A bridge between semirings

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PPS, Paris Denis Diderot & LIPN, Paris Nord

2-4 December 2013

(Work in progress)

Introduction

Semirings $\mathcal{R}$ for resource management

More and more **researche projects** are using **semirings** to model ideas of quantitative resources, we are trying to **links these approaches**.

Ghica's type system $\mathbf{T}^{\mathcal{R}}$ [GhicaSmith13]

Generalization of Girard's BLL that are substituting polynomials by element of given semirings in inferenceable type systems.

Ehrhard's model $\mathrm{Rel}_!^{\mathcal{R}}$ [CarraroEhrhardSalibra09]

Generalizations of the relational ccc $\mathrm{REL}_!$ (known for its resource management properties) that relaxe multisets exponential into semiring-indexed multisets.

Stratification of $\mathrm{Rel}_!^{\mathcal{R}}$ into a model for $\mathbf{T}^{\mathcal{R}}$

We will see that rather natural stratifications of Ehrhard's models gives rise to models for Ghica's typed calculi.

Introduction

The problem: In order to create $\mathrm{Rel}_!^{\mathcal{R}}$, Ehrhard ask for $\mathcal{R}$ to **have multiplicities**

Introducing the multiplicity

- Given any semiring $\mathcal{R}$,
- any semiring with multiplicities $\mathcal{S}$,
- a semiring onto morphism $\phi : \mathcal{S} \twoheadrightarrow \mathcal{R}$,

$\mathrm{Rel}_!^{\mathcal{S}}$ can be stratified into a model of $\mathbf{T}^{\mathcal{R}}$.

Expanding semirings to have multiplicities

Given any semiring $\mathcal{R}$ we can create

- $\mathbb{N}_f\langle\mathcal{R}\rangle$ that (nearly) has multiplicities
- $\phi : \mathbb{N}_f\langle\mathcal{R}\rangle \twoheadrightarrow \mathcal{R}$ an onto morphism.

Idea:

Resources manager type systems are modeled by stratifying ccc obtained from differential linear logic with non-usual exponential.

Definition of semirings

Semiring $\mathcal{R}$

A semiring is a structure $\mathcal{R} = (|\mathcal{R}|, +, \mathbf{0}, \cdot, \mathbf{1})$ where $|\mathcal{R}|$ is a set containing $\mathbf{0}$ and $\mathbf{1}$ and where $+$ and $\cdot$ are binaries operations verifying:

- $+$ and $\cdot$ are associative,
- $+$ is commutative,
- $+$ distribute over $\cdot$,
- $\mathbf{0}$ is neutral for $+$ and absorbing for $\cdot$,
- $\mathbf{1}$ is neutral for $\cdot$.

Elements of a semiring $\mathcal{R}$ are denoted by large Latins letters J, K...

Examples of semirings

- $\mathbb{1}$: the trivial Semiring with just one element.
- $\mathbb{P}oly$: the polynomials with their usual addition and multiplication.
- For any commutative monoid $(\mathcal{M}, +, \mathbf{0})$, the automorphisms $(\mathbf{Mon}(\mathcal{M}, \mathcal{M}), +, cst_{\mathbf{0}}, \circ, id_{\mathcal{M}})$

The semimodule $\mathcal{R}_f\langle A\rangle$

Definition of $\mathcal{R}_f\langle A\rangle$

Given a semiring $\mathcal{R}$ and a set A, the space of functions $f : A \to \mathcal{R}$ of **finite support** is denoted $\mathcal{R}_f\langle A\rangle$. It is a semimodule over $\mathcal{R}$ with:

- A commutative sum: $(f+g)(x) = f(x)+g(x)$
- An external product: $(J\cdot f)(x) = J\cdot f(x)$

Example: multisets

$\mathbb{N}_f\langle A\rangle$ correspond to finite multisets over A.

