraints satisfied by these cells up to row n:

[2]"Jenga!" means "Construct!" in Swahili. It is the name of a game created by Leslie Scott for his children in the 70s in which one dismantles block by block a tower of small wooden building blocks.

$$f_n(z) := \int\limits_{z<v_1<1} \dots \int\limits_{v_{r-1}<v_r<1} \int\limits_{0<u_\ell<z} \dots \int\limits_{0<u_1<u_2} \int\limits_{0<x<z} f_{n-1}(x)\, dx\, du_1 \dots du_\ell\, dv_r \dots dv_1.$$

Now the simple block structure of each row leads to the following simplification

$$f_n(z) = \frac{z^{\ell_n}(1-z)^{r_n}}{\ell_n! r_n!} \int_0^z f_{n-1}(x)\, dx \quad \text{and} \quad f_1(z) := \frac{z^{\ell_1}(1-z)^{r_1}}{\ell_1! r_1!}. \tag{3.1}$$

The crucial observation is now the following: The value $\int_0^1 f_n(z)\, dz$ is equal to the volume of the order polytope $\mathcal{P}$ associated to the correct fillings of $\mathcal{Y}_n$. Thus, $N! \int_0^1 f_n(z)\, dz$ is equal to the number of legitimate fillings. For more details see [4].

We thus get that the number y_n of Jenga tableaux with n rows is

$$y_n = \Big(\sum_{i=1}^{n} (\ell_i + r_i + 1) \Big)! \int_0^1 f_n(x)\, dx. \tag{3.2}$$

We now continue with some periodic patterns, that is if there exists an integer $p > 0$ such that $\ell_{i+p} = \ell_i$ and $r_{i+p} = r_i$ for all $i \geq 1$. The smallest such p is called the period. The simplest possible period is $p = 1$; this case leads to a noteworthy generating function.

Theorem 3.1 (D-finiteness of periodic Jenga tableaux with $p = 1$)**.** *The bivariate generating function $F(t,z) = \sum_{n \geq 1} f_n(z) t^n$ is D-finite in t and z. Accordingly, the counting sequence $(y_n)_{n\geq 1}$ given by Equation (3.2) of Jenga tableaux with n rows is P-recursive.*

Proof. In [3, Theorem 4.4] it was shown that $F(t,z)$ is D-finite in z for any periodic pattern with one hole. For the D-finiteness in t we use the density relations (3.1) and obtain

$$F(t,z) = t f_1(z) \exp\Big(t \int_0^z f_1(u)\, du \Big). \tag{3.3}$$

Then, taking the derivative with respect to t, we get that $F(t,z)$ is also D-finite in t:

$$t F_t(t,z) - \Big(1 + t \int_0^z f_1(u)\, du \Big) F(t,z) = 0.$$

Hence, by closure properties (Hadamard product and integration; see, e.g., [25]), the corresponding sequence $(y_n)_{n\geq 1}$ of Jenga tableaux with n rows is P-recursive. □

Note that set partitions of equal set sizes fall into the class of Theorem 3.1 as $\ell_i = m - 1$ and $r_i = 0$ for all $i \geq 0$. Let us also mention the following unexpected link.

Remark 3.2 (Link with Sheffer sequences)**.** *Considering the series expansion of $F(t,z)$ in z instead of t, Equation (3.3) shows that we have here some variant of Borel transform[3] of Sheffer sequences. Sheffer sequences are sequences of polynomials $\widehat{f}_n(t)$ having an exponential generating function of shape $\sum_{n=0}^{\infty} \widehat{f}_n(t) \frac{z^n}{n!} = A(z) \exp(tB(z))$. They play an important rôle in umbral calculus; see [24] and [25, Exercise 5.37].*

[3] The Borel transform (or the "inverse Laplace transform") of a sequence (a_n) is the sequence $(a_n/n!)$.

As further examples of Jenga shapes, the density method also gives:

Proposition 3.3. *For $r_i = 0$ for all $i \geq 1$ (see Figure 3), the number y_n of Jenga tableaux satisfies*

$$y_n = \frac{(\sum_{i=1}^{n}(\ell_i+1))!}{\prod_{i=1}^{n} \ell_i! (\sum_{j=1}^{i}(\ell_j+1))}.$$

Specializing these tableaux to periodic cases leads to some hypergeometric formulas.

