Modelling Coeffects in the relational semantic

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Introduction: Coeffects and $B_{\mathcal{S}}LL$

Coeffects

Coeffects are requirements over the environment.

Examples

Existence (file), size (stream), copies (linear), scheduling...

$\mathcal{S}$-Graded comonad

$$D(\Gamma) \vdash t : A$$

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$$x_1 : {C_1}^{I_1}, ..., x_n : {C_n}^{I_n} \vdash t : A \qquad , I \in \mathcal{S}$$

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$B_{\mathcal{S}}LL$ logics [BrGaMaZd2013] [GhSm2013]

(types) $\quad A, B, C := \alpha \mid A \otimes B \mid A \multimap B \mid A^I \qquad ,\forall I \in \mathcal{S}$

Introduction: The Results

State of the art:

A complex categorical axiomatisation

The bounded exponential situation [BrGaMaZd2013]:
parametrisation of a linear category

No existing concrete model

Our contributions:

The stratification

linear category

$\overset{(\_)^{\cdot},\mathcal{Q}}{\rightsquigarrow}$

bounded exponential situation

A class of concrete models

A relational model for $B_{\mathcal{S}}LL$ (for any $\mathcal{S}$).

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The logics $B_{\mathcal{S}}LL$

1) Formalisation
2) Examples

Categorical semantics

1) Bounded exponential situation [Brunel&al]
2) Stratification of linear categories

Concrete semantics

1) Stratifying the relational model
2) Stratifying non-free exponential

Formalisation of $B_{\mathcal{S}}LL$

Ordered semiring $(|\mathcal{S}|, +, 0, *, 1, \leq)$

- $+$ and $*$ are associative
- $+$ is commutative and distribute over $*$
- 0 is neutral for $+$ and 1 is neutral for $*$
- $+$ and $*$ are monotone for $\leq$.

Examples

$(\mathbb{B}ool, \vee, f\!f, \wedge, tt, \{f\!f \leq tt\})$

$(\mathbb{N}, +_{\mathbb{N}}, 0_{\mathbb{N}}, *_{\mathbb{N}}, 1_{\mathbb{N}}, \leq_{\mathbb{N}})$

$(\mathbb{R}^+, +_{\mathbb{R}}, 0_{\mathbb{R}}, *_{\mathbb{R}}, 1_{\mathbb{R}}, \leq_{\mathbb{R}})$

$(\mathbb{N}_{-\infty}, max, -\infty, +, 0, \leq)$

Formulas and Sequents $\Gamma \vdash A$

(types) $\quad A, B, C := \alpha \mid A \otimes B \mid A \multimap B \mid A^I \qquad , \forall I \in \mathcal{S}$

$$\frac{\Gamma \vdash B}{\Gamma, A^0 \vdash B}\ \texttt{Weak} \qquad \frac{\Gamma, A \vdash B}{\Gamma, A^1 \vdash B}\ \texttt{Der} \qquad \frac{\Gamma, A^I, A^J \vdash B}{\Gamma, A^{I+J} \vdash B}\ \texttt{Contr}$$

$$\frac{{A_1}^{I_1}, \cdots, {A_n}^{I_n} \vdash B}{{A_1}^{I_1 * J}, \cdots {A_n}^{I_n * J} \vdash B^J}\ \texttt{J-Prom} \qquad \frac{\Gamma, A^I \vdash B \qquad J \geq I}{\Gamma, A^J \vdash B}\ \mathcal{S}\texttt{wL}$$

(plus IMLL sequent rules)

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Brunel&al's categorical semantic

Ordered semiring $\mathcal{S}$ as a bimonoidal category

The preordered dual category:

Objects: $I, J, K \in \mathcal{S}$ Morphisms: $\mathcal{S}[I, J]$ singleton if $I \geq J$

The sum and product are two monoidal products

Bounded exponential situation

- a symmetric monoidal category (model of IMLL) $(\mathcal{A}, \otimes, \multimap, \mathbf{1})$,
- a bifunctor: $(\_)^{\text{-}} : \mathcal{A} \times \mathcal{S} \to \mathcal{A}$
- 6 natural transformations:

$$p^I : A^{I*J} \Longrightarrow (A^I)^J \qquad d^I : A^1 \Longrightarrow A$$

$$c^I : A^{I+J} \Longrightarrow A^I \otimes A^J \qquad w^I : A^0 \Longrightarrow \mathbf{1}$$

$$m^I : A^I \otimes B^I \Longrightarrow (A \otimes B)^I \qquad n^I : \mathbf{1} \Longrightarrow \mathbf{1}^I$$

- plus 20 commutative diagrams.