Multiset notation

For very simple functions, multiset notation may be more convenient:

$$[] := (\forall\alpha, \alpha \mapsto 0) \qquad [J\cdot\beta] := (\beta \mapsto J\ ; \forall\alpha \neq \beta, \alpha \mapsto 0)$$

The semiring $\mathbb{N}_f\langle\mathcal{M}\rangle$

$\mathbb{N}_f\langle\mathcal{M}\rangle$

Given a monoid $\mathcal{M}$, $\mathbb{N}_f\langle\mathcal{M}\rangle$ is a semiring with:

- A commutative sum: $(f+g)(x) = f(x)+g(x)$
- A product (Dirichlet convolution):

$$(f\cdot g)(x) = \sum_{\substack{y,z \\ x=y\cdot z}} f(y)\cdot g(z)$$

Example of such semirings

- $\mathbb{N} = \mathbb{N}_f\langle\mathbb{1}\rangle$
- Ghica's $\mathbb{N}_f\langle Aff_1^c\rangle$
- $\mathbb{P}oly = \mathbb{N}_f\langle\mathbb{N}_+\rangle$

Remark

This correspond to recurent semirings in implicit complexity, we will try to explain why.

Ghica's grammar and examples

Ghica's grammar

Given a semiring $\mathcal{R}$ we define typed lambda calculus:

(terms)	$\mathbf{\Lambda}$	M, N	$::=$	$x \mid \lambda x.M \mid M\ N$	
(linear types)	$\mathbf{T}_\ell^{\mathcal{R}}$	θ	$:=$	$\alpha \mid \Theta \multimap \theta$	
(types)	$\mathbf{T}^{\mathcal{R}}$	Θ	$:=$	$J{\cdot}\theta$	$J \in \mathcal{R}$

$\mathbf{T}_\ell^{\mathcal{R}}$	actual types of our terms
$\mathbf{T}^{\mathcal{R}}$	banged types that count multiplicities
$J{\cdot}\theta$	denotes that θ has to be used exactly J times

Some examples (with $\mathcal{R} = Hom(\mathbb{N})$)

$\lambda xy.y$: $0{\cdot}\theta \multimap 1{\cdot}\theta' \multimap \theta'$ $\qquad$ $\lambda xy.x(xy)$: $4{\cdot}(3{\cdot}\theta \multimap \theta) \multimap 6{\cdot}\theta \multimap \theta$

$$\lambda xy.xyy: 1{\cdot}(2{\cdot}\theta \multimap 3{\cdot}\theta \multimap \theta') \multimap 5{\cdot}\theta \multimap \theta'$$

$$\lambda xyz.x(yz): 1{\cdot}(f{\cdot}\theta \multimap \theta) \multimap f{\cdot}(g{\cdot}\theta \multimap \theta) \multimap f \circ g{\cdot}\theta \multimap \theta$$

$$\lambda xyz.y(xz): g{\cdot}(f{\cdot}\theta \multimap \theta) \multimap 1{\cdot}(g{\cdot}\theta \multimap \theta) \multimap g \circ f{\cdot}\theta \multimap \theta$$

Typing judgments

Ghica's linear type system

$$\frac{}{x{:}\mathbf{1}{\cdot}\theta \vdash x{:}\theta}\ \text{Id} \qquad \frac{\Gamma \vdash M{:}\theta}{\Gamma, x{:}\mathbf{0}{\cdot}\theta' \vdash M{:}\theta}\ \text{Weak} \qquad \frac{\Gamma, x{:}\Theta \vdash M{:}\theta'}{\Gamma \vdash \lambda x.M{:}\Theta \multimap \theta'}\ \text{Abs}$$

$$\frac{\Gamma \vdash M{:}J{\cdot}\theta \multimap \theta' \qquad \Gamma' \vdash N{:}\theta \qquad |\Gamma| = |\Gamma'|}{\Gamma + J{\cdot}\Gamma' \vdash M\ N : \theta'}\ \text{App}$$

where $|\Gamma| = |\Gamma'|$ means the equality of contexts except for the multiplicity of the types:

$$|(x_i{:}J_i{\cdot}\theta_i)_{i\leq k}| = |(y_i{:}K_i{\cdot}\theta'_i)_{i\leq k'}| \Leftrightarrow (k = k' \ \wedge\ x_i = y_i \ \wedge\ \theta_i = \theta'_i)$$
$$(x_i{:}J_i{\cdot}\theta_i)_i + (x_i{:}K_i{\cdot}\theta_i)_i := (x_i{:}(J+K){\cdot}\theta_i)_i$$

and where $\Gamma + J{\cdot}\Gamma'$ is the context obtain by applying addition and multiplication to the semiring:

$$J{\cdot}(x_i{:}K{\cdot}\theta_i)_i := (x_i{:}J{\cdot}K{\cdot}\theta_i)_i$$

Categorical model

A categorical model of linear logic: $Rel^{\mathcal{R}}$

Objects: Sets **Morphisms:** Relations

Exponential: $!A = \mathcal{R}_f\langle A\rangle$

We will see later that the semiring $\mathcal{R}$ must be a multiplicity semiring.

