Lattice Path Conference
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Presentation times for this poster:
Tuesday 1:30-2:30 pm
Thursday 6-7 pm

Weight-preserving bijections between integer partitions and a class of alternating sign trapezoids

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Alternating sign trapezoids (ASTZs)

Let $l \geq 2$. An **(n,l)-ASTZ** is a trapezoidal array of integers with n rows, l entries in the bottom row and entries -1, 0 or $+1$ such that

- the sum of the entries in each row equals 1;
- the nonzero entries alternate in sign along each row and each column;
- the top-most nonzero entry in each column is 1;
- the entries in the central $l-2$ columns sum up to 0.

Statistics on (n,l)-ASTZs

- $r :=$ # 1-columns among the n leftmost columns
- $p :=$ # 10-columns among the n leftmost columns
- $q :=$ # 10-columns among the n rightmost columns
- $\mu :=$ # -1s

$(9,4)$-ASTZ $\tilde{A}$ with $(r,p,q,\mu) = (1,1,2,2)$

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	
		0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0		
			0	0	0	0	0	0	0	0	0	0	0	1	−1	1			
				0	0	0	0	0	0	0	0	0	0	0	1				
					0	0	0	0	0	1	0	0	0	0					
						0	1	0	0	−1	0	0	1						
							0	0	0	0	0	1							
								0	0	0	1								

Column strict shifted plane partitions (CSSPPs)

A **CSSPP of class k** is a filling of a shifted Young diagram with positive integers such that

- the entries weakly decrease along each row;
- the entries strictly decrease down each column;
- the first entry of each row is k plus the corresponding row length.

Statistics on CSSPPs of class k: Let $1 \leq d \leq k$.

- $r :=$ # rows
- $p_d :=$ # parts $\pi_{i,j} = j - i + d$
- $q :=$ # 1s
- $\mu_d :=$ # parts $\pi_{i,j} \in \{2, 3, \ldots, j-i+k\} \setminus \{j-i+d\}$

CSSPP $\tilde{\pi}$ of class 3 with $(r, p_1, q, \mu_1) = (1,1,2,2)$

11	9	7	6	5	4	1	1

Joint distribution

Let $l \geq 2$ and $1 \leq d \leq l-1$. The statistics (r,p,q,μ) on (n,l)-ASTZs have the same joint distribution as the statistics (r,p_d,q,μ_d) on CSSPPs of class $l-1$ with at most n parts in the first row.

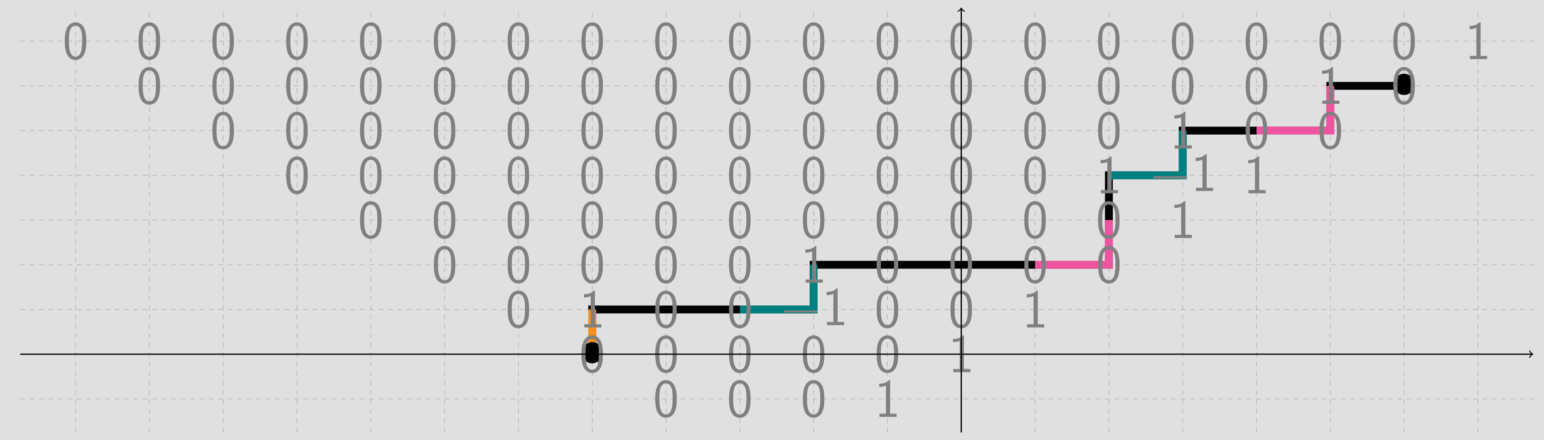
Bijective proof for $r = 1$

We construct weight-preserving bijections

$$\bigcup_{i=1}^{n} \mathcal{ASTZ}_{n,l}^{i,j} \longleftrightarrow \mathcal{CSSPP}_{n,l-1}^{j}.$$

