The resource lambda calculus is short-sighted in its relational model

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Introduction

The resource calculus ($\partial\Lambda$)

A resource sensitive extension of the pure λ-calculus with linear subterms that cannot suffer any duplication nor erasing along the reduction.

The relational model M_∞

M_∞ is a reflexive object of MRel, which is the co-Kleisli category of the well known model of linear logic Rel.

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The relational model M_∞

M_∞ is a reflexive object of MRel, which is the co-Kleisli category of the well known model of linear logic Rel.

They have good properties:

- M_∞ is a *sensible model* of $\partial\Lambda$ ($M\Uparrow \Leftrightarrow [\![M]\!] = \emptyset$).
- M_∞ is *fully abstract* for the pure λ-calculus ([Man09]).
- Both $\partial\Lambda$ and M_∞ are well-behaved w.r.t. *Taylor expansion*.

Question of [Buc&Al 11]

Is M_∞ *fully abstract* for $\partial\Lambda$?

My answer [TLCA13]

NO

Definition of $\partial\Lambda$

Grammar

$$
\begin{array}{llllllll}
\text{(terms)} & \Lambda & M, N ::= & x & \mid & \lambda x.M & \mid & M\ P \\
\text{(bags)} & \Lambda^b & P, Q ::= & [M^!] & \mid & [M] & \mid & [] \quad \mid \quad P \cdot Q \\
\text{(sums)} & \Lambda^s & \mathbb{L}, \mathbb{M} \in \mathcal{M}_f(\Lambda) & & & & &
\end{array}
$$

$[N^!]$	is a usual λ-calculus argument, used as many time as wanted.
$[N]$	is a linear argument, used exactly once.
$[]$	the absence of arguments.
$P \cdot Q$	is the union of bags, *i.e.* parallel composition of arguments.
Λ^s	possible outcomes of a computation are finite sums of terms.

Reduction (by examples)

$$(\lambda x.x\ [x])\ [u^!] \to u\ [u] \qquad (\lambda x.x\ [x])\ [u] \to 0$$

$$(\lambda x.x\ [x])\ [u, v] \to (u\ [v]) + (v\ [u]) \qquad (\lambda x.x\ [x^!])\ [u] \to u\ []$$

$$(\lambda x.x\ [x^!])\ [u, v, w] \to (u\ [v, w]) + (v\ [u, w]) + (w\ [u, v])$$

Observational semantic of $\partial\Lambda$

Reduction Strategy

outer-reduction: Reduction under any non banged context.
outer-normal forms: terms with no possible outer-reductions:

$$\lambda x_1 ... x_k . y \ P_1 \cdots P_p$$

where

$$P_i = [N_1, ..., N_n, M_1^!, ..., M_m^!]$$

with $N_1, ..., N_n$ in outer-normal form.
may-outer-convergence: converges to a sum containing a normal form:

$$M \Downarrow \quad \text{iff} \quad M \to^* \Sigma_i N_i \text{ such that } \exists N_i \in \text{ONF}$$

Example

$(\lambda x.(x\ [M^!])) + \mathbb{N}$ is a *monf* form but not $\lambda x.(x\ [(\lambda y.y)\ x])$.

Operational order ($\sqsubseteq_o$)

$$M \sqsubseteq_o N \qquad \text{iff} \qquad \forall C(\!|.|\!),\ C(\!|M|\!) \Downarrow \Rightarrow\ C(\!|N|\!) \Downarrow$$

Definition of the CCC *MRel*

A categorical model of linear logic: *Rel*

Objects: Sets **Morphisms:** Relations

Exponential: $!A = \mathcal{M}_f(A)$

Remark: this is the free exponential.

The co-kleisli category: *MRel*

Objects: Sets **Morphisms:** relations from $\mathcal{M}_f(A)$ and B

Identities: $1_A = \{([\alpha], \alpha) | \alpha \in A\}$

Composition: $f; g := \{(\Sigma_{\beta \in b} a_\beta, \alpha) | (b, \alpha) \in g, (a_\beta, \beta) \in f\}$

Cartesian product: $\&_{i \in I} A_i := \{(i, \alpha) | i \in I, \alpha \in A_i\}$

Object of morphisms: $A \Rightarrow B = \mathcal{M}_f(A) \times B$

All ccc-diagrams are given by the co-Kleisli construction from the exponential comonad.

Definition of M_∞

A reflexive object of *MRel*: M_∞

$$M_0 = \emptyset \qquad M_{n+1} = \mathcal{M}_f(M_n)^{(\omega)} \qquad M_\infty = \bigcup M_n$$

app and *abs* such that $M_\infty \underset{abs}{\overset{app}{\rightleftharpoons}} (M_\infty \Rightarrow M_\infty)$ are trivial.