The co-kleisli category: $Rel_!^{\mathcal{R}}$

Objects: Sets **Morphisms:** Relations between $\mathcal{R}_f\langle A\rangle$ and B

Identities: $1_A = \{([\mathbf{1}\cdot\alpha],\alpha) | \alpha \in A\}$

Composition: $f;g := \{(\Sigma_{\beta\in|b|} b_\beta \cdot a^\beta, \alpha) | (b,\alpha) \in g, (a^\beta,\beta) \in f\}$

Cartesian product: $\&_{i\in I} A_i := \{(i,\alpha) | i \in I, \alpha \in A_i\}$

Exponential object: $A \Rightarrow B = \mathcal{R}_f\langle A\rangle \times B$

All ccc-diagrams are given by the co-Kleisli construction from the exponential comonad (when $\mathcal{R}$ is a multiplicity semiring).

Intersection type system

Interpretation of types

$$[\![o]\!] = \mathbb{1} \qquad [\![\tau \to \tau']\!] = \mathcal{R}_f\langle [\![\tau]\!] \rangle \times [\![\tau']\!]$$

Ehrhard's typing rules

$$\frac{\alpha \lhd \tau}{x{:}[\mathbf{1}\cdot\alpha]\lhd\tau \vdash x{:}\alpha\lhd\tau}\ \text{Id} \qquad \frac{\Gamma \vdash M{:}\alpha\lhd\tau}{\Gamma, x{:}[]\lhd\tau' \vdash M{:}\alpha\lhd\tau}\ \text{Weak}$$

$$\frac{\Gamma, x{:}a\lhd\tau \vdash M{:}\beta\lhd\tau'}{\Gamma \vdash \lambda x.M{:}a\multimap\beta\lhd\tau\to\tau'}\ \text{Abs}$$

$$\frac{\Gamma \vdash M{:}a\multimap\beta\lhd\tau\to\tau' \qquad \bigwedge_{\alpha\in|a|}\ \Gamma^\alpha \vdash N{:}\alpha\lhd\tau \qquad |\Gamma| = |\Gamma^\alpha|}{\Gamma + \sum_{\alpha\in|a|} a_\alpha\cdot\Gamma^\alpha \vdash M\ N{:}\ \beta\lhd\tau'}\ \text{App}$$

$$|(x_i{:}a_i\lhd\tau_i)_{i\leq k}| = |(y_i{:}b_i\lhd\tau'_i)_{i\leq k}| \ \Leftrightarrow\ (k = k' \ \wedge\ x_i = y_i \wedge\ \tau_i = \tau'_i)$$

$$(x_i{:}a_i\lhd\tau_i)_i + (x_i{:}b_i\lhd\tau_i)_i := (x_i{:}(a_i+b_i\lhd\tau_i)_i$$

$$J\cdot(x_i{:}a_i\lhd\tau_i)_i := (x_i{:}J\cdot a_i\lhd\tau_i)_i$$

Comparison

Ehrhard's typing rules

$$\frac{}{x{:}[1\cdot\alpha] \vdash x{:}\alpha}\ \text{Id} \qquad \frac{\Gamma \vdash M{:}\alpha}{\Gamma, x{:}[] \vdash M{:}\alpha}\ \text{Weak} \qquad \frac{\Gamma, x{:}a \vdash M{:}\beta}{\Gamma \vdash \lambda x.M{:}a\multimap\beta}\ \text{Abs}$$

$$\frac{\Gamma \vdash M{:}a\multimap\beta \qquad \bigwedge_{\alpha\in|a|}\ \Gamma^\alpha \vdash N{:}\alpha \qquad |\Gamma| = |\Gamma^\alpha|}{\Gamma +\ \sum_{\alpha\in|a|} a_\alpha\cdot\Gamma^\alpha \vdash M\ N : \beta}\ \text{App}$$

$$(x_i{:}a_i)_i + (x_i{:}b_i)_i := (x_i{:}(a_i+b_i))_i \qquad J\cdot(x_i{:}a_i)_i := (x_i{:}J\cdot a_i)_i$$

