Lattice Path Conference

21-25 June 2021

Presentation times for this poster:

Tuesday 1:30-2:30 pm, 6-7 pm

Restricted Dyck Paths

Rigoberto Flórez[a] José L. Ramírez[b] Fabio A. Velandia[b] Diego Villamizar[c]

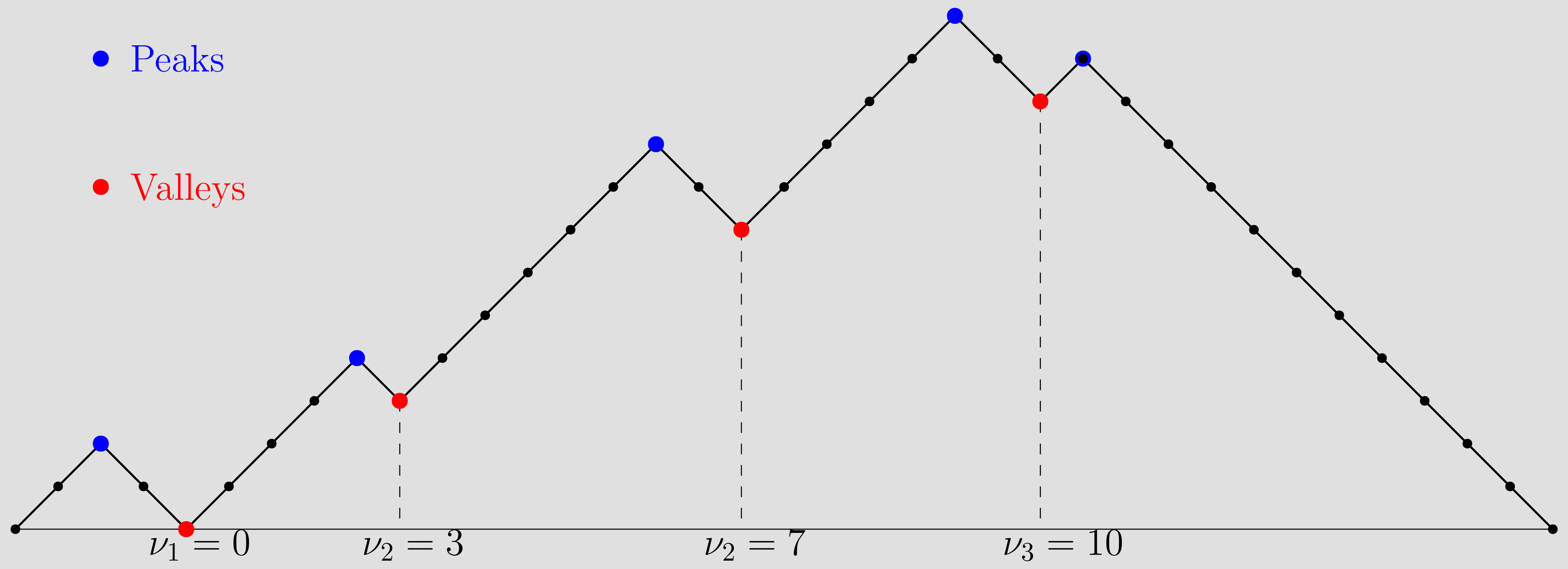
[a]Department of Mathematical Sciences, The Citadel, Charleston, SC, U.S.A.

[b]Departamento de Matemáticas, Universidad Nacional de Colombia, Colombia

[c]Department of Mathematics, Tulane University, New Orleans, U.S.A.

Restricted d-Dyck Path

The vector $\nu = (\nu_1, \nu_2, \dots, \nu_k)$, called **valley vertices**, is formed by all y-coordinates (listed from left to right) of all valleys of a Dyck path. For example, $\nu = (\nu_1, \nu_2, \nu_3, \nu_4) = (0, 3, 7, 10)$.



For a fixed $d \in \mathbb{Z}$, a Dyck path P is called **d-Dyck**, if either P has at **most one valley**, or if its valley vertex vector ν satisfies that $\nu_{i+1} - \nu_i \geq d$, where $1 \leq i < k$.

- For example, the Figure shows a 3-Dyck path of semi-length 18.

Examples

- $d \to -\infty$: (Classical) Dyck path. Enumerated by the **Catalan numbers** $C_n = \frac{1}{n+1}\binom{2n}{n}$.
- $d = 0$: Non-decreasing Dyck path. Enumerated by the **odd Fibonacci numbers**: F_{2n-1}.

