Toward dependant coeffects

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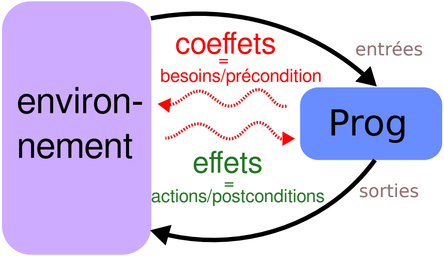
Motivations: Coeffects

Functional programing:
a program is a function.

Reality:
a program does not reduce to an input-output routine

Solution:
using (co)effects.



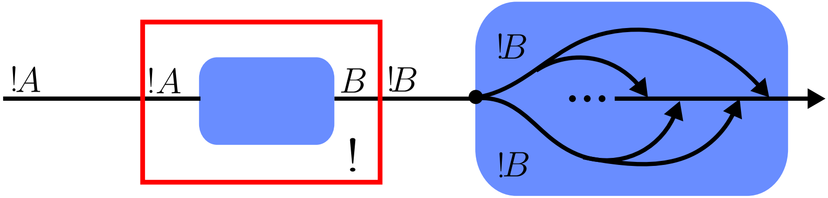
Coeffects/requirements/comonad

- file existence,
- bound over the size of a vector,
- bound over the number of copies,
- scheduling...

Effects/action/monad

- I/O,
- references,
- exceptions,
- continuations...

Linear decomposition $A \Rightarrow B := !A \multimap B$

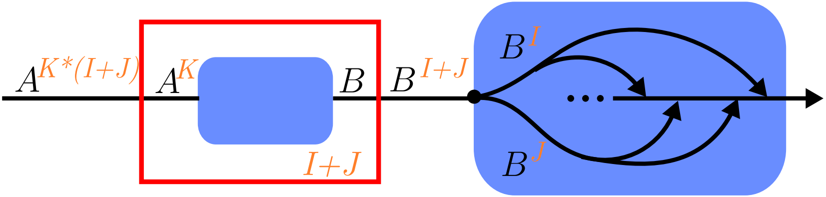


Linear logic (LL)

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ \texttt{Weak} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\ \texttt{Der}$$

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ \texttt{Contr} \qquad \frac{!A_1, \cdots, !A_n \vdash B}{!A_1, \cdots !A_n \vdash !B}\ \texttt{Prom}$$

Coeffect decomposition $A \Rightarrow B := A^I \multimap B$



Bounded logics ($B_{\mathscr{S}}LL$)

Coeffects are represented by semirings $(\mathscr{S}, +, 0, *, 1)$

$$\frac{\Gamma \vdash B}{\Gamma, A^0 \vdash B}\ \texttt{Weak} \qquad \frac{\Gamma, A \vdash B}{\Gamma, A^1 \vdash B}\ \texttt{Der}$$

$$\frac{\Gamma, A^I, A^J \vdash B}{\Gamma, A^{I+J} \vdash B}\ \texttt{Contr} \qquad \frac{A_1^{I_1}, \cdots, !A_n^{I_n} \vdash B}{A_1^{I_1*J}, \cdots A_n^{I_n*J} \vdash B^J}\ \texttt{Prom}$$

One $B_{\mathscr{S}}LL$ per semiring

Examples of captured coeffects

Number of copies

Program using 3 times its argument:

$$A^{3} \to B$$

Program with a dead input as second argument:

$$A^{2} \to B^{0} \to C$$

Program using 2 times a program itself using 1 times its argument

$$(A^{1} \to B)^{2} \to C$$

Size of a Stream

A stream of "size" $\leq n$ can be written (size-type)

$$St[n]$$

Type of a program which input and output have size 3 and 2:

$$St[3] \to St[2]$$

Program which input requirements are dependant in the output:

$$\forall n,\ St[f(n)] \to St[n]$$

written $St^{f} \to St$ for short.