Ghica's typing rules

$$\frac{}{x{:}\mathbf{1}\cdot\theta \vdash x{:}\theta}\ \text{Id} \qquad \frac{\Gamma \vdash M{:}\theta}{\Gamma, x{:}\mathbf{0}\cdot\theta' \vdash M{:}\theta}\ \text{Weak} \qquad \frac{\Gamma, x{:}\Theta \vdash M{:}\theta'}{\Gamma \vdash \lambda x.M{:}\Theta\multimap\theta'}\ \text{Abs}$$

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$$(x_i{:}J_i\cdot\theta_i)_i + (x_i{:}K_i\cdot\theta_i)_i := (x_i{:}(J+K)\cdot\theta_i)_i \qquad J\cdot(x_i{:}K\cdot\theta_i)_i := (x_i{:}J\cdot K\cdot\theta_i)_i$$

interpretation of the calculus

Intersection type system

$$\frac{\alpha \lhd \theta}{x:[\mathbf{1}\cdot\alpha] \lhd \mathbf{1}\cdot\theta \vdash x:\alpha \lhd \theta}\ \text{Id} \qquad \frac{\Gamma \vdash M:\alpha \lhd \theta}{\Gamma, x:(\!|\,|\!) \lhd \mathbf{0}\cdot\theta' \vdash M:\alpha \lhd \theta}\ \text{Weak}$$

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$$|(x_i:a_i \lhd J_i\cdot\theta_i)_{i \le k}| = |(y_i:b_i \lhd K_i\cdot\theta'_i)_{i \le k}\ \Leftrightarrow\ (k = k'\ \wedge\ x_i = y_i \wedge\ \theta_i = \theta'_i)$$

$$(x_i:a_i \lhd J_i\cdot\theta_i)_i + (x_i:b_i \lhd K_i\cdot\theta_i)_i := (x_i:(a_i + b_i \lhd (J_i + K_i)\theta_i)_i$$

$$J\cdot(x_i:a_i \lhd K_i\cdot\theta_i)_i := (x_i:J\cdot a_i \lhd (J\cdot K_i)\cdot\theta_i)_i$$

Interpretation of types

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$$[\![o]\!] = \mathbb{1} \qquad [\![\Theta \multimap \theta]\!] = [\![\Theta]\!] \times [\![\theta]\!] \qquad [\![J\cdot\theta]\!] = \mathcal{R}_J\langle [\![\theta]\!] \rangle$$

Where $\mathcal{R}_J\langle A\rangle$ are vectors of $\mathcal{R}_f\langle A\rangle$ whose wedge (in a sort of ℓ_1 norm) is exactly $J \in sR$:

$$\|a\|_{\ell_1} := \Sigma_{\alpha\in|a|} a_\alpha \qquad \mathcal{R}_J\langle A\rangle := \{a \in \mathcal{R}_f\langle A\rangle \mid \|a\|_{\ell_1} = J\}$$

Stratification

$$![\![\theta]\!] = \bigcup_{J\in\mathcal{R}} [\![J\cdot\theta]\!] \qquad J \neq K \Rightarrow [\![J\cdot\theta]\!] \cap [\![K\cdot\theta]\!] = \emptyset$$

Some intuition

Stratification

$$!\llbracket\theta\rrbracket = \bigcup_{J\in\mathcal{R}} \llbracket J\cdot\theta\rrbracket \qquad J \neq K \Rightarrow \llbracket J\cdot\theta\rrbracket \cap \llbracket K\cdot\theta\rrbracket = \emptyset$$

For $\mathrm{Order}(\tau) < 2$: $\llbracket\tau\rrbracket = \bigcup_{\mathrm{Era}(\theta)=\tau}\llbracket\theta\rrbracket$

where $\mathrm{Era}(\theta)$ is the simple type obtained by effacing semiring annotations.

For $\mathrm{Order}(\tau) \geq 2$: $\llbracket\tau\rrbracket \neq \bigcup_{\mathrm{Era}(\theta)=\tau}\llbracket\theta\rrbracket$

For exemple
$[[] \multimap *, [*] \multimap *] \multimap * \in \llbracket(* \to *) \to *\rrbracket$.