A connection with restricted words

Define $b_d(n)$ as the **number of binary words** of length n in which the 1's occur only in blocks of at least length d. This type of words is called **d-restricted**.

For example, $a_3(5) = 7$, where the 3-restricted words of length 5 are 00000, 00111, 01110, 01111, 11100, 11110, 11111.

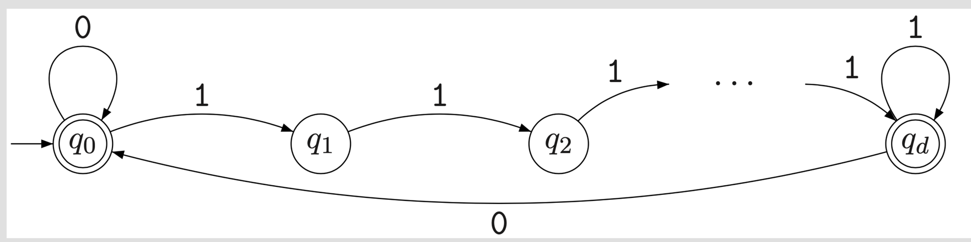


Figure: Transition diagram for the restricted binary words.

Theorem

For $d \geq 2$, the **number of d-restricted words** *is equal to the* **number of d-Dyck paths** *of semi-length $n - d + 2$.*

$$b_d(n) = |\mathcal{D}_d(n - d + 2)| = r_d(n - d + 2).$$

Case $d \geq 0$

- $\ell(P)$:= **semi-length** of P; $\rho(P)$:= **number of peaks** of P.

$$F_d(x, y) := \sum_{P \in \mathcal{D}_d} x^{\ell(P)} y^{\rho(P)}$$

Theorem

If $d \geq 0$, then we have this expression for the **generating function**

$$F_d(x, y) := 1 + \sum_{P \in \mathcal{D}_d \setminus \{\lambda\}} x^{\ell(P)} y^{\rho(P)} = 1 + \frac{xy(1 - 2x + x^2 + xy - x^{d+1}y)}{(1 - x)(1 - 2x + x^2 - x^{d+1}y)}.$$

Number of d-Dyck paths $(d \geq 1)$

$$r_d(n) = \sum_{k=0}^{\lfloor \frac{n+d-2}{d} \rfloor} \binom{n - (d-1)(k-1)}{2k}$$

Moreover, for $n > d$ we have the recurrence relation

$$r_d(n) = 2r_d(n-1) - r_d(n-2) + r_d(n-d-1),$$

with the initial values $r_d(n) = \binom{n}{2} + 1$, for $0 \leq n \leq d$.

A connection with polyominoes

We say that a **directed column convex polyomino** is **d-restricted** for $d \geq 0$, if either it is formed by exactly one or two columns or if its initial altitudes vector $A = (0, a_2, \dots, a_k)$ satisfies that $a_{i+1} - a_i \geq d$ for all $2 \leq i \leq k - 1$.