Proposition 3.4. *Consider Jenga tableaux with period p, arbitrary left sequence $(\ell_i)_{i=0}^{p}$, and right sequence $(r_i)_{i=0}^{p} = (0,\dots,0)$ (see Figure 3). Define $L := \sum_{i=1}^{p} \ell_i$. Then, the number y_n of such tableaux satisfies*

$$y_{kp+m} = y_m \left(\frac{(L+p)^L}{\prod_{i=1}^{p} \ell_i!} \right)^k \prod_{\substack{j=1 \\ j \neq \ell_1+\cdots+\ell_i+i}}^{L+p} \frac{\Gamma\left(k+\frac{j+m}{L+p}\right)}{\Gamma\left(\frac{j+m}{L+p}\right)}.$$

Accordingly, the generating function of such tableaux is the sum of p hypergeometric functions.

It is also possible to consider other shapes, such as skew Young tableaux. Next, we give such an example and thus add walls to a model analysed in [5].

Proposition 3.5. *Consider tableaux with periodic walls in a diagonal strip of width w between each column in all but the top cell; see Figure 4. Let $b_{w,n}$ be the number of such tableaux with n columns; one has*

$$b_{w,n} = \left(\frac{w^{w-2}}{(w-2)!} \right)^n \prod_{j=1}^{w-2} \frac{\Gamma\left(n+\frac{j}{w}\right)}{\Gamma\left(\frac{j}{w}\right)}.$$

Proof. The formula is obtained by a bijection (depicted in Figure 4) between this class and periodic Jenga tableaux of period $p=2$, $\ell_1 = w-2$, $\ell_2 = 0$, and $r_i = 0$, such that $b_{w,n} = a_{2n}$. □

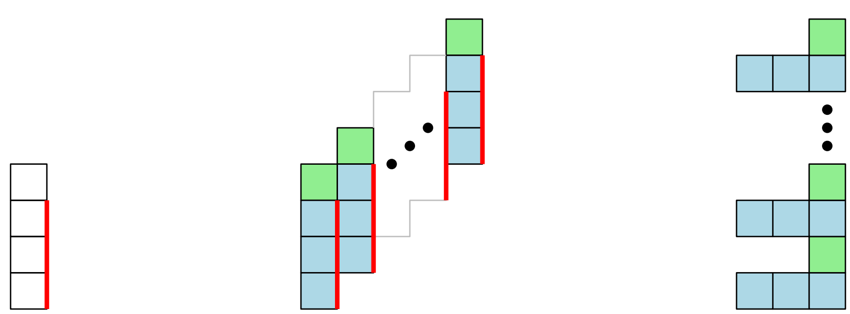


Figure 4: The building block of width 4 (left) is repeated k times and each time shifted up by one cell to form a Young tableau with periodic walls in a diagonal strip (centre). These tableaux are in bijection with periodic Jenga tableaux with period $p=2$, left sequence $(\ell_i)_{i=1}^{2} = (2,0)$, and right sequence $(r_i)_{i=1}^{2} = (0,0)$ (right).

4 Some unusual asymptotics

The density method can also be used to count and generate objects which do not have simple counting formulas. We now present such a class, which is a priori quite simple, but which however leads to rather surprising asymptotics. Thus, this class illustrates well the non-intuitive asymptotic behaviour of our objects.

6	10	14	15	17	18
3	5	9	12	13	16
2	1	7	4	11	8

	S	Y
R	X	

Figure 5: A $3 \times n$ Young tableau with walls in its first row, and the corresponding building block for each column used in the density method.

Theorem 4.1. *The number a_n of Young tableaux of length n with shape given by Figure 5 has the following asymptotics*

$$a_n = \Theta\left(n!\, 12^n e^{a_1 (3n)^{1/3}} n^{-2/3}\right), \tag{4.1}$$

where $a_1 \approx -2.338$ is the largest root of the Airy function of the first kind.

Proof (sketch). The increasing label constraints encoded in the building block of Figure 5 directly translate to the following densities

$$f_{n+1}(x,y) := x \int_0^x \int_x^y f_n(r,s)\, ds\, dr \qquad \text{and} \qquad f_1(x,y) := x(y-x).$$

Accordingly, as the initial configuration corresponding to f_1 consists of a building block without the cell R, the number of tableaux is

$$a_n = (3n+1)! \int_0^1 \int_0^y f_n(x,y)\, dx\, dy.$$

This gives the sequence `OEIS A213863`: $\{1, 7, 106, 2575, 87595, 3864040, 210455470, \dots\}$. It also counts words where each letter ℓ of the n-ary alphabet occurs 3 times and for each prefix p one has $|p|_\ell = 0$ or $|p|_\ell \geq |p|_j$ for all $j > \ell$, where $|p|_\ell$ counts the occurrences of ℓ in p. The bijection with our tableaux follows by mapping indices to rows. Formula (4.1) is then obtained by using the methods introduced in [14], i.e., sandwiching a_n between two sequences having the same asymptotics dictated by the first zero of a D-finite function (here, the Airy function satisfying $y'' - xy = 0$; see [16]). □

