Lattice Path Conference
21-25 June 2021
Presentation times for this poster:
Wednesday 1:30-2:30 pm
Thursday 6-7 pm

Patterns in Shi tableaux and Dyck paths

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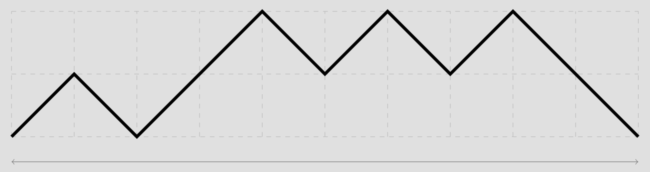
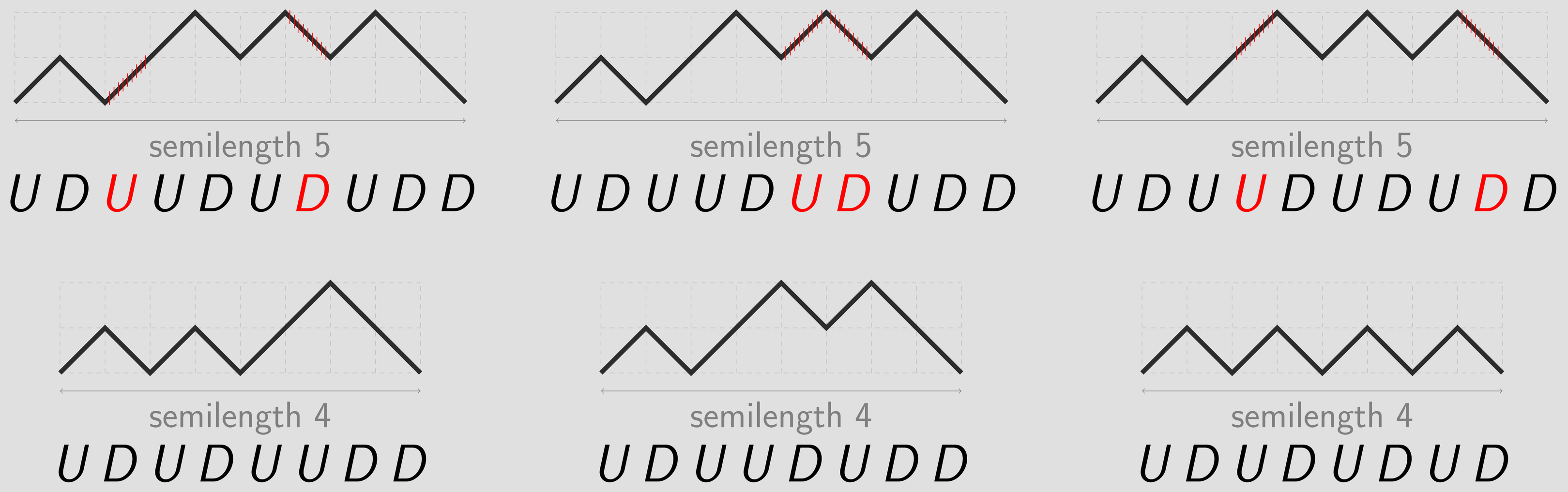
Dyck paths

- Dyck path of semilength n: a path on the plane from $(0,0)$ to $(2n,0)$ consisting of up-steps $(1,1)$ and down-steps $(1,-1)$ that never go below the x-axis