- $\mathcal{ASTZ}_{n,l}^{i,j}$: set of (n,l)-ASTZs such that the $(n+1-i)^{\text{th}}$ column is the only 1-column among the n leftmost columns and the $(n+l-2+j)^{\text{th}}$ column is the only 0-column among the n rightmost columns
- $\mathcal{CSSPP}_{n,l-1}^{j}$: set of single-row CSSPPs of class $l-1$ with $j < n$ parts

Lattice path representation for ASTZs in $\mathcal{ASTZ}_{n,l}^{i,j}$



Left turn representation

$$\begin{aligned} -2i - l + 4 \leq x_1 < x_2 < \ldots < x_{\mu+q} &\leq j - i - 1 \\ p \leq y_1 < y_2 < \ldots < y_{\mu+q} &\leq j - i - 1 \end{aligned}$$

with left turns at (x_m, y_m) such that $x_m \leq y_m$ for all $1 \leq m \leq \mu + q$.

Lattice path representation for CSSPPs in $\mathcal{CSSPP}_{n,k}^{j}$

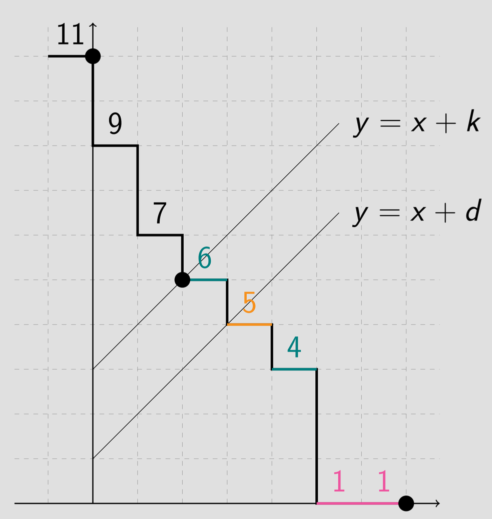


Illustration of $\mathcal{ASTZ}_{9,4}^{2,8} \ni \tilde{A} \mapsto \tilde{\pi} \in \mathcal{CSSPP}_{9,3}^{8}$ for $d = 1$