$\mathcal{M}_f(M_n)^{(\omega)}$: infinite quazi-everywhere empty lists of multisets over M_n.

This is the smallest solution of the equation $D := (!\,\&_\omega\, D)^\perp$ whose solutions are solutions of $D := (!D)^\perp \parr D$ (em i.e. $D := D \Rightarrow D$) but does not contain $\emptyset$.

Grammar of M_∞

(M_∞)	α, β	$::=$	$* \quad \mid \quad a \to \alpha$
$\mathcal{M}_f(M_\infty)$	a, b	$::=$	$[\alpha_1, \ldots, \alpha_k]$

With the equation: $\ast := [\,] \to \ast$
so that $\ast$ represent the **infinite list of empty multisets**.

Interpretation of $\partial\lambda$ into M_∞

$$[\![M]\!]^{\vec{x}} \subseteq (\mathcal{M}_f(M_\infty)^{\vec{x}} \times M_\infty) \qquad [\![P]\!]^{\vec{x}} \subseteq (\mathcal{M}_f(M_\infty)^{\vec{x}} \times \mathcal{M}_f(M_\infty))$$

Interpretation (by examples)

$$[\![x]\!]^{yx} = \{([][\alpha],\alpha)\} \qquad [\![\lambda x.x]\!] = \{[\alpha]\to\alpha\}$$

$$[\![[x^!]]\!]^{x} = \{(a,a)\} \qquad [\![[x,\lambda y.x]]\!]^{x} = \{([\alpha,\beta],[\alpha,[]\to\beta]\}$$

$$[\![\lambda y.y\ [x^!]]\!]^{x} = \{(a,[a\to\alpha]\to\alpha\} \qquad [\![\lambda y.y\ [x]]\!]^{x} = \{([\beta],[[\beta]\to\alpha]\to\alpha)\}$$

$$[\![M+N]\!]^{x} = [\![M]\!]^{x} \cup [\![N]\!]^{x} \qquad [\![[M^!,N^!]]\!] = \{a\cdot a' | a \in [\![M]\!], a' \in [\![N]\!]\}$$

$$[\![\boldsymbol{\Omega}]\!] = \emptyset \qquad [\![\boldsymbol{\Theta}]\!] = \{([\Pi_{\alpha\in a} b_\alpha\to\beta]\cdot a)\to\beta \mid (b_\alpha\to\alpha) \in [\![\boldsymbol{\Theta}]\!]\}$$

Remark: One can also see the interpretation as typing judgment into the non-idempotent type-system that is M_∞.

Full abstraction of M_∞ for $\partial\lambda$?

Sensibility

M_∞ is sensible for $\partial\lambda$:

$$M \Downarrow \quad \Leftrightarrow \quad [\![M]\!] \neq \emptyset$$

Inequational adequation

M_∞ is inequationally adequate for $\partial\lambda$

$$[\![M]\!]^{\vec{x}} \subseteq [\![N]\!]^{\vec{x}} \quad \Rightarrow \quad M \sqsubseteq_o N$$

Those are obtained easily using Taylor expansion and its properties [TLCA13].

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Failure of Inequational completeness

$$M \sqsubseteq_o N \quad \not\Rightarrow \quad [\![M]\!]^{\vec{x}} \subseteq [\![N]\!]^{\vec{x}}$$

Failure of completeness

$$M \equiv_o N \quad \not\Rightarrow \quad [\![M]\!]^{\vec{x}} = [\![N]\!]^{\vec{x}}$$

These assertions are equivalents:

$$M \sqsubseteq_o N \Leftrightarrow M{+}N \equiv_o N \qquad [\![M]\!] \subseteq [\![N]\!] \Leftrightarrow [\![M{+}N]\!] = [\![N]\!]$$

$\mathbf{A}$ such that $\lambda x.x \sqsubseteq_o \mathbf{A}$ and $[\![\lambda x.x]\!] \not\subseteq [\![\mathbf{A}]\!]$

$$\mathbf{A} := \mathbf{\Theta}\ [\mathbf{G}, \mathbf{F}^!]$$

$\mathbf{G} := \lambda uvw.w\ [(\lambda x.x)\ [v^!]]$ $\qquad$ $\mathbf{F} := \lambda uv_1v_2.u\ [(\lambda x.x)\ [v_1^!]\ [v_2^!]]$

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$\mathbf{A}$ can be seen as a non deterministic *While*