- There are many ways to define poset structures on Dyck paths

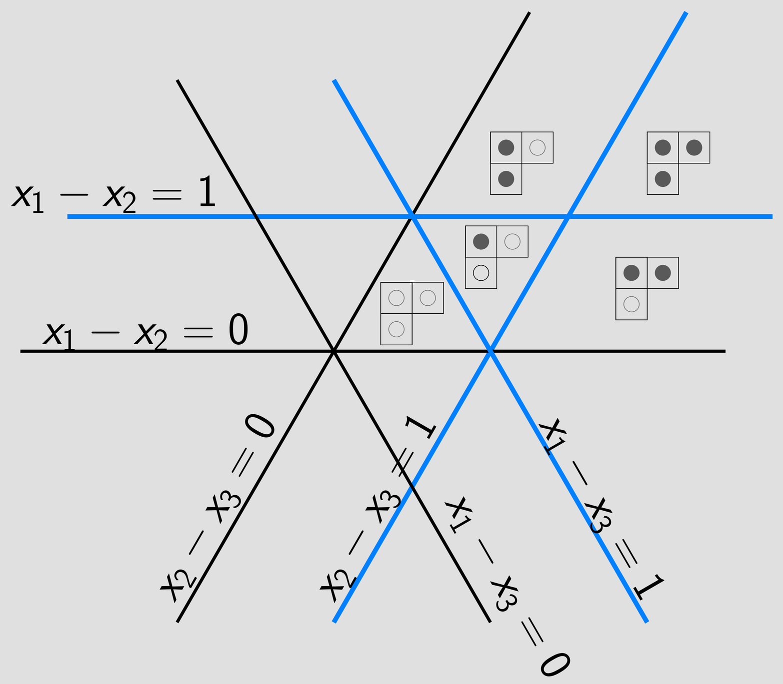


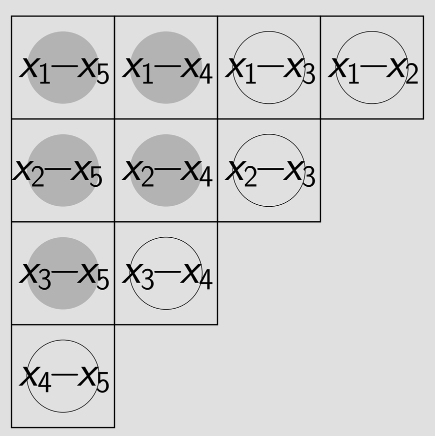
Shi Tableaux

- Shi arrangement for the root system $A_n = \mathcal{S}_{n+1}$

$$\mathcal{H} = \{x_i - x_j = 0, 1 : 1 \leq i < j \leq n+1\} \text{ in } \mathbb{R}^{n+1}$$

- regions: connected components of $\mathbb{R}^{n+1} \setminus \mathcal{H}$
- dominant regions: regions in $\bigcap_{1 \leq i < j \leq n} \{\mathbf{x} : x_i - x_j \geq 0\}$