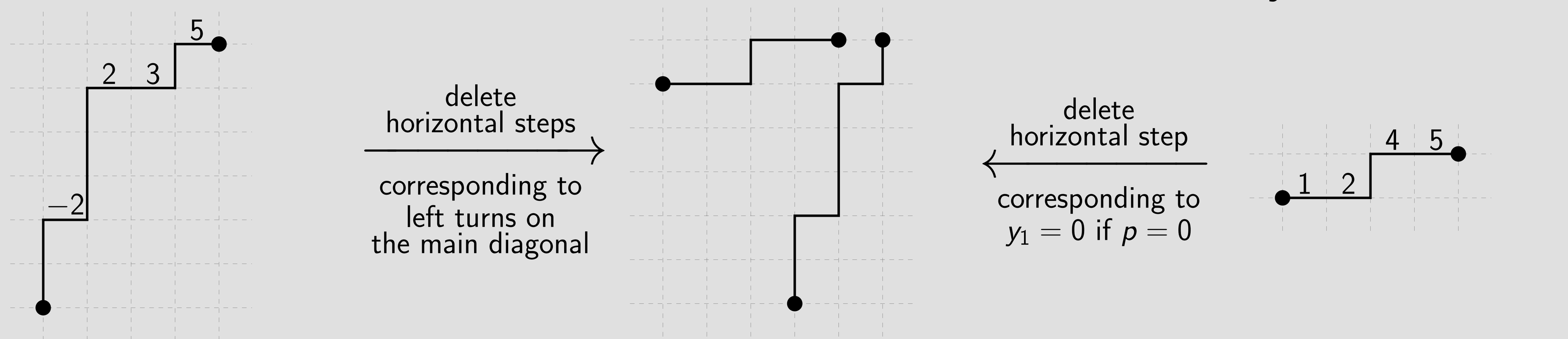
Left turn representation of $\tilde{A}$

x-coordinates: $-4 \leq -2 < \boxed{2} < 3 < \boxed{5} \leq 5$

y-coordinates: $1 \leq 1 < \boxed{2} < 4 < \boxed{5} \leq 5$

Split up the path and truncate

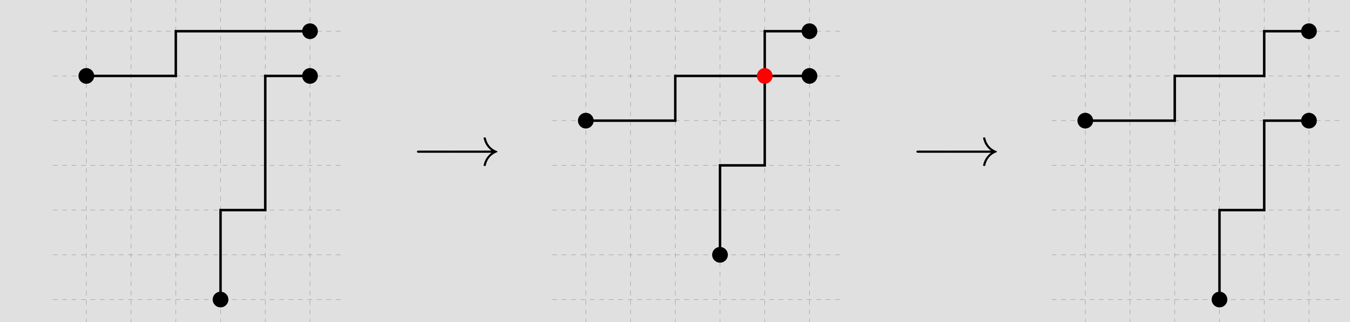
x-coordinates: y-coordinates:



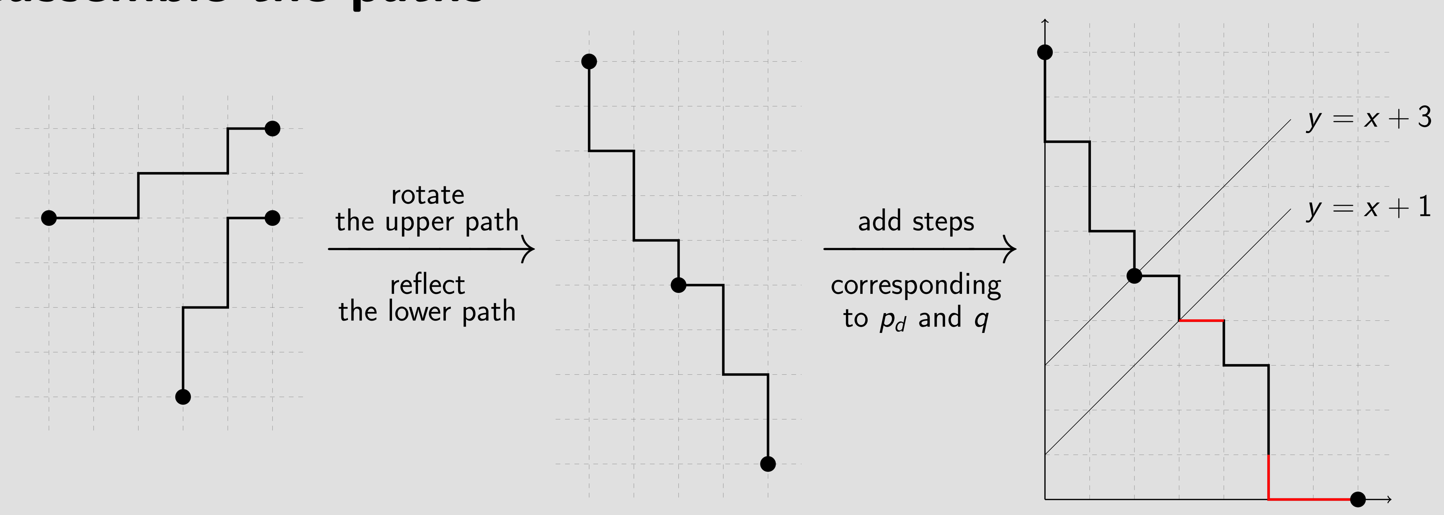
Shuffle the paths

Adjust the paths in the top right corner.

Repeat $i-1$ times: Shift the upper path down by one step and the lower path up by one step, then switch tails at the top right intersection point.



Reassemble the paths



Family of bijections

Rotating the upper path and reflecting the lower path yields four different but equally valid weight-preserving bijections.

References

[1] Höngesberg, H.: *A fourfold refined enumeration of alternating sign trapezoids*. Preprint, arXiv:2006.13388 [math.CO], 2020.

[2] Höngesberg, H.: *Weight-preserving bijections between integer partitions and a class of alternating sign trapezoids*. Preprint, arXiv:2102.07555 [math.CO], 2021.