Lattice Path Conference

21-25 June 2021

Presentation times for this poster:

Tuesday	1:30-2:30 pm, 6-7 pm
Thursday	1:30-2:30 pm

Lattice Paths with Alternating Probabilities

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Dedicated to Sri Gopal Mohanty

Sylvester's Eigenvalue Expansion Formula for M^k

Assume M is a real $n \times n$ matrix that has n known distinct eigenvalues $\omega_1, \omega_2, \ldots, \omega_n$. Then

$$M^k = A_1\omega_1^k + A_2\omega_2^k + A_3\omega_3^k + \ldots + A_n\omega_n^k \quad (1)$$

where $k = 0, 1, 2, \ldots$ and $A_1, A_2, \ldots, A_n$ are constant $n \times n$ coefficient matrices called spectral projectors (Meyer) or Frobenius covariants.

$$M^k(i,j) = A_1(i,j)\omega_1^k + A_2(i,j)\omega_2^k + A_3(i,j)\omega_3^k + \ldots + A_n(i,j)\omega_n^k$$

where $0 \le i, j \le n-1$ for $s = 1, 2, 3, \ldots, n$.

We scale the right and left eigenvectors, $\vec{R}_s$ and $\vec{L}_s$ of M by multiplying each eigenvector $\vec{L}_s$ and $\vec{R}_s$ by a constant c so that $c\vec{L}_s \cdot c\vec{R}_s = 1$. Obtaining A_s then follows by taking the scaled outer matrix product of $\vec{R}_s$ and $\vec{L}_s$, that is,

$$A_s = c^2\vec{R}_s\vec{L}_s \quad (2)$$

Theorem K (Kouachi 2006)

Suppose P is a tridiagonal matrix, having dimension $n = 2m+1$, where $m = 1, 2, 3, \ldots$. Assume that the sub- and super-diagonal entries are alternating as shown below.

$$P = \begin{bmatrix} r & p_0 & 0 & 0 & \cdots & 0 & 0 \\ q_1 & r & p_1 & 0 & \cdots & 0 & 0 \\ 0 & q_2 & r & p_0 & \cdots & 0 & 0 \\ 0 & 0 & q_1 & r & p_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & q_2 & r \end{bmatrix} \quad (3)$$

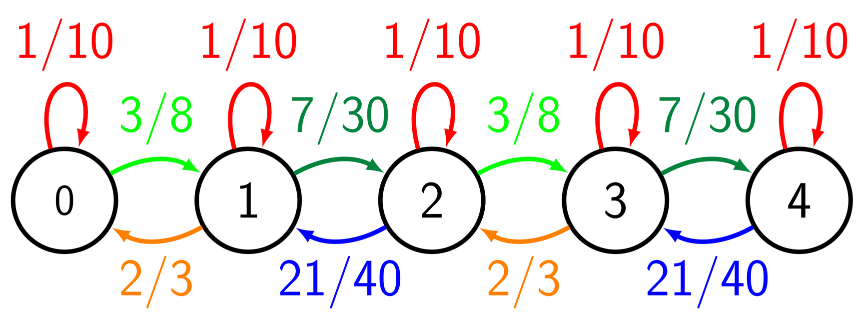
Suppose $q_1 p_0 = d_1^2$, $q_2 p_1 = d_2^2$, where $d_1 \neq 0 \neq d_2$. Then P has distinct eigenvalues given by:

$$\omega_k = \begin{cases} r + \sqrt{d_1^2 + d_2^2 + 2d_1d_2\cos(\theta_k)} & \text{if } k = 1, 2, \ldots, m \\ r - \sqrt{d_1^2 + d_2^2 + 2d_1d_2\cos(\theta_k)} & \text{if } k = m+1, m+2, \ldots, 2m \\ r & \text{if } k = n \end{cases} \quad (4)$$

Then eigenvectors $\vec{R}_s$ and $\vec{L}_s$ of P are explicitly known and we can find the spectral projectors and Sylvester Eigenvalue Expansion for P^k.