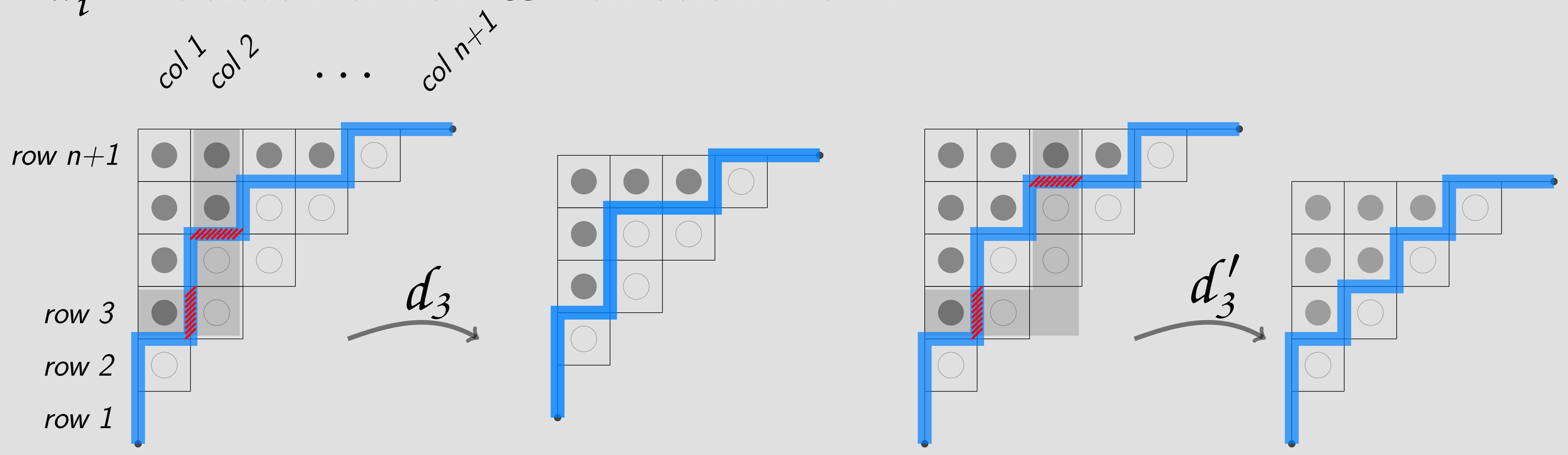




$0 \leq x_i - x_j \leq 1$ if the cell (i,j) has ●
$1 \leq x_i - x_j$ otherwise

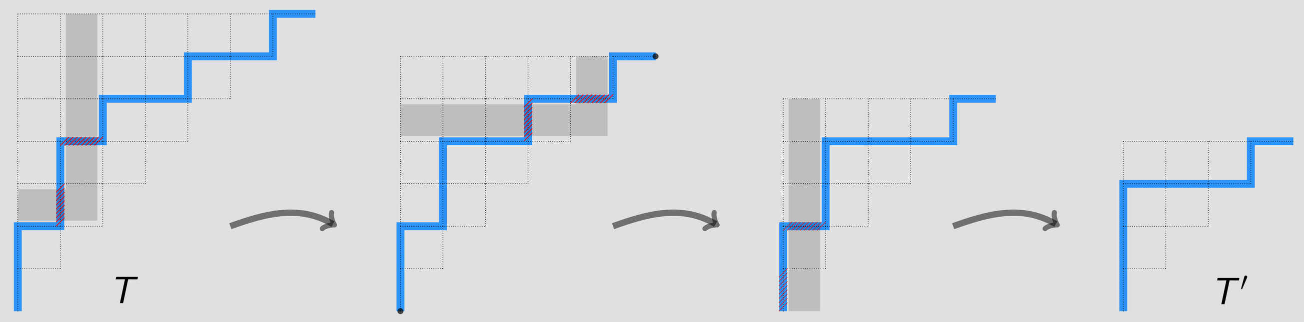
A new poset structure on the set of Shi tableaux (cover relations)

- ➠ d_i = delete i-th row & $(i-1)$-st column of T
- ➠ d'_i = delete i-th row & i-th column of T