Stratifying models of LL into models of $B_{\mathcal{S}}LL$

Stratification of a linear category $\mathcal{A}$

$$(\_)^{-} : \mathcal{A} \times \mathcal{S} \to \mathcal{A} \qquad \varrho_{A,J} : !A \Longrightarrow A^{J}$$

Such that:

- $\varrho_{A,J}$ is an epimorphism for all $a \in \mathcal{A}$ and $J \in \mathcal{S}$,
- plus 6 commuting diagrams

Def epi

$$A \overset{epi}{\to} B \overset{\phi}{\to} C = A \overset{epi}{\to} B \overset{\psi}{\to} C$$

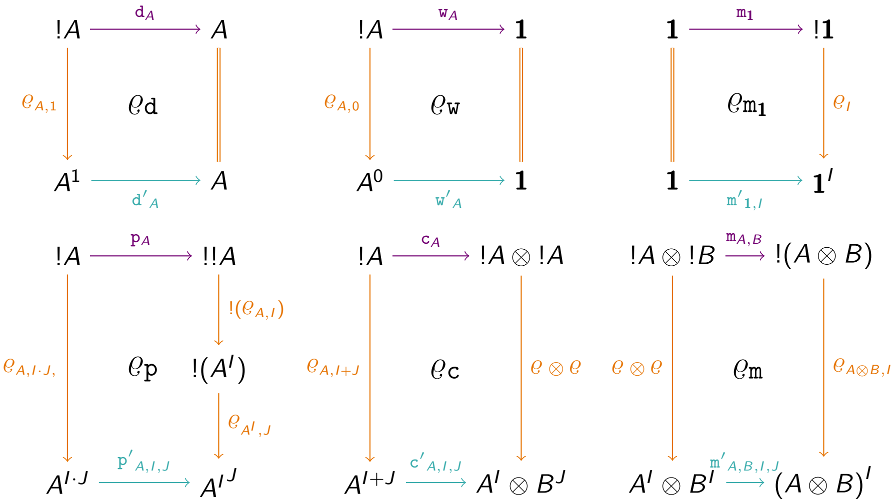
$$\Rightarrow$$

$$\phi = \psi$$

Theorem

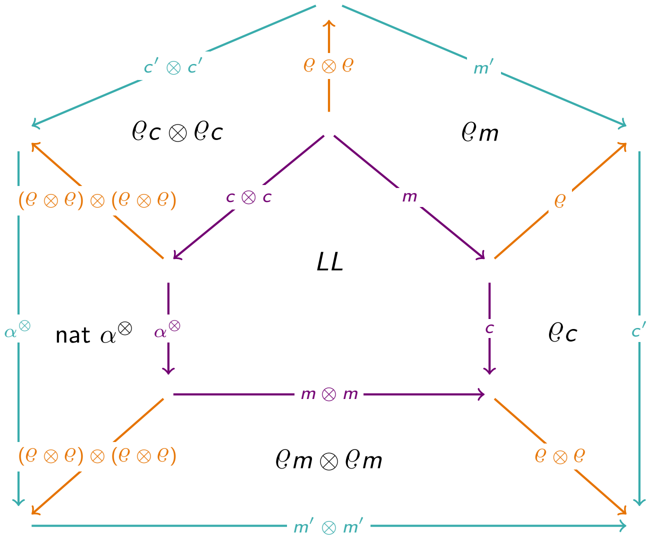
The stratification of a linear category yields a bounded exponential situation

Stratification diagrams



(d', p', w', c' and m' are uniquely determined)

An example of required diagram



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The model Rel of LL

The category *Rel* : a model of IMLL

Objects: Sets **Morphisms:** Relations

Exponential: $!A = \mathcal{M}_f(A)$

Semiring interpretation $[\![-]\!] : \mathcal{S} \to \mathcal{P}(\mathbb{N})$

$I \leq J$ implies $[\![I]\!] \subseteq [\![J]\!]$,

$[\![I]\!] \oplus [\![J]\!] \subseteq [\![I+J]\!]$, $\{0\} \subseteq [\![0_{\mathcal{S}}]\!]$,
$[\![I]\!] \odot [\![J]\!] \subseteq [\![I*J]\!]$, $\{1\} \subseteq [\![1_{\mathcal{S}}]\!]$.