Problem: When $n = 2m$, find a formula for eigenvalues for $m \geq 5$.

Transition Diagram and Transition Matrix

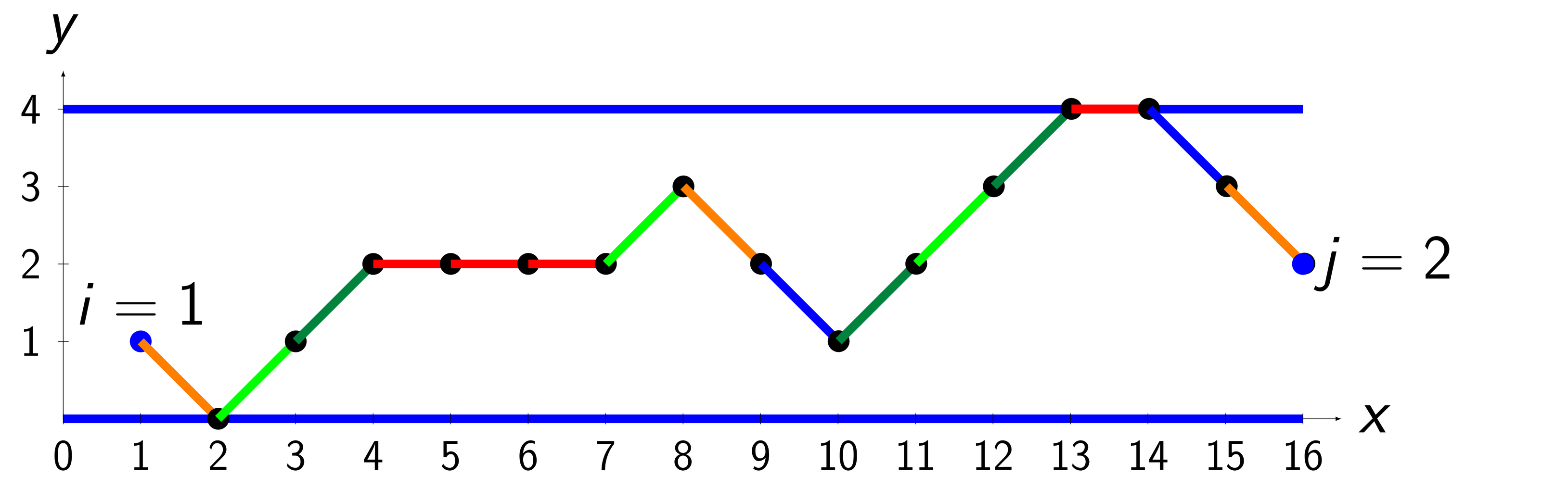


Strip Birth-Death Chain

$$P = \begin{bmatrix} \frac{1}{10} & \frac{3}{8} & 0 & 0 & 0 \\ \frac{2}{3} & \frac{1}{10} & \frac{7}{30} & 0 & 0 \\ 0 & \frac{21}{40} & \frac{1}{10} & \frac{3}{8} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{10} & \frac{7}{30} \\ 0 & 0 & 0 & \frac{21}{40} & \frac{1}{10} \end{bmatrix}$$

One Step Transition Probability Matrix

Probability of All Lattice Paths Confined to the Following Strip Going from $i = 1$ to $j = 2$ in 15 Steps



A sample path going from $i = 1$ to $j = 2$ in 15 steps staying within the strip

To find the probability of going from state $i = 1$ to $j = 2$ in 15 steps without leaving the strip, we use Kouachi's Theorem to get these eigenvalues:

$\omega_1 = \frac{\sqrt{219}}{20} + \frac{1}{10}$ $\omega_2 = \frac{\sqrt{79}}{20} + \frac{1}{10}$ $\omega_3 = \frac{1}{10} - \frac{\sqrt{219}}{20}$ $\omega_4 = \frac{1}{10} - \frac{\sqrt{79}}{20}$ $\omega_5 = \frac{1}{10}$.