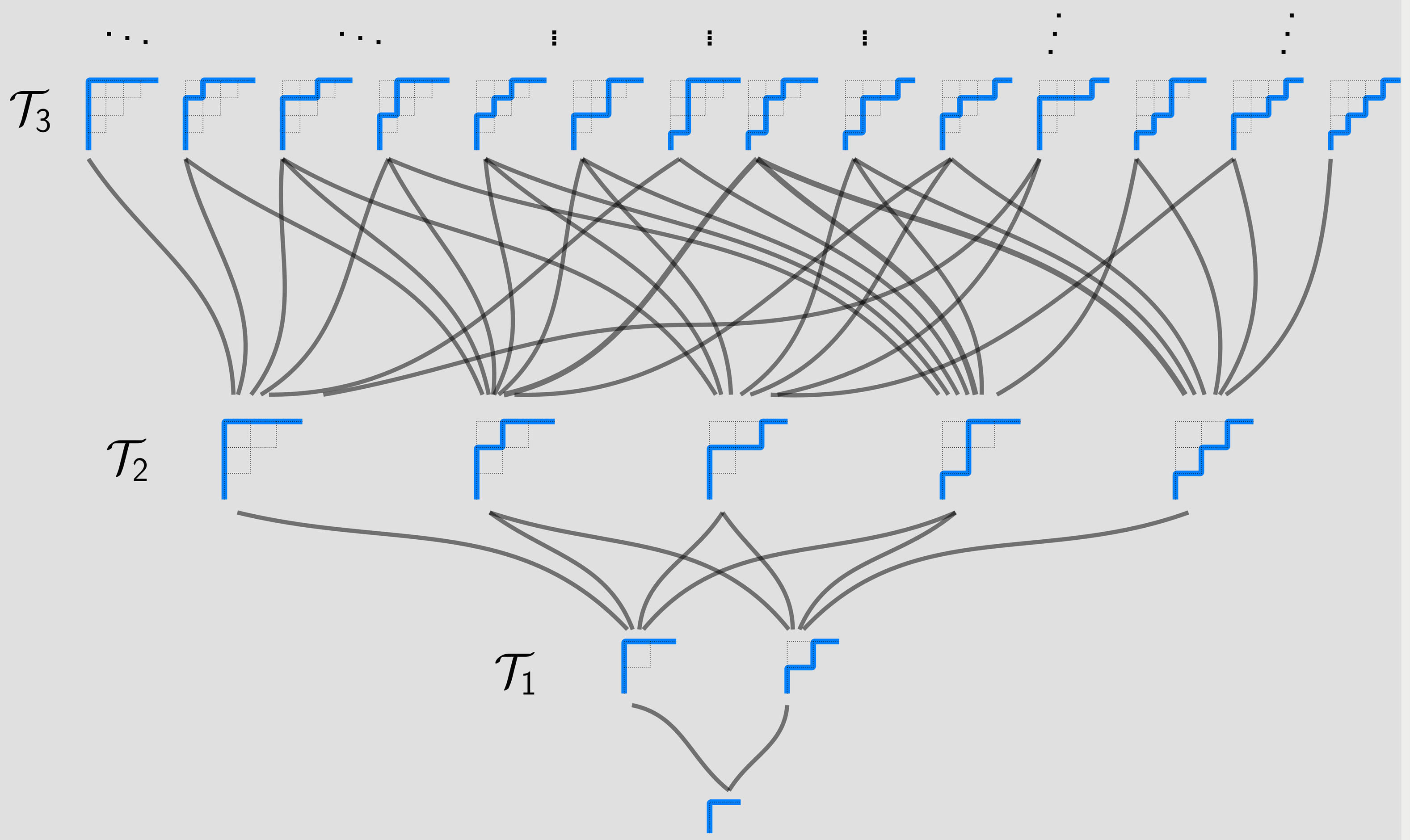


A new poset structure on the set of Shi tableaux

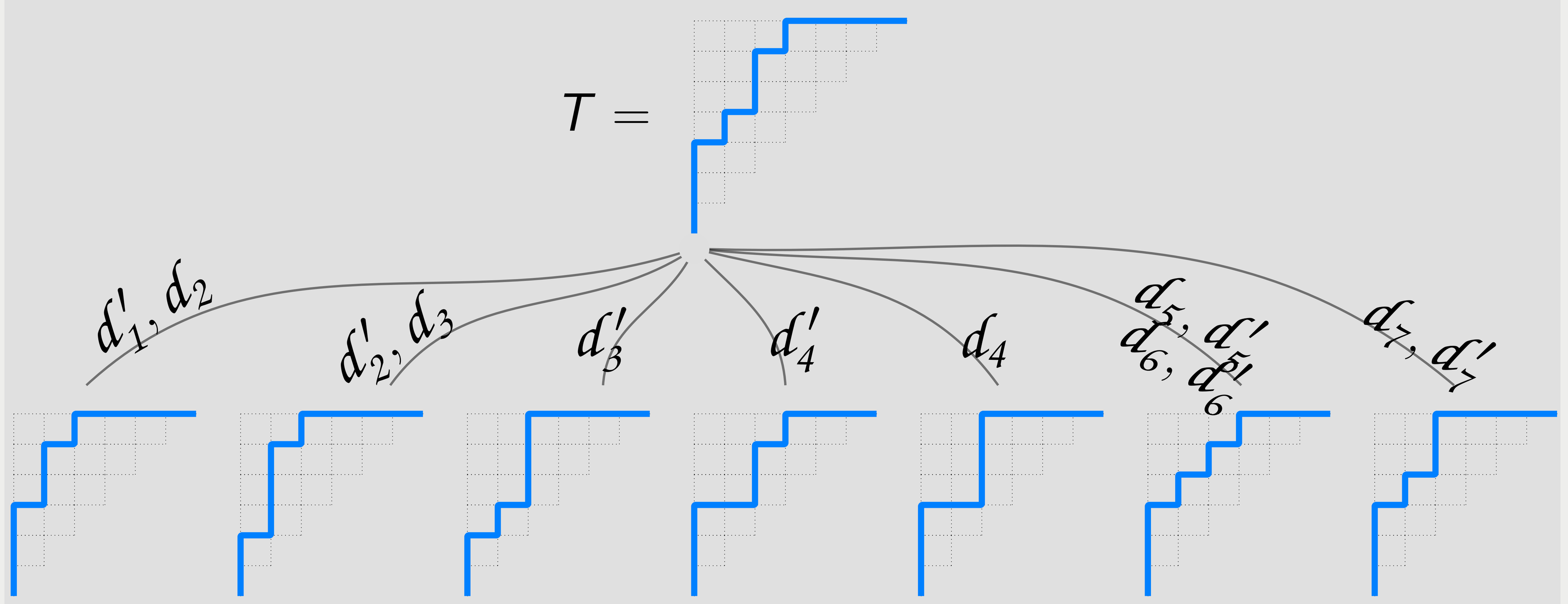
- $T' \preceq T$ if T' is obtained from T after sequence of deletions d_i, d'_j.



The first few levels of the poset



Upper and lower covers



Theorem: If T is strongly irreducible then

$$|\text{lower covers}(T)| = |peaks(T)| + |valleys(T)|$$
$$|\text{upper covers}(T)| = \text{closed expression depending on } T$$

Theorem: If T is not strongly irreducible we can compute $|\text{lower covers}(T)|/|\text{lower covers}(T)|$ w.r.t. the upper/lower covers of its irreducible components.