5 A classification of 2×2 periodic shapes

We now consider Young tableaux made of the concatenation of 2×2 blocks with walls (see Figure 1 in Section 1). This model is interesting as it leads to rather different natures of generating functions. Indeed, Table 1 hereafter summarizes the main results and groups them into four classes according to their counting sequences: simple products, algebraic, hypergeometric, or D-algebraic. Surprisingly, some of these sequences connect with classical combinatorial objects!

There are 6 possible non-trivial locations for walls in a 2×2 block (due to possible coincidences of the walls on the right when the blocks are concatenated). Thus, there are in total $2^6 = 64$ different types of building blocks. Most of these blocks come in pairs, as rotating a tableau by 180 degrees and reversing the labels gives a bijection.

First, one gets 40 blocks for which the walls create independent regions. This leads to 19 distinct sequences P1–P19, all having a simple product formula.

Second, we consider the 4 blocks without vertical walls. They lead to 3 distinct sequences A1–A3, which all have an algebraic generating function. For A1 and A2 the proof uses a bijection to Dyck paths. For A3 we decompose at the first wall that cannot be removed and get the recurrence $a_n = \mathrm{Cat}(2n) + \sum_{i=1}^{n} \mathrm{Cat}(2i-1) a_{n-i}$, which we then solve with generating functions.

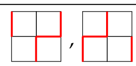
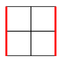
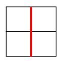
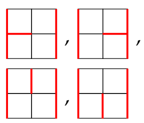
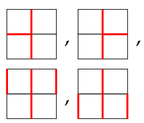
Third, we consider 14 blocks with a uniquely determined minimum or maximum. They lead to 7 distinct sequences H1–H7, all hypergeometric. The models H1–H5 are Jenga-like tableaux from Section 3 that satisfy $l_i = 0$ for all i. For the models H6 and H7 we use a recursive approach, decomposing with respect to the location of the unique minimum or maximum.

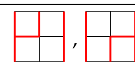
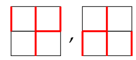
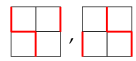
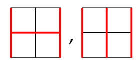
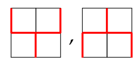
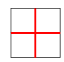
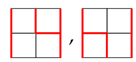
Fourth, there are three blocks which show a zig-zag-like pattern. By analogy to the known zig-zag permutations, we conjecture Z2 and Z3 to be non-D-finite. For Z1 we are able to prove that the exponential generating function is D-algebraic, and not D-finite, i.e., it satisfies a non-linear differential equation and no linear one. For this purpose we use Carlitz' theory [10] of generalized alternating permutations. Let $k_1, k_2, \dots, k_m$ be positive integers such that $k_1 + k_2 + \cdots + k_m = n$. Then, a generalized alternating permutation of type $(k_1, \dots, k_m)$ is an n-tuple $(a_1, \dots, a_n)$ such that $a_i \in \{1, \dots, n\}$ and

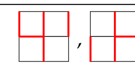
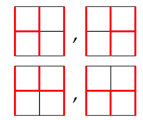
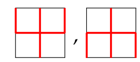
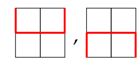
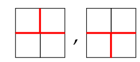
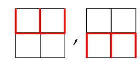
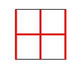
$$a_1 < \cdots < a_{k_1} > a_{k_1+1}, \quad a_{k_1+1} < \cdots < a_{k_1+k_2} > a_{k_1+k_2+1}, \quad \dots \quad a_{k_1+\cdots+k_{m-1}+1} < \cdots < a_n.$$

The type of a classical alternating permutation is thus $k_1 = \cdots = k_m = 1$, while the type of a tableau from Z1 with $m-1$ blocks is $k_1 = 3$, $k_2 = \cdots = k_{m-1} = 4$, and $k_m = 1$. Then, the claimed closed form of $A(t) = \sum_{n \geq 0} a_n \frac{t^n}{n!}$ for Z1 follows from a generalization of [10, Equation (1.11)] taking into account the different behaviours at the beginning *and* the end of the permutation. Thus, we get

$$A(t) = \frac{F_{4,3}(t) F_{4,1}(t)}{F_{4,0}(t)} + F_{4,0}(t) \qquad \text{where} \qquad F_{k,r}(t) = \sum_{n \geq 0} (-1)^n \frac{t^{nk+r}}{(nk+r)!}.$$