Singleton degeneration

In fact linear types a so restrictive that all types interpretation are singleton. But this doese not hold for natural generalisations.

A semiring with "some" properties?

A semiring $\mathcal{M}$ is said to have multiplicities if:

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(MS3) $\mathcal{M}$ has additive splitting properties:

$$n_1 + n_2 = p_1 + p_2 \ \Rightarrow \exists r_{11}, r_{12}, r_{21}, r_{22}, p_i = r_{1i} + r_{2i}, n_i = r_{i1} + r2i$$

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(MS4) $\mathcal{M}$ has multiplicative splitting property:

$$n_1 + n_2 = pm \ \Rightarrow \exists p_1, p_2, m_{11}, m_{12}, m_{21}, m_{22}, p = p_1 + p_2,$$

$$m = m_{11} + m_{12} = m_{21} + m_{22}, n_i = p_1 m_{1i} + p_2 m_{2i}$$

A semiring onto morphism:

Given $\phi : \mathcal{S} \twoheadrightarrow \mathcal{R}$, we can interpret $\mathbf{T}^{\mathcal{R}}$ in a stratified version of $\mathrm{REL}_!^{\mathcal{S}}$:

Interpretation of types

$$[\![o]\!] = \mathbb{1} \qquad [\![\Theta \multimap \theta]\!] = [\![\Theta]\!] \times [\![\theta]\!] \qquad [\![J \cdot \theta]\!] = \mathcal{S}_J\langle [\![\theta]\!] \rangle$$

Where $\mathcal{S}_J\langle A\rangle$ are vectors of $\mathcal{S}_f\langle A\rangle$ whose wedge is in $\phi^{-1}(J)$:

$$\|a\|_{\ell_1} := \Sigma_{\alpha\in|a|} a_\alpha \qquad \mathcal{S}_J\langle A\rangle := \{a \in \mathcal{S}_f\langle A\rangle \mid \phi(\|a\|_{\ell_1}) = J\}$$

Example

For $\mathcal{R} = \mathbb{1}$ and $\mathcal{S} = \mathbb{N}$: gives the interpretation of simple types in $\mathrm{REL}_!$.

For $\mathcal{R} = \mathbb{Z}/2\mathbb{Z}$ and $\mathcal{S} = \mathbb{N}$: if θ is a type of $\mathbf{T}_{\mathbb{Z}/2\mathbb{Z}}$ without any 1 then $\alpha \in [\![\theta]\!]$ iff $\alpha \in [\![\mathit{Eff}(\theta)]\!]$ and every bags inside θ are of even size.

A semiring onto morphism: interpretation of the calculus

Intersection type system

$$\frac{\alpha \lhd \theta}{x{:}[\mathbf{1}\cdot\alpha]\lhd\mathbf{1}\cdot\theta \vdash x{:}\alpha\lhd\theta}\ \text{Id} \qquad \frac{\Gamma \vdash M{:}\alpha\lhd\theta}{\Gamma, x{:}(\!|\,|\!)\lhd\mathbf{0}\cdot\theta' \vdash M{:}\alpha\lhd\theta}\ \text{Weak}$$

$$\frac{\Gamma, x{:}a\lhd\Theta \vdash M{:}\beta\lhd\theta'}{\Gamma \vdash \lambda x.M{:}a\multimap\beta\lhd\Theta\multimap\theta}\ \text{Abs}$$

$$\frac{\Gamma \vdash M{:}a\multimap\beta\lhd J\cdot\theta\multimap\theta' \qquad \bigwedge_{\alpha\in|a|}\ \Gamma^\alpha \vdash N{:}\alpha\lhd\theta \qquad |\Gamma| = |\Gamma^\alpha|}{\Gamma +\ \Sigma_{\alpha\in|a|} a_\alpha\cdot\Gamma^\alpha \vdash M\ N{:}\ \beta\lhd\theta'}\ \text{App}$$

$$|(x_i{:}a_i\lhd J_i\cdot\theta_i)_{i\leq k}| = |(y_i{:}b_i\lhd J'_i\cdot\theta'_i)_{i\leq k} \ \Leftrightarrow\ (k = k' \ \wedge\ x_i = y_i \wedge\ \theta_i = \theta'_i)$$

$$(x_i{:}a_i\lhd J_i\cdot\theta_i)_i + (x_i{:}b_i\lhd J'_i\cdot\theta_i)_i := (x_i{:}(a_i+b_i\lhd(J_i+J'_i)\theta_i)_i$$

$$K\cdot(x_i{:}a_i\lhd J_i\cdot\theta_i)_i := (x_i{:}K\cdot a_i\lhd(\phi(K)\cdot J_i)\cdot\theta_i)_i$$

