Lattice Path Conference
21-25 June 2021
Presentation times for this poster:
Tuesday 1:30-2:30 pm
Thursday 6-7 pm

Time between the maximum and the minimum of a stochastic process

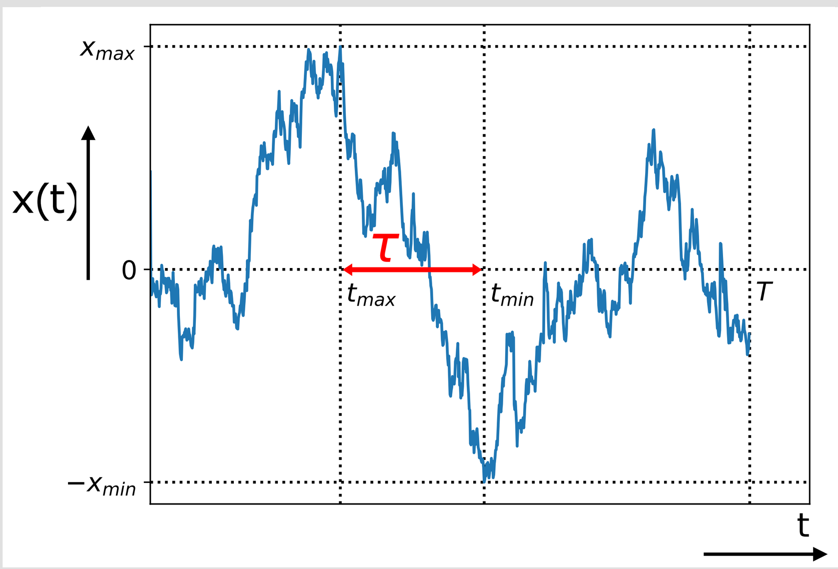
Francesco Mori,[a] Satya N. Majumdar[a] Gregory Schehr [b]

[a] LPTMS, CNRS, Université Paris-Saclay, 91405 Orsay, France
[b]Sorbonne Université, Laboratoire de Physique Théorique et Hautes Energies, CNRS, UMR 7589, 4 Place Jussieu, 75252 Paris Cedex 05, France

Introduction

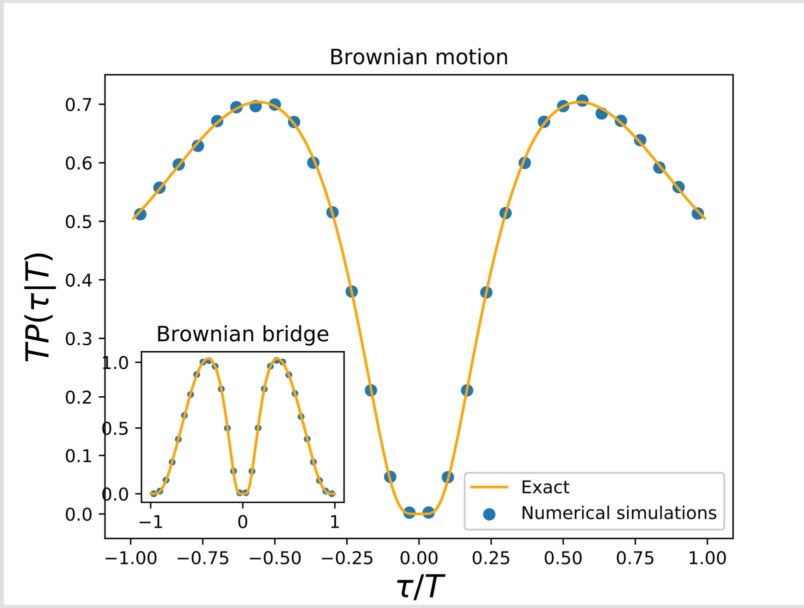
The properties of **extremes of a stochastic process** are of fundamental importance in describing a plethora of natural and artificial phenomena. Applications can be found in: computer science, finance, climate studies, physics, etc. A natural question in finance is: "*How long should one wait between the buying and the selling of a stock?*" A good strategy would be to buy when the price is minimal and then to sell when it is maximal. **Main question**: how long does it take to go from the global maximum to the global minimum of a generic one-dimensional stochastic process?

Problem statement



- Consider a one-dimensional **Brownian motion** $x(t)$ during a time interval $[0,T]$. The position of the Brownian particle evolves according to the Langevin equation
$$\dot{x}(t)=\eta(t)$$
where $\eta(t)$ is Gaussian white noise.
- Define $t_{\max}$ (or $t_{\min}$) as the time at which the global maximum (or minimum) is reached. The probability distribution of $t_{\max}$ is known in literature:
$$P(t_{\max}|T)=\frac{1}{\pi\sqrt{t_{\max}(T-t_{\max})}}\,.$$
$t_{\min}$ has the same distribution by symmetry.
- **How long does it take to go from the global maximum to the global minimum?**
$$\tau=t_{\min}-t_{\max}$$
What is the distribution of τ?
Note that $-T\leq\tau\leq T$.
- Non-trivial problem: **strong correlations** between $t_{\max}$ and $t_{\min}$.
- What is the distribution of τ for a **Brownian bridge** (a periodic Brownian motion of period T)?

