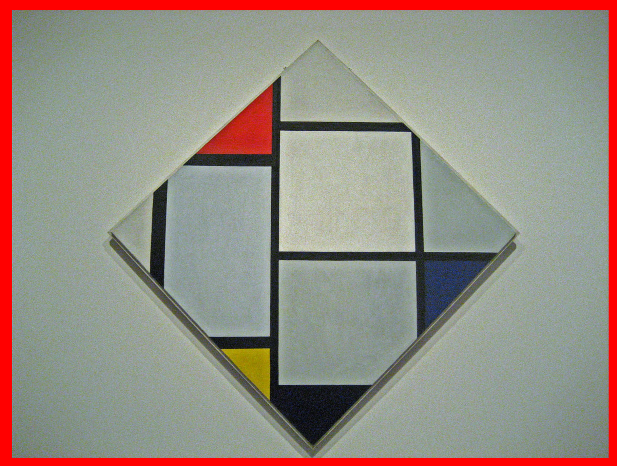
CIRM - Lattice Paths, Combinatorics and Interactions
21-25 June 2021

Presentation times for this poster:

Tuesday	6-7 pm
Thursday	6-7 pm



Mondrian - Tableau No. IV

From Lattice Paths to Standard Young Tableaux

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Standard Young Tableaux

- **Partition**: $\lambda = (\lambda_1, \ldots, \lambda_r)$, where $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$. Represent by **Young diagrams**
- **Standard Young tableau** of shape λ: Each row and column strictly increasing, using $\{1, 2, \ldots, n\}$, where $n = \sum_i \lambda_i$