We can calculate the spectral projectors:

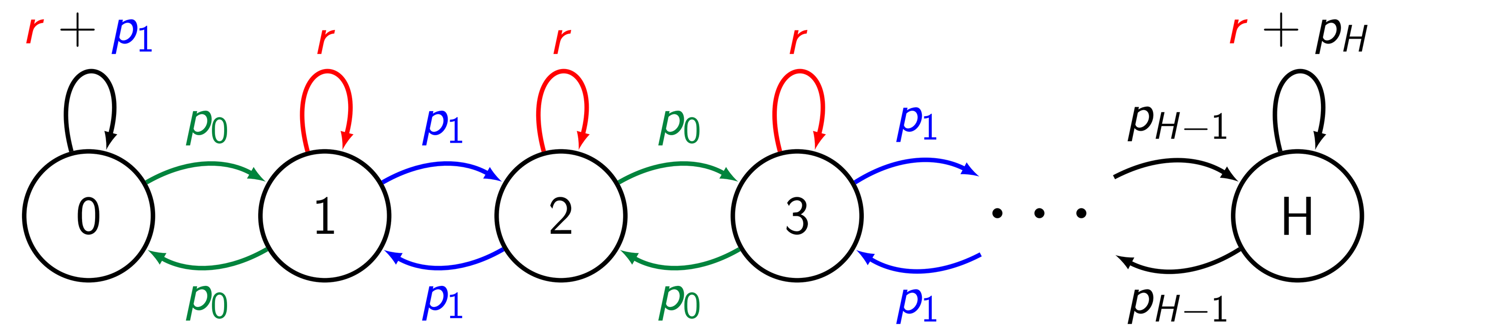
$$P^{15}(1,2) = A_1(1,2)\omega_1^{15} + A_2(1,2)\omega_2^{15} + A_3(1,2)\omega_3^{15} + A_4(1,2)\omega_4^{15} + A_5(1,2)\omega_5^{15}$$

$$= \frac{17\sqrt{219}}{1314}\left(\frac{\sqrt{219}}{20} + \frac{1}{10}\right)^{15} - \frac{\sqrt{79}}{158}\left(\frac{\sqrt{79}}{20} + \frac{1}{10}\right)^{15}$$

$$- \frac{17\sqrt{219}}{1314}\left(\frac{1}{10} - \frac{\sqrt{219}}{20}\right)^{15} + \frac{\sqrt{79}}{158}\left(\frac{1}{10} - \frac{\sqrt{79}}{20}\right)^{15} + 0\left(\frac{1}{10}\right)^{15} \approx 0.0142$$

Theorem K: Special Case p's $= q$'s

Suppose a birth-death chain has the following state transition diagram and transition matrix as shown below:



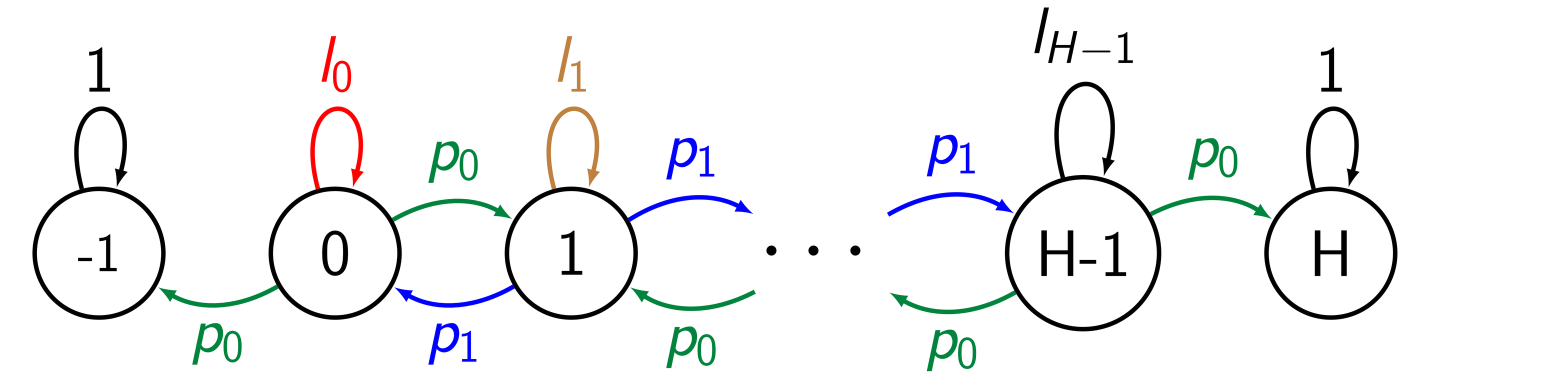
for H a natural number, $0 < p_0,\ p_1 < 1$, $0 \le r < 1$, and $r + p_0 + p_1 = 1$.