Our issue: The Dependence

Polymorphism

Question: How to represent

$$\forall i,j,\ A[f(i,j)] \to B[g(i,j)] \to C[i]$$

Solution: Adding free variables:

$$\vdash_{j\in\mathbb{N}} A^{f(\_,j)} \to B^{g(\_,j)} \to C$$

Dependence

Question: How to represent

$$\forall i,\ \Big(\forall k, B[g(i,k)] \to C[k]\Big) \to D[i]$$

Solution: Adding binders:

$$\vdash (B^{k:=g(i,\_)} \to C)^{i:=\_} \to D$$

What about the multiplicity example?

Ho to represent a function $(A \to B \to C) \to D$ which use twice its argument but differently, once with the type $A^3 \multimap B^3 \multimap C$ and once with the type $A^4 \multimap B^4 \multimap C$?

Solution:

$$(A^x \multimap B^x \multimap C)^{x\in[2,3]} \multimap D$$

Our issue: The Dependence

Polymorphism

Question: How to represent

$$\forall i,j,\ A[f(i,j)] \to B[g(i,j)] \to C[i]$$

Solution: Adding free variables:

$$\vdash_{j\in\mathbb{N}} A^{f(\_,j)} \to B^{g(\_,j)} \to C$$

Dependence

Question: How to represent

$$\forall i,\ \Big(\forall k, B[g(i,k)] \to C[k]\Big) \to D[i]$$

Solution: Adding binders:

$$\vdash (B^{k:=g(i,\_)} \to C)^{i:=\_} \to D$$

What about the multiplicity example?

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A^2 \multimap B^2 \vdash A^2 \multimap B^2}\ \mathtt{Ax} \qquad \dfrac{}{A^3 \multimap B^3 \vdash A^3 \multimap B^3}\ \mathtt{Ax}}{(A^2 \multimap B^2),(A^3 \multimap B^3) \vdash (A^2 \multimap B^2) \otimes (A^3 \multimap B^3)}\ \otimes\mathtt{R}}{(A^2 \multimap A^2),(A^x \multimap B^x)^{x\in[3]} \vdash (A^2 \multimap B^2) \otimes (A^3 \multimap B^3)}\ \mathtt{Der}}{(A^x \multimap B^x)^{x\in[2]},(A^x \multimap B^x)^{x\in[3]} \vdash (A^2 \multimap B^2) \otimes (A^3 \multimap B^3)}\ \mathtt{Der}}{(A^x \multimap B^x)^{x\in[2,3]} \vdash_{\mathbb{1}} (A^2 \multimap B^2) \otimes (A^3 \multimap B^3)}\ \mathtt{Contr}$$

Logics with dependant coeffects in the literature

Indexed linear logic [Bucciarelli,Ehrhard'01]

Indexes are (inj) functions $f : \rho \to \sigma$

$$\frac{A^{g} \vdash_{\rho} B}{A^{f \circ g} \vdash_{\sigma} B^{f}}\ \textbf{Prom}$$

Dereliction restricted to f bijective

$$\frac{f_*(A) \vdash_{\sigma} B}{A^{f} \vdash_{\sigma} B}\ \textbf{Der}$$

A surprising contraction defined for $f : \rho \to \sigma$ and $g : \tau \to \sigma$

$$\frac{(A|_{\rho})^{f}, (A|_{\tau})^{g} \vdash_{\sigma} B}{A^{f \cup g} \vdash_{\sigma} B}\ \textbf{Contr}$$

Bounded linear logic [Girard,Scedrov,Scott'92]

The indexes are bounded binders $x \leq p$ with p a polynomial

The same kind of dereliction:

$$\frac{A[x := 0] \vdash_{\rho} B}{A^{x \leq 0} \vdash_{\rho} B}\ \textbf{Der}$$

Somme substitutive contraction:

$$\frac{A^{x \leq p}, (A[x := p + x])^{x \leq q} \vdash_{\rho} B}{A^{x \leq p+q} \vdash_{\sigma} B}\ \textbf{Contr}$$