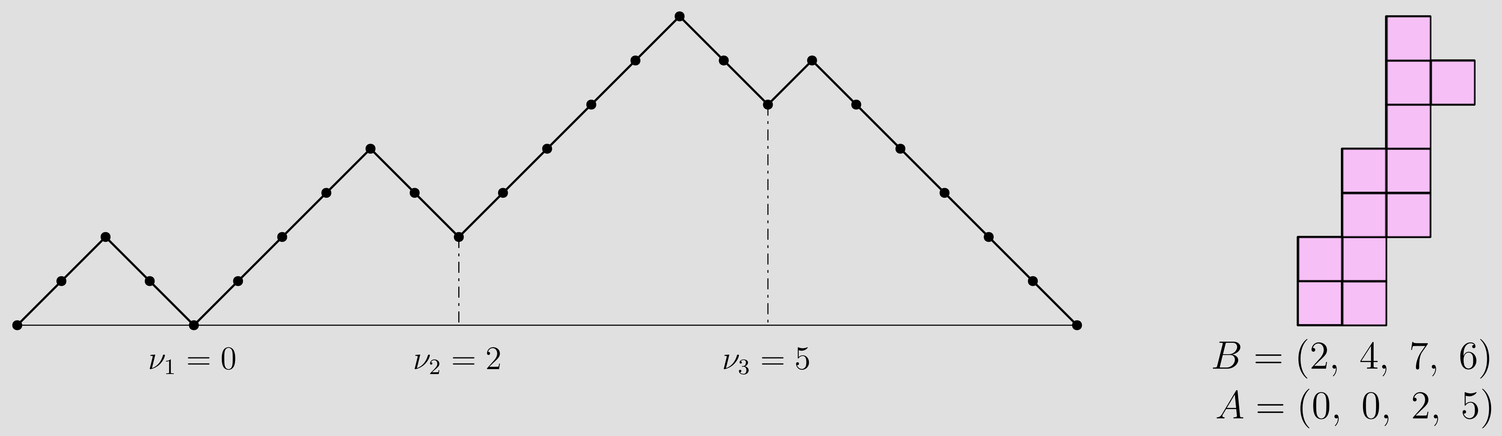
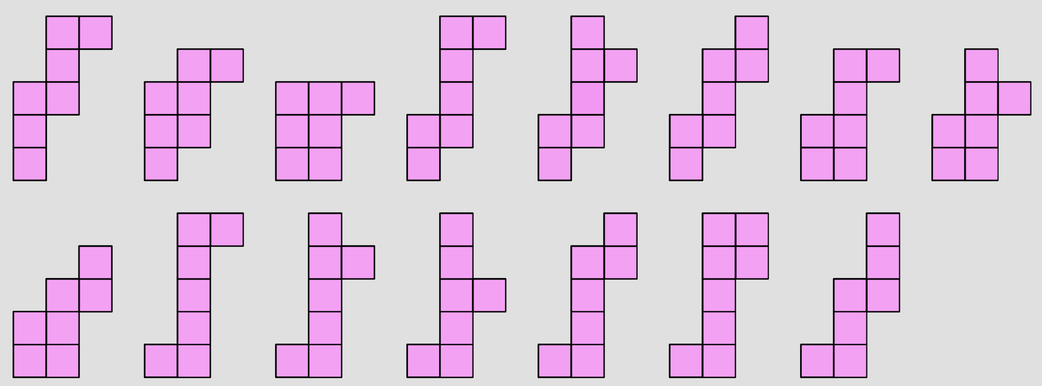


Figure: A bijection between dccp polyomino and non-decreasing Dyck path.



Case $d < 0$

Let $L_e(x, y)$ denote the generating function, where $e := |d|$, defined by

$$L_e(x, y) := \sum_{P \in \mathcal{D}_d} x^{\ell(P)} y^{\rho(P)} = \sum_{P \in \mathcal{D}_{-e}} x^{\ell(P)} y^{\rho(P)}$$

Theorem

If $d \geq 0$, then $L_e(x, y)$ satisfies the functional equation

$$L_e(x, y) = xy + xL_e(x, y) + xS_e(x, y)L_e(x, y),$$

where $S_e(x)$ satisfies the algebraic equation

$$(1 - xS_e(x, y))^e(y + (1 - y)xS_e(x, y)) - S_e(x, y)(1 - xS_e(x, y))^{e+1} - \frac{x^{e+2}y}{1 - x} S_e(x, y) = 0.$$

Theorem

If $1 \leq k \leq |d| + 3$, then the k-th coefficient of the generating function $L_e(x, 1)$ coincides with the Catalan number C_k.

The first few values for the sequence $r_d(n)$

$$\{r_{-1}(n)\}_{n \geq 1} = \{1, 2, 5, 14, 41, 123, 375, 1157, 3603, \dots\},$$
$$\{r_{-2}(n)\}_{n \geq 1} = \{1, 2, 5, 14, 42, 131, 419, 1365, 4511 \dots\},$$
$$\{r_{-3}(n)\}_{n \geq 1} = \{1, 2, 5, 14, 42, 132, 428, 1419, 4785 \dots\},$$
$$\{r_{-4}(n)\}_{n \geq 1} = \{1, 2, 5, 14, 42, 132, 429, 1429, 4850, \dots\}.$$

Theorem (Case $d = -1$)

The **bivariate generating function** *$L_{d=-1}(x, y)$ is given by*

$$\frac{(x-1)y\left(1 - x(2+y) - \sqrt{(1 - x - 2xy - 2x^2y + x^2y^2 - x^3y^2)/(1-x)}\right)}{2(1 - 2x + x^2 - 2xy + x^2y)}$$

$$L_{d=-1}(x, 1) = \frac{-1 + 4x - 3x^2 + \sqrt{1 - 4x + 2x^2 + x^4}}{2(1 - 4x + 2x^2)}$$

Conclusions

- For $d \geq 0$, we obtain rational generating functions.
- For $d < 0$, we obtain algebraic (non-rational) generating functions.
- Odd Fibonacci Numbers $\to$ Catalan numbers.