$$P = \begin{bmatrix} r+p_1 & p_0 & 0 & \cdots & 0 & 0 & 0 \\ p_0 & r & p_1 & \cdots & 0 & 0 & 0 \\ 0 & p_1 & r & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & r & p_0 & 0 \\ 0 & 0 & 0 & \cdots & p_0 & r & p_{H-1} \\ 0 & 0 & 0 & \cdots & 0 & p_{H-1} & r+p_H \end{bmatrix} \qquad p_{H-1} = \begin{cases} p_1 & \text{when } H \text{ is even} \\ p_0 & \text{when } H \text{ is odd} \end{cases} \qquad p_H = \begin{cases} p_0 & \text{when } H \text{ is even} \\ p_1 & \text{when } H \text{ is odd} \end{cases}$$

Then the eigenvalues of P have an explicit formula that depends on H. The Sylvester Eigenvalue Expansion for P^k has a closed formula for all natural numbers H.

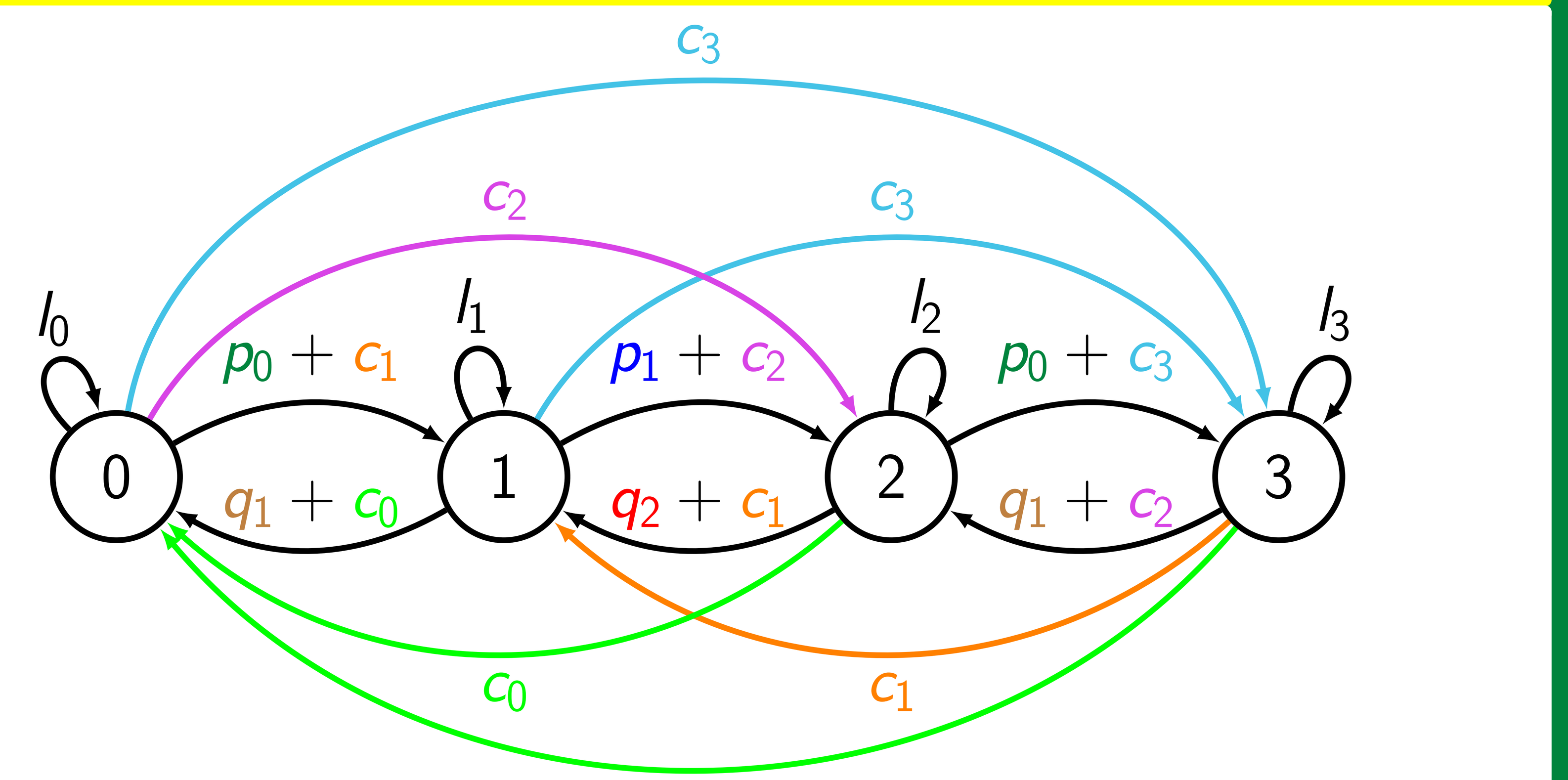
Corollary: Finite Time Gambler's Ruin Problem