Rel : a model of $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$ (given $[\![-]\!] : \mathcal{S} \to \mathcal{P}(\mathbb{N})$)

Bounded exponential:
$A^J = \{[a_1, ..., a_n] \in \mathcal{M}_f(A) \mid n \in [\![J]\!]\}$
epi transformation:
$\varrho_{J,A} = \{(u, u) \mid u \in A^J\}$

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Exemples of semiring interpretations

$(\mathbb{B}ool, ff \leq tt)$	$[\![ff]\!] := \{0\}$, $[\![tt]\!] := \mathbb{N}$	$A^{ff} = \{[]\}$	$A^{tt} = !A$
$(\mathbb{B}ool, id)$	$[\![ff]\!] := \{0\}$, $[\![tt]\!] := [1, \infty[$	$A^{ff} = \{[]\}$	$A^{tt} = !A - \{[]\}$
$(\mathbb{N}, id)$	$[\![n]\!] := \{n\}$	$A^n = \{[a_1, ..., a_n]\}$	
$(\mathbb{N}, \leq_\mathbb{N})$	$[\![n]\!] := [0, n]$	$A^n = \{[a_1, ..., a_m] \mid m \leq n\}$	

The model $\mathrm{Rel}^{\mathcal{R}}$ of LL [Carraro&Al]

Multiplicity semiring $\mathcal{R}$

A semiring with coalgebraic constraints (associativity of additive splitting...)

$Rel^{\mathcal{R}}$: a model of ILL

Exponential: $!A = \mathcal{R}_f\langle A\rangle$

the set of finitely supported functions from A to $\mathcal{R}$.

(It is a generalization of $\mathrm{Rel}^{\mathbb{N}}$ since $\mathcal{M}_f(A) = \mathbb{N}_f\langle A\rangle$.)

Interpretation $[\![-]\!] : \mathcal{S} \to \mathcal{P}(\mathcal{R})$

Similar to the interpretation in $\mathcal{P}(\mathbb{N})$.

Universal interpretation

$[\![.]\!] : \mathcal{S} \to \mathcal{P}(\mathcal{M}_f(\mathcal{S}_*))$

$Rel^{\mathcal{R}}$: a model of $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$

Bounded exponential: $A^J = \{u \in \mathcal{R}_f\langle A\rangle \mid \Sigma_{a\in A} u(a) \in [\![J]\!]\}$

epimorphism: $\varrho_{J,A} = \{(u,u) \mid u \in A^J\}$

Theorems and conclusion

Stratification

Stratifying models of LL gives models of $B_{\mathcal{S}}$LL.

Universal construction

Any bounded logic $B_{\mathcal{S}}$LL has as model a stratification of $\mathrm{REL}^{\mathcal{M}_f(\mathcal{S}_*)}$.

(but it may have other more accurate relational models)

The internal semirings $\mathcal{P}(\mathbb{N})$ and $\mathcal{P}(\mathcal{R})$

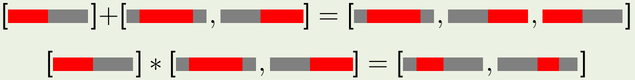
More details in my thesis.

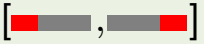
Examples and interest

Counting resources

- $\mathbb{N}$: if $t : A^{n} \multimap U$ then a CbNAM on $(t\ s)$ will evaluate at most n times the program s.
- $\mathbb{P}oly$: if t is typable then $(t\ s)$ has a polynomial head reduction on the size on s.
- $\mathbb{R}^{+}$: if $t : A^{r} \multimap U$ then the expected value over the number of evaluations of s in the execution of $(t\ s)$ is at most r (in presence of probabilistic operations).

A more complicated example: Ghica&Smith's



This computes the sequentially of an execution (with scheduling op).
A term $t : A$$\multimap B$ will use its argument two times: once during the first third of the execution time and once during the last third.

The operations $\oplus$ and $\odot$

The cs-semiring $\mathcal{P}(\mathbb{N})$

$$p \oplus q = \{m + n \mid \forall m \in p,\ \forall n \in q\}$$
$$p \odot q = \{n_1 + ... + n_m \mid \forall m \in p,\ \forall n_1, ..., n_m \in q\}$$

Not a full semiring

$$(\{1\}\oplus\{1\}) \odot \{1,2\} = \{2,4\}, \quad (\{1\}\odot\{1,2\}) \oplus (\{1\}\odot\{1,2\}) = \{2,3,4\}$$

The cs-semiring $\mathcal{P}(\mathcal{R})$

$$\alpha \oplus \beta := \{p + q \mid p \in \alpha, q \in \beta\},$$
$$\alpha \odot \beta := \left\{\sum_{i=1}^{h} p_i \cdot q_i \mid h \geq 0, \sum_{i=1}^{h} q_i \in \beta, \forall i \leq h, p_i \in \alpha\right\}.$$

Multiplicity [Carraro&Al]

A semiring $\mathcal{R}$ has multiplicities if

(MS1) it is positive: $p+q = 0 \;\Rightarrow\; p = q = 0$

(MS2) it is discreet: $p+q = 1 \;\Rightarrow\; p = 0$ or $q = 0$

(MS3) it has additive splitting properties.

