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Presentation times for this poster:
Wednesday 1:30–2:30 pm
Thursday 6–7 pm

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Down-step statistics in generalized Dyck paths

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A family of generalized Dyck paths

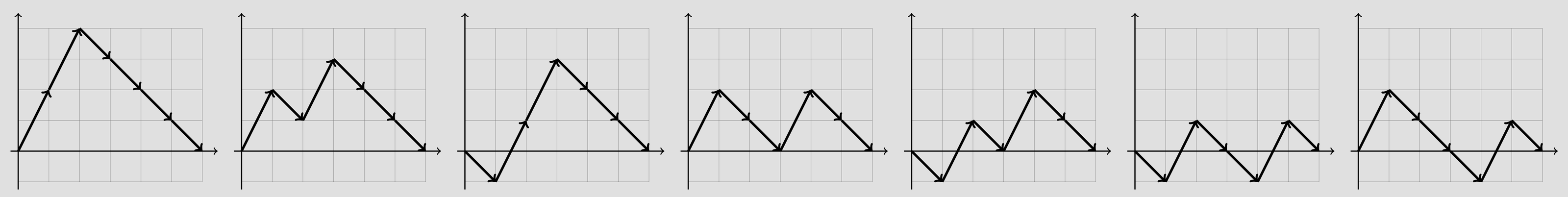
k_t-Dyck paths

- Let $k \in \mathbb{N}$ and $0 \le t \le k-1$, $t \in \mathbb{Z}$.
- Step set: $\mathcal{S} = \{(1,k), (1,-1)\}$.
- Length $(k+1)n$: starts at $(0,0)$, ends at $(0,(k+1)n)$, $n \in \mathbb{N}$.
- Restrictions: paths stay (weakly) above $y = -t$.

Down-steps between pairs of up-steps

$s_{n,t,r}$ is the total number of down-steps between the r-th and the $(r+1)$-th up-steps in all k_t-Dyck paths of length $(k+1)n$.

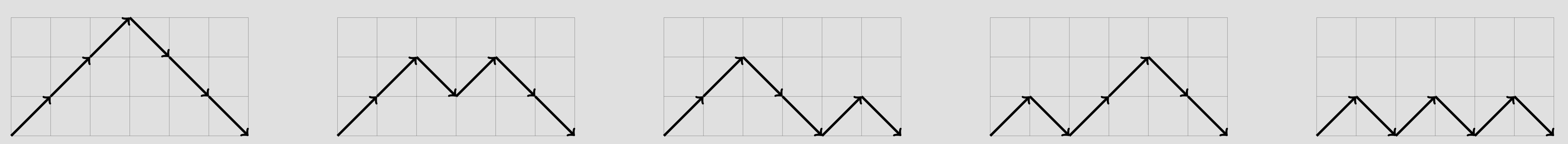
For the seven 2_1-Dyck paths of length 6:
$s_{2,1,0} = 3$ $s_{2,1,1} = 9$ $s_{2,1,2} = 16$



(0,0,4) (0,1,3) (1,0,3) (0,2,2) (1,1,2) (1,2,1) (0,3,1)

Coding theory connection

- Error correcting codes introduce redundancy to allow for error correction.
- Some of the redundant bits in a convolutional code can be removed without (i.e. punctured convolutional codes).
- Binary matrices are used to represent which bits are removed/kept.
- Two binary matrices produce the same output if they are obtainable from each other by a cyclic shift of columns.
- Equivalence classes of $(k+1) \times n$ binary matrices with $n+1$ ones are bijective to k_t-Dyck paths of length $(k+1)n$, with $0 \le t \le k-1$.
- The special case of $k = 1$ leads to a new interpretation of the Catalan numbers in terms of equivalence classes of binary matrices.



$\begin{pmatrix}1&1&0\\1&1&0\end{pmatrix}$ $\begin{pmatrix}1&1&1\\1&0&0\end{pmatrix}$ $\begin{pmatrix}1&1&0\\1&0&1\end{pmatrix}$ $\begin{pmatrix}1&0&1\\1&1&0\end{pmatrix}$ $\begin{pmatrix}1&0&0\\1&1&1\end{pmatrix}$

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Enumerative results

Let $0 \le r \le n$ be a non-negative integer, then:

1. The total number of down-steps before the first up-step in all k_t-Dyck paths of length $(k+1)n$ is given by
$$s_{n,t,0} = \sum_{j=0}^{t-1} \frac{j+1}{(k+1)n+j+1}\binom{(k+1)n+j+1}{n}.$$
2. The total number of down-steps between the first and second up-step in all k_0-Dyck paths is given by
$$s_{n,0,1} = \frac{k}{n}\binom{(k+1)(n-1)}{n-1}.$$
3. For all positive integers r with $1 \le r \le n$, the following recurrence relation holds:
$$s_{n,t,r} = s_{n,t,r-1} + \frac{t+1}{(k+1)r+t+1}\binom{(k+1)r+t+1}{r} \times (s_{n-r+1,0,1} - t[\![r=n]\!]).$$

Proofs via bijective and generating function methods.