But a weird promotion:

$$\frac{(A[y := \sum_{w \leq x} q(w) + z])^{z \leq q(x)} \vdash_{\rho;x} B}{A^{y \leq \sum_{w \leq p} q(w)} \vdash_{\rho} B^{x \leq p}}\ \textbf{Prom}$$

Global rewriting: Why and How

$$\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma \vdash_{\rho,i} A}}{\Gamma^{i:=f} \vdash_\rho A^{i:=f}}\ \textsf{Prom} \qquad \dfrac{\dfrac{\Pi_2}{A\{f/i\} \vdash_\rho B}}{A^{i:=f} \vdash_\rho B}\ \textsf{Der}}{\Gamma^{i:=1} \vdash_\rho B}\ \textsf{Cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Pi_1\{f/i\}}{\Gamma\{f/i\} \vdash_\rho A\{f/i\}} \qquad \dfrac{\Pi_2}{A\{f/i\} \vdash_\rho B}}{\Gamma\{f/i\} \vdash_\rho B}\ \textsf{Cut}}{\Gamma^{i:=f} \vdash_\rho B}\ \textsf{Der}$$

Our result

A dependant logic for coeffects...

- True dependence
- parametrised by a "dependant semiring"
- global rewriting
- "contains" BLL and IndLL

... Modeled by slicing linear categories

Any linear category

- naturally embed a "dependant semiring"
- can be sliced to model a dependant logic for coeffects

Oidification of the semiring

Oidification: Generalising an algebraic structure by associating to elements a source and a target
$\rightsquigarrow$ algebras becomes categories
$\rightsquigarrow$ the original structures are the one-point categories

Monoids $\mapsto$ Categories Groups $\mapsto$ Groupoids

Coeffects are morphisms

- IndLL: $f : \sigma \to \rho$ so that:

$$\frac{\vdash_\sigma B}{\vdash_\rho B^f}\ \textsf{Prom}$$

- BLL: $x \leq p$ from $\mathbb{N}^\rho$ to a segment in $\mathbb{N}$

$$\frac{\vdash_{\rho,x} B}{\vdash_\rho B^{x\leq p}}\ \textsf{Prom}$$

- Size types: $x := f$ from $\mathbb{N}^\rho$ to $\mathbb{N}$

$$\frac{\vdash_{\rho,x} B}{\vdash_\rho B^{x:=f}}\ \textsf{Prom}$$

- multiplicities: $\vec{x} \in M$ from $\mathbb{N}^\rho$ to a multiset in $\mathbb{N}^n$

$$\frac{\vdash_{\rho,\vec{x}} B}{\vdash_\rho B^{\vec{x}\in M}}\ \textsf{Prom}$$

Multiplications/diggings are compositions

IndLL:

$$\frac{(A^{f})^{g} \vdash_\sigma B}{A^{g\circ f} \vdash_\sigma B}\ \text{Digg}$$

Size-types:

$$\frac{(A^{x:=f})^{y:=g} \vdash_\sigma B}{A^{(x,y):=(f\circ g,g)} \vdash_\sigma B}$$

Multiplicities:

$$\frac{(A^{x\in M(y)})^{y\in N} \vdash_\sigma B}{A^{(x,y)\in\sum_{y\in N} map(x\mapsto(x,y))(M(y))} \vdash_\sigma B}$$

Issue: Derelictions should be the identities

IndLL:

$$\frac{f_*(A) \vdash_\sigma B}{A^{f} \vdash_\sigma B}\ \text{Der}$$

is different from

$$\frac{A \vdash_\sigma B}{A^{id} \vdash_\sigma B}\ \text{Der}$$

Multiplicities:

$$\frac{(A^3 \multimap B^3) \vdash C}{(A^x \multimap B^x)^{x\in[3]} \vdash C}\ \mathtt{Der}$$

is different from

$$\frac{(A^y \multimap B^y) \vdash_y C}{(A^x \multimap B^x)^{x\in[y]} \vdash_y C}\ \mathtt{Der}$$

$\rightsquigarrow$ We need global rewriting!!