Class	Shape	Formula
P1		$4\frac{(4n)!}{24^n}$
P2		$\frac{(4n)!}{12^n}$
P3		$3\frac{(4n)!}{12^n}$
P4		$\frac{(4n)!}{8^n}$
P5		$4\frac{(4n)!}{8^n}$

Class	Shape	Formula
P6		$\frac{(4n)!}{6^n}$
P7		$3\frac{(4n)!}{6^n}$
P8		$\frac{8}{5}(4n)!\left(\frac{5}{24}\right)^n$
P9		$\frac{(4n)!}{4^n}$
P10		$2\frac{(4n)!}{4^n}$
P11		$4\frac{(4n)!}{4^n}$
P12		$\frac{(4n)!}{3^n}$

Class	Shape	Formula
P13		$6\frac{(4n)!}{3^n}$
P14		$\frac{(4n)!}{2^n}$
P15		$2\frac{(4n)!}{2^n}$
P16		$\frac{(4n)!}{(2n)!2^n}$
P17		$2\frac{(4n)!}{(2n)!2^n}$
P18		$\frac{(4n)!}{(2n)!}$
P19		$(4n)!$

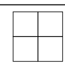
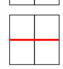
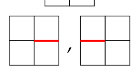
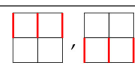
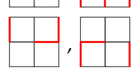
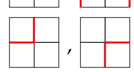
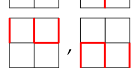
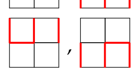
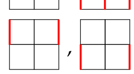
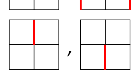
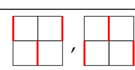
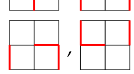
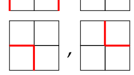
Class	Shape	Sequence	OEIS
A1		$\mathrm{Cat}(2n) = \frac{1}{2n+1}\binom{4n}{2n}$	A048990
A2		$\binom{4n}{2n}$	A001448
A3		$2^{2n+1}\,\mathrm{Cat}(n) - \mathrm{Cat}(2n+1)$	A079489
H1		$\prod_{i=1}^{n}(4i-1)(4i-3)$	A101485
H2		$\prod_{i=1}^{n}(2i-1)(4i-1)$	A159605
H3		$2^{n+1}n!\prod_{i=1}^{n}(4i-3)$	$2^{n+1}\cdot$A084943
H4		$\binom{4n}{n}\prod_{i=1}^{n}(3i-1)$	$\binom{4n}{n}\cdot$A008544
H5		$\binom{4n}{n}\prod_{i=1}^{n}(3i-2)$	$\binom{4n}{n}\cdot$A007559
H6		$2^n n!\prod_{i=1}^{n}(4i-3)$	$n!\cdot$A084948
H7		$\prod_{i=1}^{n}(2i-1)(4i-1)$	A159605
Z1		$\frac{\cos(t/\sqrt{2})^2+\cosh(t/\sqrt{2})^2}{2\cos(t/\sqrt{2})\cosh(t/\sqrt{2})}$	related to A211212
Z2		?	???
Z3		?	???

Table 1: The 64 different models of 2×2 blocks for tableaux with periodic walls grouped into 4 different classes: (P) simple products, (A) algebraic, (H) hypergeometric, (Z) zig-zag. The length n is equal to the number of repeated blocks. The model Z1 is D-algebraic and not D-finite, which is what we conjecture for the models Z2 and Z3.

A pleasant feature of the density method approach is that it is automatable. See our `Jenga` Maple package dedicated to the enumeration of tableaux with walls, thus allowing our readers to play with the examples of their choice!

In conclusion, we have seen that Young tableaux with walls are a rich model, leading (via the density method) to new varieties of recurrences, interesting *per se*, mixing finite differences and differential operators (challenging the current state of the art in computer algebra and holonomy theory!), and surprising asymptotics (challenging the current state of the art in analytic combinatorics!).

Acknowledgments: We thank the two anonymous referees for their kind feedback.

References

[1] Désiré André. Sur les permutations alternées. *Journal de mathématiques pures et appliquées*, 7:167–184, 1881. Long version of several articles by André in 1879.