$$p_1+q_1 = p_2+q_2 \;\Rightarrow\; \exists (r_{ij})_{1\leq i,j\leq 2}, p_i = r_{i1}+r_{i2},\; q_j = r_{1j}+r_{2j}.$$

(MS4) it has multiplicative k-ary splitting property:

$$q_1+q_2 = rp \;\Rightarrow \exists k, r_1, \dots, r_k, p_{1,1}, \dots, p_{1,k}, p_{2,1}, \dots, p_{2,k},$$

$$r = \sum_{j\leq k} r_j,\; q_i = \sum_{j\leq k} r_j p_{i,j},\; \forall j \leq k, p = p_{1,j}+p_{2,j}$$

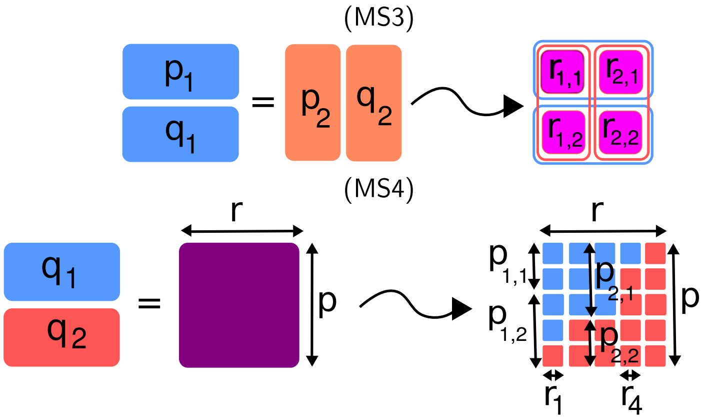
Examples

- $\mathbb{N}$
- $\bar{\mathbb{N}}$
- $\mathcal{M}_f(\mathbb{M})$
- $Poly/_{(X^2=0)}$

$\mathcal{R}$ morally behave like $\mathbb{N}$

- $\mathcal{R}$ contain $\mathbb{N}$ as sub semiring
- The sum in $\mathcal{R}$ is the one of $\mathbb{N}$

Multiplicity [Carraro&Al]



The model $\mathrm{Rel}^{\mathcal{R}}$ of LL [Carraro&Al]

Definition of $\mathcal{R}_f\langle A\rangle$

Given a semiring $\mathcal{R}$ and a set A, the space of functions $f : A \to \mathcal{R}$ of **finite support** is denoted $\mathcal{R}_f\langle A\rangle$. It is a semimodule over $\mathcal{R}$ with:

- A commutative sum: $(f+g)(x) = f(x)+g(x)$
- An external product: $(I\cdot f)(x) = I\cdot f(x)$

$Rel^{\mathcal{R}}$: a model of MELL

Exponential: $!A = \mathcal{R}_f\langle A\rangle$

$$\delta_A = \{(u, V) \mid V \in !!A, u = \Sigma_{v\in !A} V(v)\cdot v\} \qquad \epsilon_A = \{([1\cdot a], a) \mid a \in A\}$$

$$c_A = \{(u, (v, w) \mid u = v+w \in !A\} \quad w_A = \{([], *)\} \quad n = \{(*, [J\cdot *]) | J\in\mathcal{R}\}$$

$$m_{A,B} = \{((u, v), w) \mid \forall a, u(a) = \Sigma_{b\in B} w(a, b), \forall b, v(b) = \Sigma_{a\in A} w(a, b)\}$$

The semiring $\mathcal{M}_f(\mathbb{M})$

Semiring $\mathcal{M}_f(\mathbb{M})$

Given a monoid $\mathbb{M}$, $\mathcal{M}_f(\mathbb{M})$ is a semiring with:

- A commutative sum: $(f+g)(x) = f(x)+g(x)$
- A product (Dirichlet convolution):

$$(f \cdot g)(x) = \sum_{\substack{y,z \\ x=y\cdot z}} f(y)\cdot g(z)$$

Example of such semirings

- $\mathbb{N} = \mathcal{M}_f(\mathbb{B}ool)$
- Ghica's $\mathcal{M}_f(Aff_1^c)$
- $\mathbb{P}oly = \mathcal{M}_f(\mathbb{N}_+)$

$\mathcal{M}_f(\mathbb{M})$ is a multiplicity semiring

Stratifying $Rel^{\mathcal{M}_f(\mathcal{S}_*)}$ into a model of $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$

$$[\![\cdot]\!] : \begin{pmatrix} \mathcal{S} & \rightarrow & \mathcal{M}_f(\mathcal{S}_*) \\ J & \mapsto & \left\{ [K_1, ..., K_n] \;\middle|\; \Sigma_i K_i \leq J \right\} \end{pmatrix}$$