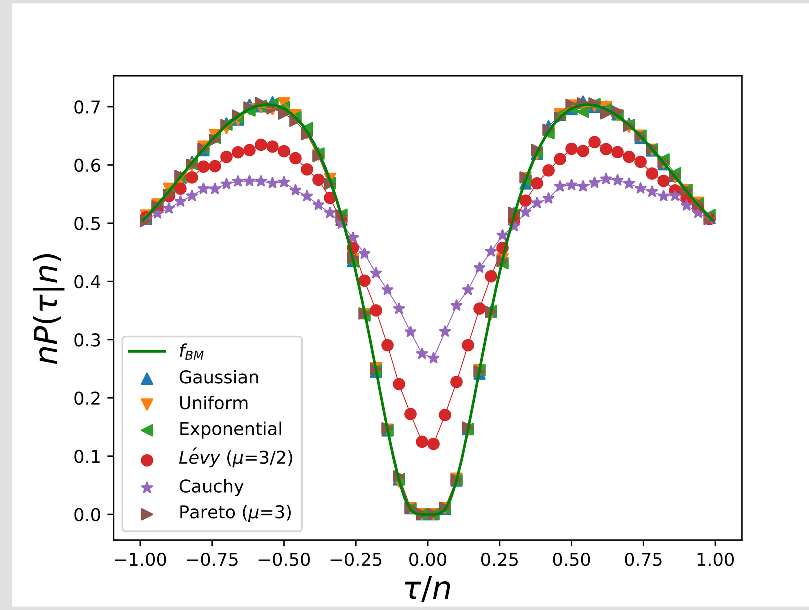
Results



Brownian motion Using a path integral technique we compute
$$P(\tau|T)=\frac{1}{T}f_{\rm BM}\left(\frac{\tau}{T}\right)\,,$$
where
$$f_{\rm BM}(y)=\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{|y|}\tanh^2\left(\frac{m\pi}{2}\sqrt{\frac{|y|}{1-|y|}}\right)\,.$$
$f_{\rm BM}(y)$ is symmetric: $f_{\rm BM}(y)=f_{\rm BM}(-y)$ (see Fig 2). Asymptotic behaviors:
$$f_{\rm BM}(y)\underset{y\to0}{\approx}\frac{8}{y^2}e^{-\frac{\pi}{\sqrt{y}}},\qquad f_{\rm BM}(y)\underset{y\to1}{\approx}\frac{1}{2}.$$
Brownian bridge We show that
$$f_{\rm BB}(y)=\sum_{m,n=1}^{\infty}\frac{3(-1)^{m+n}m^2n^2(1-|y|)}{[m^2|y|+n^2(1-|y|)]^{5/2}}\,,$$
which is again symmetric around $y=0$ (see the inset of Fig. 2). Asymptotic behaviors
$$f_{\rm BB}(y)\underset{y\to0}{\approx}\frac{\sqrt{2}\pi^2}{y^{\frac{9}{2}}}e^{-\frac{\pi}{\sqrt{y}}}\,,$$
$$f_{\rm BB}(y)\underset{y\to1}{\approx}\frac{\sqrt{2}\,\pi^2}{(1-y)^{\frac{7}{2}}}e^{-\frac{\pi}{\sqrt{1-y}}}.$$
Covariance of $t_{\min}$ and $t_{\max}$
We compute exactly the covariance
$$\mathrm{cov}(t_{\min},t_{\max})=\langle t_{\min}t_{\max}\rangle-\langle t_{\min}\rangle\langle t_{\max}\rangle$$
In the two cases we get:
$$\mathrm{cov}_{\rm BM}(t_{\min},t_{\max})=-\frac{7\zeta(3)-6}{32}T^2\approx-0.0754\;T^2$$
$$\mathrm{cov}_{\rm BB}(t_{\min},t_{\max})=-\frac{\pi^2-9}{36}T^2\approx-0.0241\;T^2\,,$$
In both cases $t_{\max}$ and $t_{\min}$ are **anti-correlated**.

Applications

Random walks We demonstrate that the scaling function $f_{\rm BM}$ is **universal** in the sense of **Central Limit Theorem**. Consider a random walk of n steps
$$x_k=x_{k-1}+\eta_k\,,\quad x_0=0\,,$$
where $\eta_k\underset{i.i.d.}{\sim}p(\eta)$, $p(\eta)$ is symmetric and it has variance σ^2. We show that
- if σ^2 is finite, the scaling function of τ converges to $f_{\rm BM}$ in the large n limit
- if σ^2 diverges, the scaling function of τ depends on the tail behavior of $p(\eta)$



Fluctuating interfaces Consider the stationary state of a **Kardar-Parisi-Zhang** (KPZ) [3] interface in one-dimension on a finite substrate of size L. Let $H(x,t)$ be the height of the interface as a function of the position x and time t. $H(x,t)$ evolves according to
$$\frac{\partial H(x,t)}{\partial t}=\frac{\partial^2H(x,t)}{\partial x^2}+\lambda\left(\frac{\partial H(x,t)}{\partial x}\right)^2+\eta(x,t)\,,$$
where $\lambda\geq0$ and $\eta(x,t)$ is Gaussian white noise. We define τ as the position-difference between the minimal and maximal heights of the stationary interface $H(x)$. We show that
- $P(\tau|L)=\frac{1}{L}f_{\rm BM}(\frac{\tau}{L})$ for FBC
- $P(\tau|L)=\frac{1}{L}f_{\rm BB}(\frac{\tau}{L})$ for PBC

References

[1] F. Mori, S. N. Majumdar and G. Schehr, Phys. Rev. Lett. 123, 200201 (2019).

[2] F. Mori, S. N. Majumdar and G. Schehr, Phys. Rev. E 101, 052111 (2020).

[3] M. Kardar, and G. Parisi and Y. Zhang, Phys. Rev. Lett. 56, 889 (1986).