Asymptotic results (Average/Variance)

As $n \to \infty$,

1. The expected number of down-steps before the first up-step in k_t-Dyck paths of length $(k+1)n$ is
$$\mathbb{E}X_{n,t,0} = \frac{k}{t+1}\left(\frac{k^{t+1}}{(k+1)^t} - k + t\right) + O\left(\frac{1}{n}\right).$$
2. For fixed $1 \le r < n$, the expected number of down-steps between the r-th and $(r+1)$-th up-steps in k_t-Dyck paths of length $(k+1)n$ is
$$\mathbb{E}X_{n,t,r} = \frac{k^{t+2}}{(k+1)^t}\sum_{j=1}^{r}\frac{1}{(k+1)j+t+1}\binom{(k+1)j+t+1}{j}\left(\frac{k^k}{(k+1)^{k+1}}\right)^j + \frac{k}{t+1}\left(\frac{k^{t+1}}{(k+1)^t} - k + t\right) + O\left(\frac{1}{n}\right).$$
3. The average number of down-steps after the last up-step in k_t-Dyck paths of length $(k+1)n$ is
$$\mathbb{E}X_{n,t,n} = \frac{(k+1)^{k+1}}{k^k} - (t+1) + O\left(\frac{1}{n}\right).$$

Given values of k, t, r, n, there is an algorithm to determine the variance.
Code: `https://gitlab.aau.at/behackl/kt-dyck-downstep-code`.

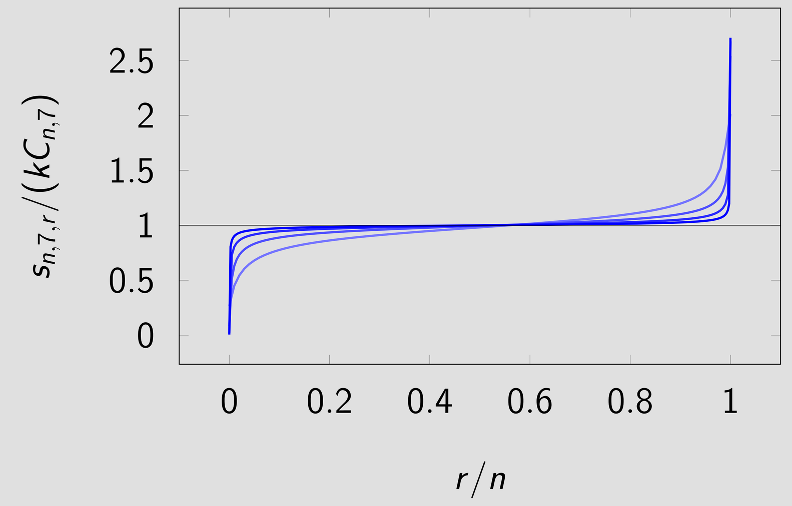
Limiting behaviour

The following limiting behaviour is observed:

- $\lim_{k\to\infty}\lim_{n\to\infty}\frac{\mathbb{E}X_{n,0,r}}{k} = \sum_{j=1}^{r}\frac{j^{j-1}}{j!}\frac{1}{e^j}$
- $\lim_{k\to\infty}\lim_{n\to\infty}\frac{\mathbb{E}X_{n,k,r}}{k} = \sum_{j=0}^{r}\frac{(j+1)^j}{(j+1)!}\frac{1}{e^{j+1}}$
- $\lim_{k\to\infty}\lim_{n\to\infty}\frac{\mathbb{E}X_{n,t,n}}{k} = e - \frac{2t+2-e}{2k} + O(\frac{1}{k^2})$
- $\lim_{k\to\infty}\lim_{n\to\infty}\frac{\mathbb{E}X_{n,t,n-r}}{k} = e - \sum_{j=1}^{r}\frac{(j-1)^{j-1}}{j!}\frac{1}{e^{j-1}}$

Note the relationship of the above to the Lambert W function,
$$-W(-x) = \sum_{j\ge1}\frac{j^{j-1}}{j!}x^j.$$

This is used to explain why for $0 < \beta < 1$, $\lim_{k\to\infty}\lim_{n\to\infty}\frac{s_{n,t,[\beta n]}}{kC_{n,t}} = 1$.
Example: $t = 7$,
$(k, n) \in \{(10, 100), (40, 400), (160, 1600), (640, 6400)\}$



Conclusions

- A bijective approach to the problem gives a recursive formula for the number of down-steps, while a generating function approach leads to a closed form formula.
- Results from both methods leads to several combinatorial identities.
- Asymptotic results were obtained, and unexpected limiting behaviours explained.
- A relationship between coding theory and k_t-Dyck paths was established, along with a Cycle-lemma type result, and a new interpretation for the Catalan numbers.
- Further work, i.e. $k \le t$ and asymptotics, is in progress.

References

[1] Asinowski, A.; Hackl, B; Selkirk, S.: *Down-step statistics in generalized Dyck paths*. arXiv:2007.15562, 2020.