An action to represent rewritings

A dependant semiring is

- A set environments $\sigma, \rho \in \mathscr{E}$
- A category $\mathscr{C}$ with elements of $\mathscr{E}$ as objects and coeffects $f, p, M\ldots$ as morphisms,
- A category $\mathscr{R}$ with elements of $\mathscr{E}$ as objects and rewriting transformations $\iota, \pi\ldots$ as morphisms,
- An action $\ltimes : \mathscr{C} \times \mathscr{R} \to \mathscr{C}$

Such that BlaBlaBla...

$$\frac{\iota(A) \vdash_\sigma B}{A^{id \ltimes \iota} \vdash_\sigma B}\ \textbf{Der}$$

the notation $\iota(A)$ is a rewriting over A!

$$\iota(A^f) := A^{f \ltimes \iota} \qquad \iota(A \multimap B) := \iota(A) \multimap \iota(B) \qquad \iota(A \otimes B) := \iota(A) \otimes \iota(B)$$

The enigma of the contraction

A dependant semiring is

- ...BlaBlaBla
- A Cartesian structure $(+, \pi_1, \pi_2)$ over $\mathscr{R}$,
- A tensorial product $+$ over (sliced categories of) $\mathscr{C}$.

Such that BlaBlaBla...

If $f : \rho \to \sigma$ and $g : \tau \to \sigma$ then $f + g : \rho + \tau \to \sigma$ and

$$\frac{\pi_1(A)^f \otimes \pi_2(A)^g \vdash_\sigma B}{A^{f+g} \vdash_\sigma B}\ \textbf{Contr}$$

A dependant semiring inside linear categories

$\mathscr{L}$: A linear category $\qquad$ $\mathtt{m}_{\mathbb{1}},\mathtt{m}_{\rho,\rho}$: monoidality of !

The environments $\mathscr{E}$

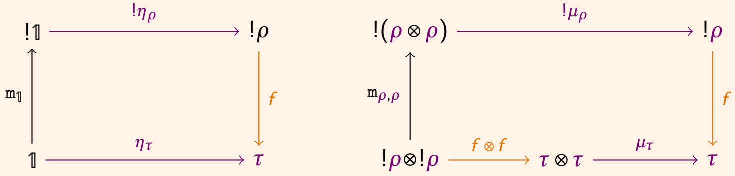
The $\otimes$-monoids $(\rho,\mu_\rho,\eta_\rho)$ over $\mathscr{L}$.

The rewriting category $\mathscr{R}$

The category of $\otimes$-monoid morphisms.

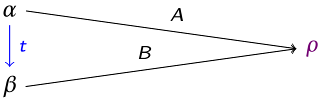
The coeffect category $\mathscr{C}$

The Kleisli category of monoid morphisms for the monad $(\rho,\mu,\eta)\mapsto(!\rho,(m_{\rho,\rho};\mu),(m_{\perp};\eta))$.
Spelling out, the morphisms are the $f:\rho\to\tau$ commuting:



Slicing linear categories

- Formula A of domain ρ are morphisms $A : \alpha \to \rho$
- Programs (or derivations) $x : A \vdash_\rho t : B$ of domain ρ are morphisms $t : \alpha \to \beta$ such that:



- Exponentials and rewritings are defined by:

$$A^{f} := {!\alpha} \xrightarrow{!A} {!\rho} \xrightarrow{f} \tau \qquad\qquad \iota(A) := \alpha \xrightarrow{A} \rho \xrightarrow{\iota} \tau$$

- Tensor product is defined by:

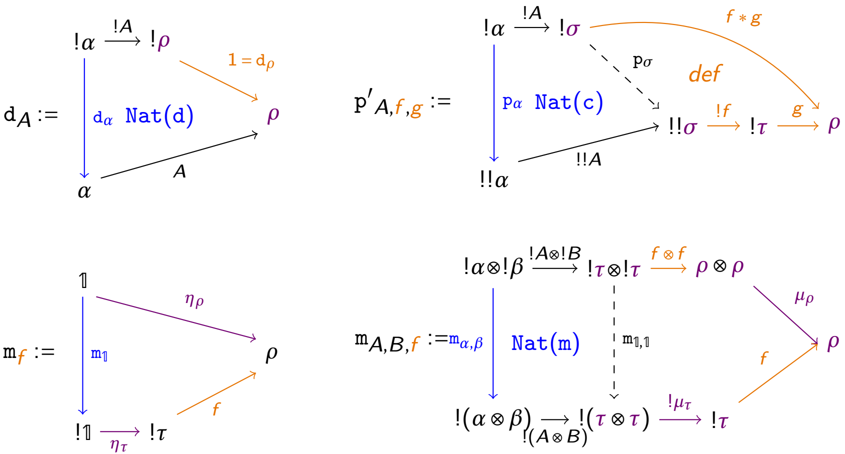
$$A \widetilde{\otimes} B := \alpha \otimes \beta \xrightarrow{A \otimes B} \rho \otimes \rho \xrightarrow{\mu_\rho} \rho$$

- Cartesian coproduct and product are defined by:

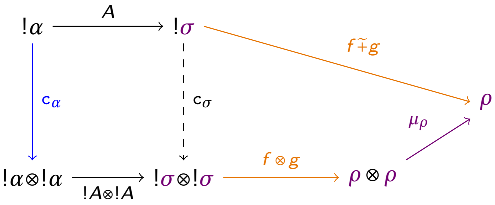
$$A \widetilde{\oplus} B := \alpha \oplus \beta \xrightarrow{\{A, B\}} \rho \qquad\qquad A \,\&\, B := \alpha \,\&\, \beta \xrightarrow{A \& B} \rho \,\&\, \tau =: \rho + \tau$$

However, the monoidal closure has to be required...

Exponential formulas



Two different contractions

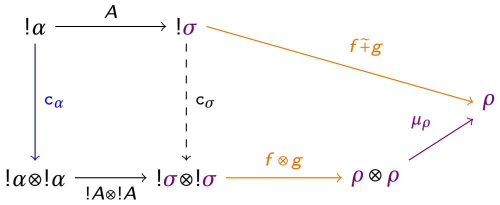


$$\frac{A^{f}, A^{g} \vdash_{\rho} B}{A^{f \tilde{+} g} \vdash_{\rho} B}$$

with

$$f, g : \sigma \to \rho$$

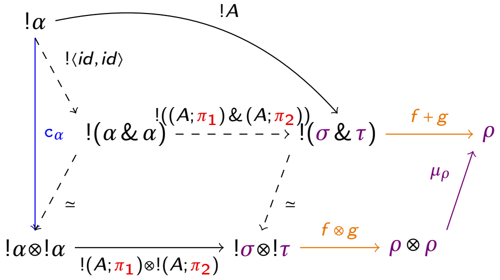
Two different contractions



$$\frac{A^f, A^g \vdash_\rho B}{A^{f \tilde{+} g} \vdash_\rho B}$$

with

$f, g : \sigma \to \rho$



$$\frac{\pi_1(A)^f, \pi_2(A)^g \vdash_\rho B}{A^{f+g} \vdash_\rho B}$$

with

$f : \sigma \to \rho \quad g : \tau \to \rho$

Conclusion

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Done / Work in Progress

- Ordering and relaxation
- Embedding all our examples
- Defining important fragments

Future work

- Generic realisability techniques
- Generic inference techniques
- A logic for dependant effects