G is the stopping condition, and **F** the continuation condition:

$$\mathbf{A} \equiv_\beta \mathbf{G}[\mathbf{A}^!] + \mathbf{F}[\mathbf{A}^!] \equiv_\beta \mathbf{G}[\mathbf{A}^!] + \mathbf{F}[\mathbf{G}[\mathbf{A}^!]] + \mathbf{F}^2[\mathbf{A}^!] \equiv_\beta \cdots$$

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$\mathbf{A}$ as an infinite sum

$\mathbf{A} \simeq \Sigma_{n\geq 1}\mathbf{B}_n$ with for $n \geq 1$:

$$\mathbf{B}_n = \lambda v_1 \ldots v_n w.w\ [(\lambda x.x)\ [v_1^!]\ [v_2^!]\ \cdots\ [v_n^!]]$$

$\mathbf{G}[\mathbf{A}^!] \to_o^* \mathbf{B}_1$ For all i, $\mathbf{F}\ [\mathbf{B}_i^!] \to_o^* \mathbf{B}_{i+1}$

$$\mathbf{A} \equiv_\beta \mathbf{B}_1 + \mathbf{F}[\mathbf{A}] \equiv_\beta \mathbf{B}_1 + \mathbf{B}_2 + \mathbf{F}^2[\mathbf{A}] \equiv_\beta (\Sigma_{n=1}^k \mathbf{B}_n) + \mathbf{F}^k[\mathbf{A}]$$

$\lambda x.x$ and $\Sigma_i B_i$

$\mathbf{A} \simeq \Sigma_{n\geq 1}\mathbf{B}_n$ with for $n \geq 1$: $\mathbf{B}_n = \lambda v_1 \dots v_n w.w\ [(\lambda x.x)\ [v_1^!]\ [v_2^!]\ \cdots\ [v_n^!]]$

$\lambda x.x \sqsubseteq_o \Sigma_i B_i$

If $(\lambda x.x)\ P_1 \cdots P_k$ converges then:

$$\mathbf{B}_k\ P_1 \cdots P_k \to^* \lambda w.w\ [(\lambda x.x)\ P_1 \cdots\ P_k]$$

Remark: We are using a context lemma for $\partial\lambda$ [TLCA13]

$[\![\lambda x.x]\!] \not\subseteq \Sigma_i B_i$

$[*]{\rightarrow}* \in [\![\lambda x.x]\!]$, but for every n, $[*]{\rightarrow}* \notin [\![\mathbf{B}_n]\!]$:

$$[*]{\rightarrow}* \in [\![B_n]\!] \Leftrightarrow ([*][] \cdots [][], *) \in [\![w\ [(\lambda x.x)[v_1^!]\ [v_2^!] \cdots [v_n^!]]\!]^{v_1 \dots v_n w}$$

What is impossible.

Clues for the full abstraction

- M_∞ is fully abstract for the λ-calculus ([Manzonetto09])
- M_∞ is fully abstract for the purely linear fragment ([Buc &Al 12])
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Our result, $\mathbf{A} := \mathbf{\Theta}\ [\mathbf{G}, \mathbf{F}^!]$

We construct a counter-example as combination of an untyped fixpoint and a may non deterministic argument to create a term of infinite range whose behavior will not respect the semantics.

The short-sightedness

Observations in $\partial\lambda$

They are performed by finite applicative contexts, thus **their range is finite**.

Observations in M_∞

They are performed by elements of M_∞ that are infinite lists, thus **their range are infinite**.

Range of $\partial\lambda$

$\partial\lambda$ contains **terms of infinite range** that can outrange the operational observations but not the denotational one.

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Over-approximation?

Over-approximations in denotational semantics are **too strong**.

Must/probabilistic?

Are must and probabilistic convergences **more natural**?

In any case the problem of finding a fully abstract model of $\partial\lambda$ is reopen.

Conclusion

A term of infinite range

$\mathbf{A} = \mathbf{\Theta}\ [G, F^!]$: A non deterministic while that can *choose* its head variable

M_∞ is not inequationally fully abstract for $\partial\lambda$

$(\lambda x.x) \sqsubseteq_o \mathbf{A}$ but $[\![\lambda x.x]\!] \not\subseteq [\![\mathbf{A}]\!]$

M_∞ is not fully abstract for $\partial\lambda$

$(\mathbf{A}+\lambda x.x) \equiv_o \mathbf{A}$ but $[\![\mathbf{A}+\lambda x.x]\!] \neq [\![\mathbf{A}]\!]$

Questions?