Assuming the conditions of the previous theorem and $r > |p_1 - p_0|$, then the Duality Theorem provides explicitly known eigenvalues for any $H \in \mathbb{N}$.



where $l_0 = r + p_1 - p_0$ and $l_1 = r + p_0 - p_1$; l_0 and l_1 then alternate until the absorbing state at H. The Sylvester Eigenvalue Expansion for the transition matrix corresponding to the preceding diagram has a closed formula for all natural numbers H, see related problems in [2] and [5].

General Markov Chain with Alternating Entries, $m = 2n$



$l_0 = r + q_1 + c_0$ $\quad l_1 = r + q_2 - q_1 + c_1$

$l_2 = r + q_1 - q_2 + c_2$ $\quad l_3 = r + p_0 + c_3$ $\quad 0 < q_1,\ q_2,\ r,\ p_0,\ p_1 < 1$

$q_1 + r + p_0 + c_0 + c_1 + c_2 + c_3 = 1$ $\quad q_2 + r + p_1 + c_0 + c_1 + c_2 + c_3 = 1$

References

[1] Felsner, S.; Heldt, D.: *Lattice path enumeration and Toeplitz matrices*, Journal of Integer Sequences, 18, pp. 1–16, 2015.

[2] Hunter, B. et al.: *Gambler's ruin with catastrophes and windfalls*, Journal of Statistical Theory and Practice, no. 2, pp. 199–219, 2008.

[3] Kouachi, S.: *Eigenvalues and Eigenvectors of Tridiagonal Matrices*, Electronic Journal of Linear Algebra, Vol. 15, pp. 115-133, 2006 *Eigenvalues and Eigenvectors of Some Tridiagonal Matrices with Non-Constant Diagonal Entries*, Applicationes Mathematicae 35, Vol. 1, pp. 107–120, 2008.

[4] Krinik et al.: *Explicit Transient Probabilities of Various Markov Models*, AMS Contemporary Mathematics Series volume 774 to appear in 2021.

[5] Lorek, P.: *Generalized Gambler's Ruin Problem: Explicit Formulas via Siegmund Duality*, Methodol Comput Appl Probab, 19, pp. 603-613, 2017.