Ex: $\lambda = (5, 4, 1)$

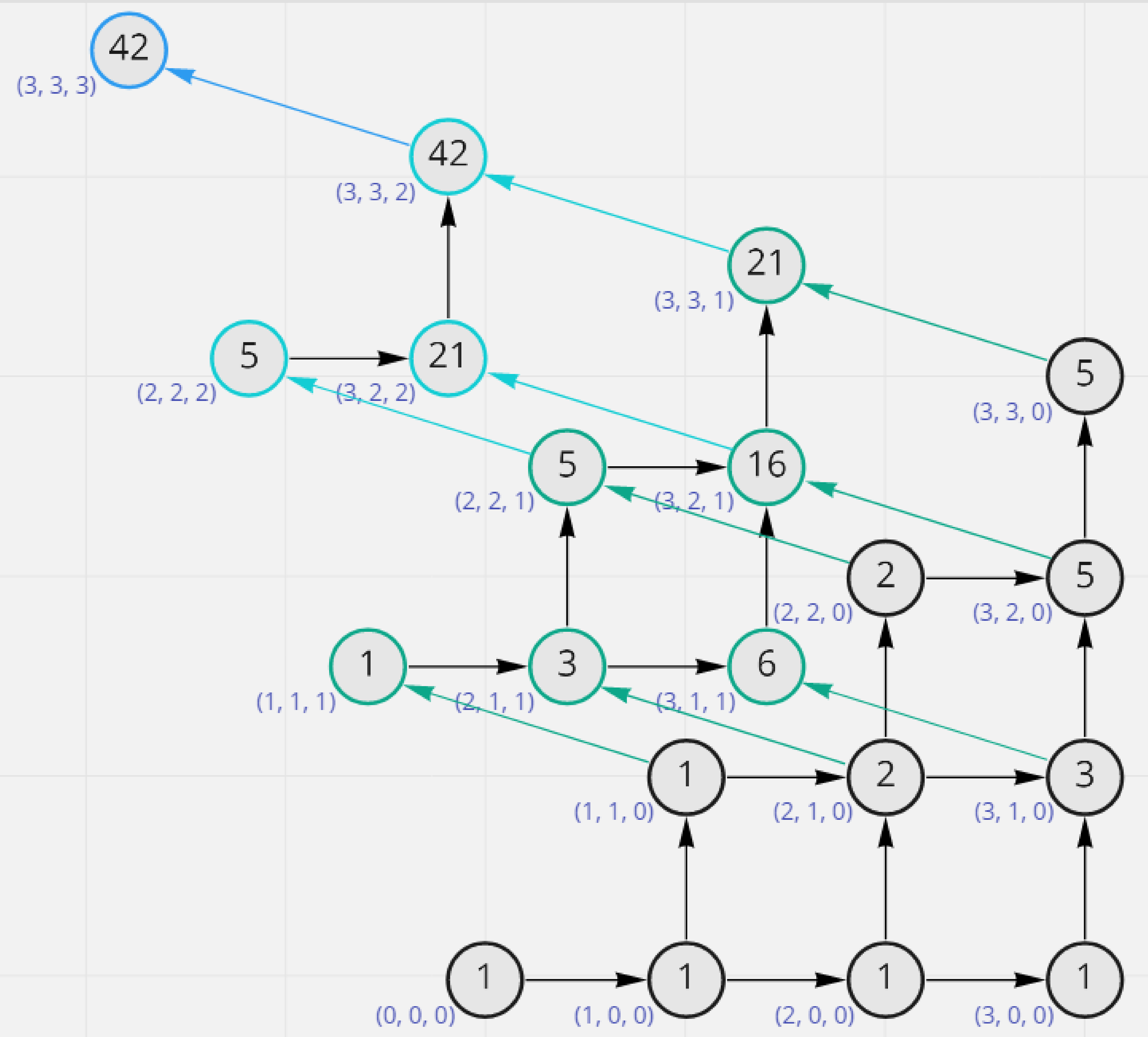
1	2	4	7	8
3	5	6	9	
10				

- $f^\lambda = \#$ of distinct standard Young tableaux of shape λ

Lattice Path Interpretation

- Lattice: $\Lambda^r = \{\lambda \text{ up to height } r\} \subseteq \mathbb{N}_0^r$
- Paths in Λ^r:
  1. Start at origin
  2. Move set: $\{\delta_i = (0, \ldots, \underset{i}{1}, \ldots, 0) \mid 1 \leq i \leq r\}$
- $f^\lambda = \#$ of paths ending at λ.

Example: Lattice Λ^3, with $\lambda_i \leq 3$



Vertex Numbers and Admissible States

- **Vertex Numbers**: $v_n(\mathbf{x}) = \#$ paths ending at $\mathbf{x} = (x_1, \ldots, x_r)$ of length n.
- Extend domain of v_n to all of $\mathbb{N}_0^r$ by *Σ-admissibility*.
- A **Σ-admissible** state v must satisfy:

$$v(\sigma\mathbf{x}) = (-1)^{|\sigma|} v(\mathbf{x}),$$

for all $\sigma \in \Sigma_r$, where Σ_r is the **symmetric group** on r letters, and $|\sigma|$ is the **signature** of σ.

- Properties of Σ-admissibility
  1. If $\mathbf{y} = \sigma\mathbf{x}$, where σ is a transposition, then $v(\mathbf{x}) + v(\mathbf{y}) = 0$.
  2. $v(\mathbf{x}) = 0$ if $\mathbf{x}$ has any repeated entries.
- **Note**, our definition of v_n corresponds to f^λ via

$$f^\lambda = v_n(\lambda + \mathbf{r}),$$

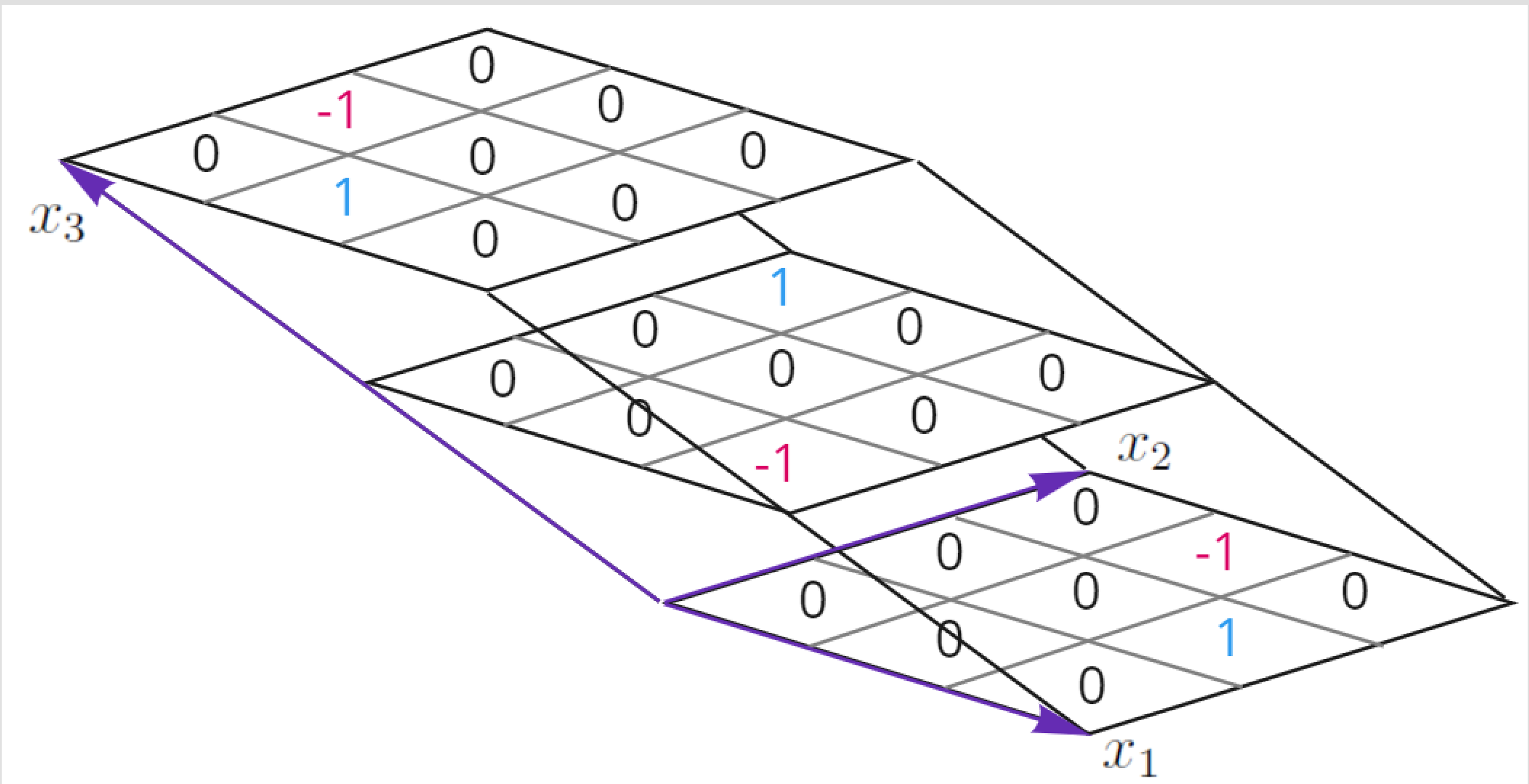
where $n = \sum_i \lambda_i$, and $\mathbf{r} = (r-1, r-2, \ldots, 0)$.