[2] Cyril Banderier, Mireille Bousquet-Mélou, Alain Denise, Philippe Flajolet, Danièle Gardy, and Dominique Gouyou-Beauchamps. Generating functions for generating trees. *Discrete Math.*, 246(1-3):29–55, 2002. A preliminary version appeared in the proceedings of FPSAC'99.

[3] Cyril Banderier, Philippe Marchal, and Michael Wallner. Rectangular Young tableaux with local decreases and the density method for uniform random generation. In *GASCom 2018, CEUR Workshop Proceedings. Vol. 2113.*, pages 60–68, 2018.

[4] Cyril Banderier, Philippe Marchal, and Michael Wallner. Periodic Pólya urns, the density method and asymptotics of Young tableaux. *Annals of Probability*, 48(4):1821–1965, 2020.

[5] Yuliy Baryshnikov and Dan Romik. Enumeration formulas for Young tableaux in a diagonal strip. *Israel J. Math.*, 178:157–186, 2010.

[6] Nicolas Basset. Counting and generating permutations in regular classes. *Algorithmica*, 76(4):989–1034, 2016.

[7] Frits Beukers, Johan A. C. Kolk, and Eugenio Calabi. Sums of generalized harmonic series and volumes. *Nieuw Arch. Wiskd., IV. Ser.*, 11(3):217–224, 1993.

[8] Miklós Bóna. *Combinatorics of Permutations*. CRC/Chapman & Hall, 3rd edition, 2022.

[9] Graham Brightwell and Peter Winkler. Counting linear extensions. *Order*, 8(3):225–242, 1991. A preliminary version appeared in the proceedings of STOC'91.

[10] Leonard Carlitz. Permutations with prescribed pattern. *Math. Nachr.*, 58:31–53, 1973.

[11] Samuel Dittmer and Igor Pak. Counting linear extensions of restricted posets. *Electron. J. Comb.*, 27, 2020.

[12] Philippe Duchon, Philippe Flajolet, Guy Louchard, and Gilles Schaeffer. Random sampling from Boltzmann principles. In *Automata, languages and programming. Proceedings of ICALP 2002.*, pages 501–513. Springer, 2002.

[13] Noam D. Elkies. On the sums $\sum_{k=-\infty}^{\infty}(4k+1)^{-n}$. *Amer. Math. Monthly*, 110(7):561–573, 2003.

[14] Andrew Elvey Price, Wenjie Fang, and Michael Wallner. Compacted binary trees admit a stretched exponential. *J. Combin. Theory Ser. A*, 177:105306, 40, 2021.

[15] Philippe Flajolet, Paul Zimmermann, and Bernard van Cutsem. A calculus for the random generation of labelled combinatorial structures. *Theor. Comput. Sci.*, 132(1-2):1–35, 1994.

[16] Michael Fuchs, Guan-Ru Yu, and Louxin Zhang. On the asymptotic growth of the number of tree-child networks. *Eur. J. Comb.*, 93:103278, 20, 2021.

[17] Mark Jerrum. *Counting, Sampling and Integrating: Algorithms and Complexity*. Birkhäuser, 2003.

[18] Donald E. Knuth. *The Art of Computer Programming. Volume 4A. Combinatorial Algorithms, Part 1.* Addison-Wesley, 2011.

[19] Jean-Marc Luck. On the frequencies of patterns of rises and falls. *Physica A*, 407:252–275, 2014.

[20] Philippe Marchal. Rectangular Young tableaux and the Jacobi ensemble. *Discrete Math. Theor. Comput. Sci. Proceedings*, BC:839–850, 2016.

[21] Philippe Marchal. The density method and permutations with a prescribed descent set. In *GASCom 2018, CEUR Workshop Proceedings. Vol. 2113.*, pages 179–186, 2018.

[22] Albert Nijenhuis and Herbert S. Wilf. *Combinatorial Algorithms for Computers and Calculators*. Academic Press, 1978. 2nd ed. (1st edition in 1975).

[23] Igor Pak. Hook length formula and geometric combinatorics. *Sém. Lothar. Combin.*, 46:Art. B46f, 13, 2001.

[24] Gian-Carlo Rota. *Finite Operator Calculus. With the collaboration of P. Doubilet, C. Greene, D. Kahaner, A. Odlyzko and R. Stanley*. Academic Press, 1975.

[25] Richard P. Stanley. *Enumerative Combinatorics. Volume 2*. Cambridge University Press, 1999.

[26] Richard P. Stanley. A survey of alternating permutations. In *Combinatorics and graphs*, pages 165–196. American Mathematical Society (AMS), 2010.