Patern avoidance

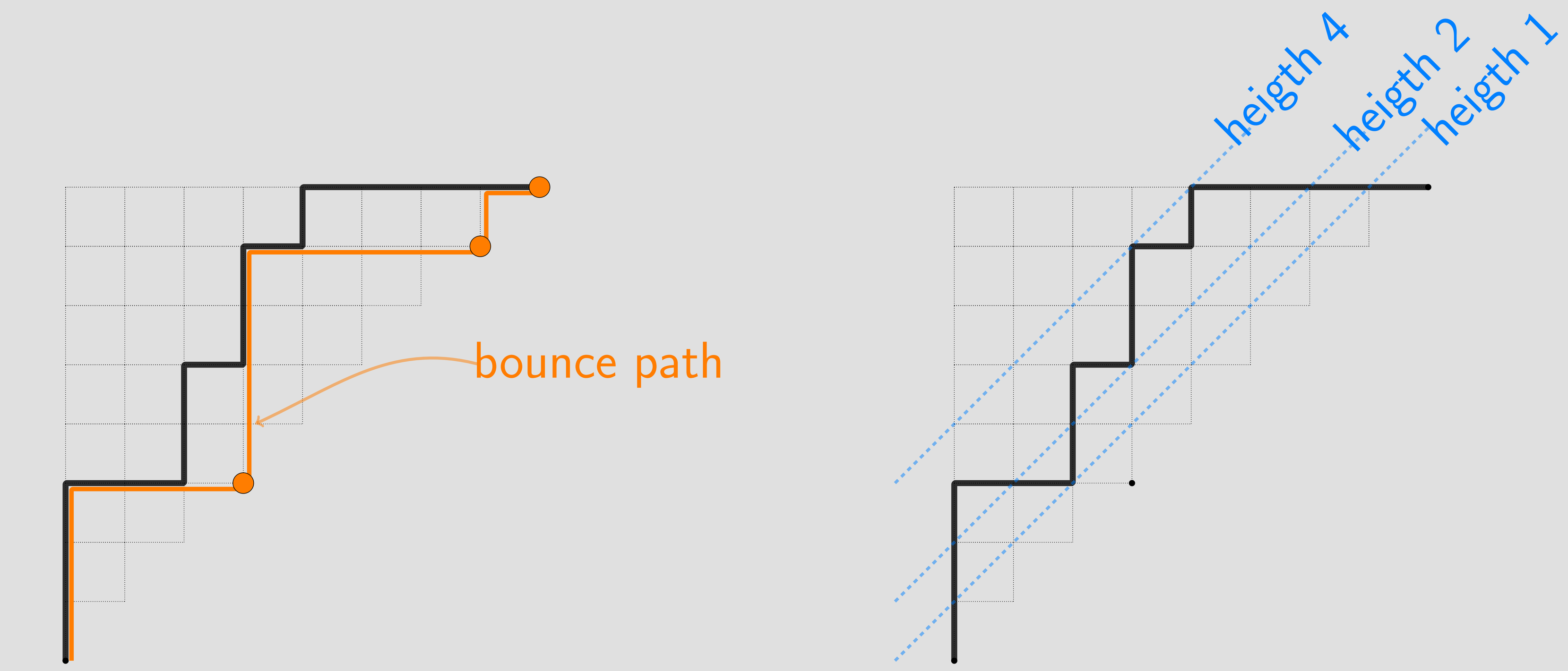
Notation: $T^k_{\circ\circ} = U^{k+1}D^{k+1}$, $T^k_{\bullet\bullet} = (UD)^{k+1}$, $T^k_{\bullet\circ} = U^kDUD^k$

Theorem

① $T \in \mathcal{T}_n$ avoids $T^k_{\bullet\bullet}$ iff its *bounce path* has at most k *return points*

② T avoids $T^k_{\circ\circ}$ iff its *height is at most 2*

③ T avoids $T^k_{\bullet\circ}$ iff it has *no valley at height* $\geq k$



Shi tableaux of type A	Sequence	OEIS[...]	Pairs of permutations
$\lvert Av_n(T_{\circ\circ})\rvert, \lvert Av_n(T_{\bullet\bullet})\rvert$	2^n	A000079	$\lvert Av_{n+1}(132,123)\rvert$
$\lvert Av_n(T_{\bullet\circ})\rvert, \lvert Av_n(T_{\bullet\circ})\rvert, \lvert Av_n(T_{\bullet\bullet})\rvert$	$\binom{n}{2}+n+1$	A000124	$\lvert Av_{n+1}(132,321)\rvert$
$\lvert Av_n(T^k_{\circ\circ})\rvert, \lvert Av_n(T^k_{\bullet\bullet})\rvert, \lvert Av_n(T^k_{\bullet\circ})\rvert$	$\lvert\mathcal{H}(n+1,k)\rvert$	A080934	$\lvert Av_{n+1}(132, 12\ldots k)\rvert$

Open problems

① Find an explanation for (or quantify) the above phenomenon linking pattern avoidance in Shi tableaux to permutations avoiding a pair of patterns.

② The actions d_i, d'_i have a strong resemblance to the deletion and contraction operations on hyperplane arrangements. Could our poset structure explained in this setting?

③ Study the poset of Shi tableaux, its Möbius function and topology

References

[1] A. Bacher, A. Bernini, L. Ferrari, B. Gunby, R. Pinzani, and J. West, The Dyck pattern poset, Discrete Math. 321 (2014), 12–23

[2] C.Krattenthaler, Permutations with Restricted Patterns and Dyck Paths, Adv. Appl. Math. 27, Issues 2–3, (2001), 510–530

[3] A. Spiridonov, Pattern-avoidance in binary fillings of grid shapes, Discrete Math. Theor. Comput. Sci. Proc. (2008),677–690