Initial State

- Set $v_0(\mathbf{r}) = 1 \quad = f^{(0,\ldots,0)}$
- Set $v_0(\mathbf{x}) = 0$ for all $\mathbf{x} \neq \mathbf{r}$ such that $x_1 > x_2 > \cdots > x_r$
- Extend v_0 to all of $\mathbb{N}_0$ by Σ-admissibility
- **Generating function (OGF)** for v_0:

$$V_0(\mathbf{x}) = \sum_{\sigma \in \Sigma_r} (-1)^{|\sigma|} \mathbf{x}^{\sigma\mathbf{r}} = \prod_{i<j} (x_i - x_j)$$

Example: v_0 for $r = 3$



Transitions

- Given a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, let

$$R_i(\lambda) = (\lambda_1, \ldots, \lambda_i - 1, \ldots, \lambda_r)$$

- With the convention that $f^\lambda = 0$ if λ does not correspond to a valid partition,

$$f^\lambda = \sum_{i=1}^{r} f^{R_i(\lambda)}$$

- Vertex numbers v_n are *shifted* by $\mathbf{r}$ with respect to λ, but an analogous formula applies. In terms of OGF:

$$V_{n+1} = TV_n,$$

where $T = x_1 + x_2 + \cdots + x_r$.

- **Note**, T preserves Σ-admissibility.
- $\implies V_n = T^n V_0$ is an OGF for v_n.
- Shift back by dividing by $\mathbf{x}^\mathbf{r}$

Main Theorem

With T, V_0 as above,

$$f^\lambda = \left[\frac{T^n V_0}{\mathbf{x}^\mathbf{r}}\right]_\lambda,$$

where $[P(\mathbf{x})]_\lambda$ is the coefficient of $\mathbf{x}^\lambda$ in P.

Example: $r = 3$, $n = 4$

$$T^4 V_0/(x_1^2 x_2) = x_1^4 + 3x_1^3 x_2 + 2x_1^2 x_2^2 + 3x_1^2 x_2 x_3 + (\text{other terms}^*)$$

$$= f^{(4,0,0)}\mathbf{x}^{(4,0,0)} + f^{(3,1,0)}\mathbf{x}^{(3,1,0)} + f^{(2,2,0)}\mathbf{x}^{(2,2,0)} + f^{(2,1,1)}\mathbf{x}^{(2,1,1)} + \cdots$$

*Exponents of the *other terms* do not correspond to a valid partition λ.

References

[1] Ault, S.; Kicey, C. *Counting Lattice Paths Using Fourier Methods*. Birkhäuser/Springer (2019).

This work was inspired by the recent talk by Robert Donley, *Vandermonde convolution for ranked posets*, May 25, 2021 - CANT Conference