Structuring a semiring

Recalls on $\mathbb{N}_f\langle\mathcal{R}\rangle$

$\mathbb{N}_f\langle\mathcal{R}\rangle$ is a semiring with

- A commutative sum: $(f+g)(x) = f(x)+g(x)$
- A product (Dirichlet convolution):

$$(f\cdot g)(x) = \sum_{\substack{y,z \\ x=y\cdot z}} f(y)\cdot g(z)$$

The onto morphism

We can define $\phi : \mathbb{N}_f\langle\mathcal{R}\rangle \twoheadrightarrow \mathcal{R}$ by:

$$\phi(f) = \Sigma_{J\in\mathcal{R}} f(J)\cdot J$$

where $n\cdot J$ is a macro for $J + \underbrace{\cdots}_{n \text{ times}} + J$

Does $\mathbb{N}_f\langle\mathcal{R}\rangle$ have multiplicities?

(Com) $\mathbb{N}_f\langle\mathcal{R}\rangle$ is not necessarily commutative: iff $\mathcal{M}$ is commutative

(MS1) $\mathbb{N}_f\langle\mathcal{R}\rangle$ is positive: The sum is the one of $\mathbb{N}$

(MS2) $\mathbb{N}_f\langle\mathcal{R}\rangle$ is discreet: The sum is the one of $\mathbb{N}$

(MS3) $\mathbb{N}_f\langle\mathcal{R}\rangle$ has additive splitting properties:The sum is the one of $\mathbb{N}$

(MS4) $\mathbb{N}_f\langle\mathcal{R}\rangle$ does not have multiplicative splitting property:

Does $\mathbb{N}_f\langle\mathcal{R}\rangle$ have multiplicities?

But $\mathbb{N}_f\langle\mathcal{R}\rangle$ has generalised multiplicites, *i.e.* respect:

(MS1) $\mathbb{N}_f\langle\mathcal{R}\rangle$ is positive.

(MS2) $\mathbb{N}_f\langle\mathcal{R}\rangle$ is discreet.

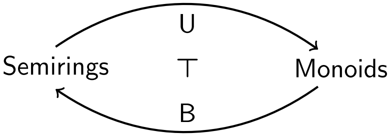
(MS3) $\mathbb{N}_f\langle\mathcal{R}\rangle$ has additive splitting properties.

(MS4') $\mathbb{N}_f\langle\mathcal{R}\rangle$ has multiplicative k-ary splitting property:

$$n_1+n_2 = pm \;\Rightarrow \exists k, p_1, p_2, m_{11}, m_{12}, m_{21}, m_{22}, p = \sum_{j\leq k} p_j,$$

$$n_i = \sum_{j\leq k} p_j m_{ji}, \forall j \leq k, m = m_{j1}+mj2$$

Forgetful adjunction



Where U is the forgetful functor and it adjunction B generate from $\mathcal{M}$ the free semimodule of basis $\mathcal{M}$ with the multiplication
$(\Sigma_i n_i \cdot a_i) \times_{B(\mathcal{M})} (\Sigma_j m_j \cdot b_j) = \Sigma_{i,j} n_i m_j \cdot (a_i \times_{\mathcal{M}} b_j)$
We have then the properties:

$$Im(B) \subseteq MSR^*$$

$$\Phi = der(BU)_{\mathcal{R}} : (BU(\mathcal{R}) \twoheadrightarrow \mathcal{R})$$

The conjecture and conclusion

Conjecture

If $\mathcal{R}$ has generalised multipliities then $\mathrm{REL}^{\mathcal{R}}$ is a model of linear logic.

If this conjecture apear to be tru we would have:

General theorem

Every typing system $\mathbf{T}^{\mathcal{R}}$ has a model obtain by stratification of a certain $\mathrm{REL}_{!}^{\mathcal{S}}$.